Outline

1. General Info

2. 5.5 Exponential Random Variables

3. 5.6 Other Absolutely Continuous Random Variables
HW6 is due Friday, 10/16, at noon. Please submit your HW via the course Moodle page. Make make that your HW is uploaded successfully.

27 of you took Test 1. The median score for Test 1 is 85. Solution to Test 1 is on my homepage now.
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For any $\lambda > 0$, the function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is a probability density. It is called an exponential density with parameter $\lambda$.

A random variable $X$ is called an exponential random variable with parameter $\lambda > 0$ if it is an absolutely continuous random variable whose density is an exponential density with parameter $\lambda$. 
For any $\lambda > 0$, the function

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A random variable $X$ is called an exponential random variable with parameter $\lambda > 0$ if it is an absolutely continuous random variable whose density is an exponential density with parameter $\lambda$. 
If $X$ is an exponential random variable with parameter $\lambda > 0$, then for any $x \geq 0$,

$$P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x},$$

$$P(X > x) = e^{-\lambda x}.$$ 

Thus the distribution function of $X$ is

$$F(x) = \begin{cases} 
1 - e^{-\lambda x}, & x \geq 0 \\
0, & x < 0. 
\end{cases}$$

If $X$ is an exponential random variable with parameter $\lambda > 0$, then

$$E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.$$
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$$E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.$$
\[ E[X] = \int_0^\infty x \lambda e^{-\lambda x} \, dx = \int_0^\infty xd(-e^{-\lambda x}) \]
\[ = -xe^{-\lambda x}\bigg|_0^\infty + \int_0^\infty e^{-\lambda x} \, dx \]
\[ = 0 - \frac{1}{\lambda}e^{-\lambda x}\bigg|_0^\infty = \frac{1}{\lambda}, \]

For \( n > 1 \),
\[ E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} \, dx = \int_0^\infty x^n d(-e^{-\lambda x}) \]
\[ = -x^n e^{-\lambda x}\bigg|_0^\infty + \int_0^\infty nx^{n-1} e^{-\lambda x} \, dx \]
\[ = \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} \, dx = \frac{n}{\lambda} E[X^{n-1}]. \]
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Example

Suppose that the length $X$ of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/5$. Find the probability that the phone call will (a) last more than 5 minutes; (b) last between 5 and 10 minutes.

\[
P(X > 5) = \int_5^\infty \frac{1}{5} e^{-x/5} dx = e^{-1}.
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\[
P(5 \leq X \leq 10) = \int_5^{10} \frac{1}{5} e^{-x/5} dx = e^{-1} - e^{-2}.
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Suppose $X$ is an exponential random variable with parameter $\lambda > 0$. For any $s, t > 0$,

$$P(X > s + t | X > t) = \frac{P(X > s + t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

This property is called the memoryless property. Any exponential random variable satisfies the memoryless property.

The memoryless property is equivalent to

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$$P(X > s + t) = P(X > s)P(X > t), \quad s, t > 0.$$
It can be shown that if $g$ is a non-negative right continuous function on $(0, \infty)$ taking values in $(0, 1)$ such that

$$g(s + t) = g(s)g(t), \quad s, t > 0,$$

then there exists $\lambda > 0$ such that

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Thus if a random variable satisfies the memoryless property, it must be an exponential random variable. Thus exponential random variables are very important in applications.
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If the lifetime of a certain object, like a light bulb or computer chip, has the memoryless property, then we can use the exponential distribution to model the lifetime. The lifetime of a car usually does not satisfy the memoryless property, thus it is not reasonable to use an exponential random variable to model the lifetime of a car.

**Example 2**

Suppose that $X$ an exponential random variable with parameter $\lambda > 0$. Define a new random variable $Y$ as follows: $Y = n$ when $X \in (n - 1, n]$, $n = 1, 2, \ldots$. Find the mass function of $Y$. 
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For $n = 1, 2, \ldots$,

$$P(Y = n) = P(n - 1 < X \leq n) = e^{-\lambda(n-1)} - e^{-\lambda n}$$
$$= e^{-\lambda(n-1)}(1 - e^{-\lambda}).$$

Thus $Y$ is a geometric random variable with parameter $p = 1 - e^{-\lambda}$.

Geometric random variables are the discrete counterpart of exponential random variables. Exponential random variables are the continuous counterpart of geometric random variables.
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Example 3

Suppose that the lifetime $X$ of a light bulb in months is an exponential random variable with parameter $\lambda = 1/12$. If the light bulb has been working for 12 months, find the probability that it will work for another 12 months.

By the memoryless property

$$P(X > 24|X > 12) = P(X > 12) = e^{-1}.$$
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For any $\alpha > 0$, we define

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$ 

For any $\alpha > 0$, $\Gamma(\alpha) \in (0, \infty)$. But we do not know the value of $\Gamma(\alpha)$ in general. We do know that $\Gamma(1) = 1$.

We claim that, for any $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$. Combining this with $\Gamma(1) = 1$, we immediately get $\Gamma(n) = (n - 1)!$ for all $n \geq 1$. 
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\[ \Gamma(\alpha + 1) = \int_0^\infty y^{\alpha} e^{-y} \, dy = \int_0^\infty y^{\alpha} d(-e^{-y}) \]
\[ = - \left. y^{\alpha} e^{-y} \right|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y} \, dy \]
\[ = \alpha \Gamma(\alpha). \]

By using a simple change of variables, one can check that, for any \( \alpha > 0 \) and \( \lambda > 0 \), the function

\[ f(x) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}, & x \geq 0 \\
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is a probability density. It is called a Gamma density with parameters \((\alpha, \lambda)\).
\[ \Gamma(\alpha + 1) = \int_0^\infty y^\alpha e^{-y} dy = \int_0^\infty y^\alpha d(-e^{-y}) \]

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