9th Homework Set — Solutions
Chapter 6

Problem 6.38 (a) $P\{X = i, Y = j\} = \frac{1}{5i}$ for $i = 1, \ldots, 5$ and $j = 1, \ldots, i$, 0 otherwise.

$P\{X = i, Y = j\}$	Y=1	Y=2	Y=3	Y=4	Y=5	$P\{X = i\}$
X=1	$\frac{1}{5}$	0	0	0	0	$\frac{1}{5}$
X=2	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	$\frac{1}{5}$
X=3	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	0	0	$\frac{1}{5}$
X=4	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{5}$
X=5	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{5}$
$P\{Y = j\}$	$\frac{137}{300}$	$\frac{77}{300}$	$\frac{47}{300}$	$\frac{9}{100}$	$\frac{1}{25}$	1

(b) $P\{X = i \mid Y = j\} = \frac{\frac{1}{5i}}{\sum_{k=i}^{5} \frac{1}{5k}}$

$P\{X = i \mid Y = j\}$	Y=1	Y=2	Y=3	Y=4	Y=5
$X = 1$	$\frac{60}{137}$	0	0	0	0
$X = 2$	$\frac{30}{137}$	$\frac{30}{77}$	0	0	0
$X = 3$	$\frac{20}{137}$	$\frac{20}{77}$	$\frac{20}{47}$	0	0
$X = 4$	$\frac{15}{137}$	$\frac{15}{77}$	$\frac{15}{47}$	$\frac{5}{9}$	0
$X = 5$	$\frac{12}{137}$	$\frac{12}{77}$	$\frac{12}{47}$	$\frac{4}{9}$	1

(c) No.

Problem 6.40

$p(i,j)$	$j = 1$	$j = 2$	
$i = 1$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
$i = 2$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
	$\frac{1}{4}$	$\frac{3}{4}$	1

(a)

$P\{X = i \mid Y = j\}$	$j = 1$	$j = 2$
$i = 1$	$\frac{1}{2}$	$\frac{1}{3}$
$i = 2$	$\frac{1}{2}$	$\frac{2}{3}$

(b) No.

(c) $P\{XY \leq 3\} = 1 - p(2,2) = \frac{1}{2}$

$P\{X + Y > 2\} = 1 - p(1,1) = \frac{7}{8}$

$P\left\{\frac{X}{Y} > 1\right\} = p(2,1) = \frac{1}{8}$

Problem 6.41 Let X and Y be jointly continuous with density function $f(x,y) = xe^{-x(y+1)}$ for $x > 0, y > 0$. Note that $f_X(x) = \int_0^\infty f(x,y)dy = e^{-x}$ for $x > 0$, and $f_Y(y) = \int_0^\infty f(x,y)dx = \frac{1}{(y+1)^2}$ for $y > 0$.

(a) $f_{X|Y}(x|y) = (y+1)^2 xe^{-x(y+1)}$ for $x > 0, y > 0$, 0 otherwise, and $f_{Y|X}(y|x) = xe^{-xy}$ for $x > 0, y > 0$.

(b) Let $Z = XY$. Then for $a > 0$,

$$F_Z(a) = P\{XY < a\} = \int_0^\infty \int_0^{\frac{a}{x}} xe^{-x(y+1)}dydx = 1 - e^{-a}.$$

Hence, $f_Z(a) = \frac{d}{da}F_Z(a) = e^{-a}$ for $a > 0$, 0 otherwise.

Problem 6.42 Let X and Y be jointly continuous with density function

$$f(x,y) = c(x^2 - y^2)e^{-x}$$

for $0 \le x < \infty, -x \le y \le x$. For $x > 0$, we have

$$f_X(x) = \int_{-x}^{x} c(x^2 - y^2)e^{-x}dy = \frac{4c}{3}x^3 e^{-x}.$$

Hence, $f_{Y|X}(y|x) = \frac{3}{4}\frac{x^2-y^2}{x^3}$ for $-x < y < x$, 0 otherwise. We conclude that

$$F_{Y|X}(y|x) = \begin{cases} 0 & y \le -x \\ \frac{3}{4}\int_{-x}^{y} \frac{x^2-y^2}{x^3}dy = \frac{1}{4}\left(\frac{y(3x^2-y^2)}{x^3} + 2\right) & -x < y < x \\ 1 & x \le y. \end{cases}$$

Problem 6.48 Let $X_1, \ldots, X_5$ be independent exponential random variables with parameter λ.

(a)

$$\begin{aligned} P\{\min(X_1,\ldots,X_5) \le a\} &= 1 - P\{\min(X_1,\ldots,X_5) > a\} \\ &= 1 - P\{X_1 > a, \ldots, X_5 > a\} \\ &= 1 - P\{X_1 > a\} \cdots P\{X_5 > a\} \\ &= \begin{cases} 1 - (e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned}
P\{\max(X_1,\dots,X_5)\le a\} &= P\{X_1\le a,\dots,X_5\le a\}\\
&= P\{X_1\le a\}\cdots P\{X_5\le a\}\\
&= \begin{cases}(1-e^{-\lambda a})^5 & a>0\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Chapter 7

Problem 7.5 If (X,Y) is the location of the accident, then X and Y are uniform random variables on $(-\frac{3}{2},\frac{3}{2})$. Let $D=|X|+|Y|$. Then

$$\begin{aligned}
E[D] &= E[|X|]+E[|Y|]=2E[|X|]\\
&= 2\int_{-\frac{3}{2}}^{\frac{3}{2}}\frac{|x|}{3}dx=\frac{4}{3}\int_0^{\frac{3}{2}}xdx\\
&= \frac{4}{3}\cdot\frac{9}{8}=\frac{3}{2}.
\end{aligned}$$

Problem 7.6 Let X_i be the outcome of the i-th roll of the die, for $i=1,\dots,10$, and note that $E[X_i]=\frac{7}{2}$. Let $X=X_1+\cdots+X_{10}$. Now $E[X]=E[X_1]+\cdots+E[X_{10}]=10E[X_1]=35$.

Problem 7.7 (a) Let X_i be one if both A and B choose the i-th object, for $i=1,\dots,10$. Then $E[X_i]=P\{X_i=1\}=\left(\frac{3}{10}\right)^2=\frac{9}{100}$. Now, the expected number of objects chosen by both A and B is $E[X_1]+\cdots+E[X_{10}]=10E[X_1]=0.9$.

(b) Let Y_i be one if neither A nor B choose the i-th object. Then $E[Y_i]=P\{Y_i=1\}=\left(\frac{7}{10}\right)^2=\frac{49}{100}$, so that $E[Y_1+\cdots+Y_{10}]=10E[Y_1]=4.9$.

(c) Let Z_i be one if either A or B (but not both) chooses the i-th object. Then $E[Z_i]=P\{Z_i=1\}=2\frac{3}{10}\frac{7}{10}=\frac{21}{50}$. Now, $E[Z_1+\cdots+Z_{10}]=10E[Z_1]=\frac{21}{5}=4.2$.

Problem 7.8 Following the hint, let X_1 be one if the i-th arrival sits at a previously unoccupied table. Then $E[X_i]=P\{X_i=1\}=(1-p)^{i-1}$, so that

$$E[X_1+\cdots+X_N]=\sum_{i=1}^{N}(1-p)^{i-1}=\frac{1-(1-p)^N}{1-(1-p)}=\frac{1-(1-p)^N}{p}.$$

Problem 7.11 Let X_i be one if the i-th outcome differs from the $(i-1)$-th outcome, for $i = 2, \ldots, n$. We have $E[X_i] = P\{X_i = 1\} = 2p(1-p)$, so that $E[X_2 + \cdots + X_n] = 2(n-1)p(1-p)$.

Problem 7.18 Let X_i be one if the i-th card is a match, for $i = 1, \ldots, 13$, and let $X = X_1 + \cdots + X_{52}$. Then $P\{X_i = 1\} = \frac{1}{13}$, so that $E[X] = 52E[X_1] = \frac{52}{13} = 4$.

Problem 7.19

(a) If X is the number of insects caught before a type 1 catch, then $(X+1)$ is geometric with parameter P_1, so that $E[X] = \frac{1}{P_1} - 1$.

(b) Let Y_i be one if an insect of type i is caught before an insect of type 1, for $i = 2, \ldots, r$. Then $Y = Y_2 + \cdots + Y_r$ is the number of insects caught before an insect of type 1. We have $E[Y_i] = P\{Y_i = 1\} = \frac{P_i}{P_i + P_1}$, so that

$$E[Y] = \sum_{i=2}^{r} \frac{P_i}{P_i + P_1}.$$

Problem 7.21

(a) Let X be the number of days of the year that are birthdays of exactly 3 people. For $i = 1, \ldots, 365$, let $X_i = 1$ if the i-day is the birthday of eaxctly 3 people and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$. Since for each i,

$$EX_i = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97},$$

we get that

$$EX = 365 \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}.$$

(b) Let Y be the number of distinct birthdays. For $i = 1, \ldots, 365$, let $Y_i = 1$ if the i-day is someone's birthday and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$. Since for each i,

$$EY_i = P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \left(\frac{364}{365}\right)^{100},$$

we get that

$$EY = 365 \left[1 - \left(\frac{364}{365}\right)^{100}\right].$$