Problem 4.84 (a) For $i = 1, \ldots, 5$, let $X_i = 1$ if the $i$-th box is empty and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_5$ is the number of empty boxes. For $i = 1, \ldots, 5$,

$$E[X_i] = P(X_i = 1) = (1 - p_i)^{10}.$$  

Thus

$$E[X] = E[X_1] + \cdots + E[X_5] = \sum_{i=1}^{5} (1 - p_i)^{10}.$$  

(b) For $i = 1, \ldots, 5$, let $Y_i = 1$ if the $i$-th box has exactly 1 ball and $Y_i = 0$ otherwise. Then $Y = Y_1 + \cdots + Y_5$ is the number of boxes that have exactly 1 ball. For $i = 1, \ldots, 5$,

$$E[Y_i] = P(Y_i = 1) = 10p_i (1 - p_i)^9.$$  

Thus

$$E[Y] = E[Y_1] + \cdots + E[Y_5] = \sum_{i=1}^{5} 10p_i (1 - p_i)^9.$$  

Problem 4.85 For $i = 1, \ldots, k$, let $X_i = 1$ if the $i$-th type appear at least once in the set of $n$ coupons. Then $X = X_1 + \cdots + X_k$ is the number of distinct types that appear in this set. For $i = 1, \ldots, k$,

$$E[X_i] = P(X_i = 1) = 1 - P(X_i = 0) = 1 - (1 - p_i)^{n}.$$  

Thus

$$E[X] = E[X_1] + \cdots + E[X_k] = k - \sum_{i=1}^{k} (1 - p_i)^{n}.$$  

Chapter 5

Problem 5.1 (a) We have $1 = \int_{-1}^{1} c(1 - x^2)dx = cx \left(1 - \frac{x^2}{3}\right)|_{-1}^{1} = \frac{4}{3} c$, so that $c = \frac{3}{4}$.
(b) We have \( \int_{-1}^{x} f(y) \, dy = \frac{3}{4}y \left( 1 - \frac{y^2}{3} \right) \big|_{-1}^{x} = \frac{1}{2} + \frac{3}{4}x \left( 1 - \frac{x^2}{3} \right) \) if \(-1 \leq x \leq 1\). Hence,

\[
F(x) = \begin{cases} 
0 & x < -1, \\
\frac{1}{2} + \frac{3}{4}x \left( 1 - \frac{x^2}{3} \right) & -1 \leq x \leq 1, \\
1 & x > 1.
\end{cases}
\]

Problem 5.2 Determine \( C \):

\[
\int_{0}^{\infty} xe^{-\frac{x}{2}} \, dx = -2xe^{-\frac{x}{2}} \big|_{0}^{\infty} + \int_{0}^{\infty} 2e^{-\frac{x}{2}} \, dx = (-2x - 4)e^{-\frac{x}{2}} \big|_{0}^{\infty} = 4,
\]

so that \( C = \frac{1}{4} \).

Now, we have \( P \{ X \geq 5 \} = \int_{5}^{\infty} \frac{1}{4}xe^{-\frac{x}{2}} = -\left( \frac{5}{2} + 1 \right) e^{-\frac{5}{2}} \big|_{5}^{\infty} = \frac{7}{2} e^{-\frac{5}{2}} \).

Problem 5.4 (a) \( P \{ X > 20 \} = \int_{20}^{\infty} \frac{10}{x^2} \, dx = -\frac{10}{x} \big|_{20}^{\infty} = \frac{1}{2} \).

(b)

\[
F(x) = \begin{cases} 
0 & x < 10, \\
1 - \frac{10}{x} & x \geq 10.
\end{cases}
\]

(c) Let’s assume that lifetimes of the six devices are independent of each other. Let \( p = 1 - F(15) \). Then the desired probability is

\[
\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.
\]

Problem 5.5 We want to find \( C \) such that \( F(C) \geq 0.99 \). We have \( F(C) = \int_{0}^{C} 5(1 - x)^4 \, dx = -5(1 - x)^5 \big|_{0}^{C} = 1 - (1 - C)^5 \). We want \( 1 - (1 - C)^5 \geq 0.99 \), i.e., \( (1 - C)^5 \leq 0.01 \), hence \( C \geq 1 - (0.01)^{1/5} \).

Problem 5.6 (a)

\[
E[X] = \int_{-\infty}^{\infty} xf(x) \, dx = \frac{1}{4} \int_{0}^{\infty} x^2 e^{-\frac{x}{2}} \, dx = \frac{1}{4} (-2x^2 - 8x - 16) e^{-\frac{x}{2}} \big|_{0}^{\infty} = 4
\]

(b) \( E[X] = \int_{-1}^{1} c(1 - x^2) \, dx = 0 \) by symmetry
Problem 5.10 (a) Let $X$ be uniform on $[0, 60]$. Then

\[
P(\text{passenger goes to } A) = P\{5 \leq X < 15\} + P\{20 \leq X < 30\} P\{35 \leq X < 45\} + P\{50 \leq X < 60\} = \frac{2}{3}.
\]

(b) Same as above.

Problem 5.12 If service stations are located in $A$, $B$, and the center, then the distance between two service stations is 50 miles, so that the expected distance from a service station at the time of a breakdown is

\[
\frac{1}{50} \left( \int_0^{25} x \, dx + \int_{25}^{50} (50 - x) \, dx \right) = \frac{1}{50} \left( \frac{25^2}{2} + 25 \cdot 50 - \frac{50^2}{2} + \frac{25^2}{2} \right) = 12.5.
\]

If the service stations are located at mile 25, 50, and 75, then the expected distance from a station at the time of a breakdown is

\[
\frac{1}{50} \left( \int_0^{25} x \, dx + \int_{25}^{37.5} (x - 25) \, dx + \int_{37.5}^{50} (50 - x) \, dx \right)
\]

\[
= \frac{1}{50} \left( \frac{25^2}{2} + 2 \cdot \frac{12.5^2}{2} \right) = 9.375.
\]

The second strategy is more efficient.

Problem 5.13 (a) $P\{X > 10\} = \frac{2}{3}$

(b) $P\{X > 25|X > 15\} = \frac{P\{X > 25\}}{P\{X > 15\}} = \frac{2}{\frac{5}{2}} = \frac{4}{5} = \frac{1}{3}$.

Problem 5.15 (a) $P\{X > 5\} = P\left\{ \frac{X-10}{6} > \frac{5-10}{6} \right\} = 1 - \Phi\left(\frac{-5}{6}\right) = \Phi\left(\frac{5}{6}\right) = 0.7977$

(b)

\[
P\{4 < X < 16\} = P\left\{ -1 < \frac{X - 10}{6} < 1 \right\} = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827
\]
\[ P \{ X < 8 \} = P \left\{ \frac{X - 10}{6} < -\frac{1}{3} \right\} \]
\[ = \Phi \left( -\frac{1}{3} \right) = 1 - \Phi \left( \frac{1}{3} \right) = 0.3695 \]

(d) \[ P \{ X < 20 \} = P \left\{ \frac{X - 10}{6} < -\frac{10}{6} \right\} = \Phi \left( \frac{5}{3} \right) = 0.9522 \]

(e) \[ P \{ X > 16 \} = P \left\{ \frac{X - 10}{6} > 1 \right\} = 1 - \Phi (1) = 0.1587 \]

Problem 5.18 We have \[ P \{ X > 9 \} = P \left\{ \frac{X - 5}{\sigma} > \frac{4}{\sigma} \right\} = 1 - \Phi \left( \frac{4}{\sigma} \right) = 0.2, \]
so that \[ \Phi \left( \frac{4}{\sigma} \right) = 0.8, \]
and hence \[ \frac{4}{\sigma} = 0.85. \] This implies that \[ \sigma = 4.7059, \]
so that the variance is \[ \sigma^2 = 22.145. \]

Problem 5.21 Let \( X \) be a normal random variable with \( \mu = 71 \) and \( \sigma^2 = 6.25. \)
Then \[ P \{ X > 74 \} = P \left\{ \frac{X - 71}{2.5} > \frac{3}{2.5} \right\} = 1 - \Phi \left( \frac{6}{5} \right) = 0.1151. \] Moreover,
\[ P \{ X > 77 \mid X \geq 72 \} = \frac{P \left\{ \frac{X - 71}{2.5} > \frac{6}{2.5} \right\}}{P \left\{ \frac{X - 71}{2.5} \geq \frac{3}{2.5} \right\}} = \frac{1 - \Phi \left( \frac{12}{5} \right)}{1 - \Phi \left( \frac{6}{5} \right)} = 0.024. \]

Problem 5.22 Let \( X \) be normal with \( \mu = 0.9 \) and \( \sigma = 0.003. \)

(a) \[ P \{ |X - 0.9| > 0.005 \} = P \left\{ \frac{|X - 0.9|}{0.003} > \frac{5}{3} \right\} = 2 - 2\Phi \left( \frac{5}{3} \right) = 0.095. \]

(b) We want \[ P \left\{ \frac{|X - 0.9|}{\sigma} > 0.005 \right\} = 2 - 2\Phi \left( \frac{0.005}{\sigma} \right) \leq 0.01, \]
so that \( \Phi \left( \frac{0.005}{\sigma} \right) \geq 0.995, \)
and hence \( \frac{0.005}{\sigma} \geq 2.58, \)
and \( \sigma = 0.0019. \)

Problem 5.23 Let \( X \) be the number of times the number six appears.
\[ P \{ 149.5 < X < 200.5 \} \]
\[ = P \left\{ \frac{149.5 - 1000}{\sqrt{5000/36}} < \frac{X - 1000}{\sqrt{5000/36}} < \frac{200.5 - 5000}{\sqrt{5000/36}} \right\} \]
\[ = \Phi (2.87) + \Phi (1.46) - 1 = 0.9258. \]

\[ P \{ X < 149.5 \} = P \left\{ \frac{X - 800}{\sqrt{230/25}} < \frac{149.5 - 800}{\sqrt{320025}} \right\} = 1 - \Phi (0.92) = 0.1762. \]

Problem 5.25 Let \( X \) be a binomial random variable with \( p = 0.05 \) and \( n = 150. \)
Then \[ P \{ X \leq 10 \} = P \{ X \leq 10.5 \} = P \left\{ \frac{X - 7.5}{\sqrt{7.125}} \leq \frac{10.5 - 7.5}{\sqrt{7.125}} \right\} = \Phi (1.1239) = 0.8695, \]
using DeMoivre-Laplace.
Problem 5.28 Let $X$ be the number of lefthanders. Then $X$ is binomial with $p = 0.12$ and $n = 200$. Then

$$P \{X \geq 20\} = P \{X > 19.5\}$$

$$= P \left\{ \frac{X - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}} > \frac{19.5 - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}} \right\}$$

$$= 1 - \Phi(-0.9792) = \Phi(0.9792) = 0.8363.$$