Sixth Homework Set — Solutions

Chapter 4

Problem 4.84 (a) For $i = 1, \ldots, 5$, let $X_i = 1$ if the i-th box is empty and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_5$ is the number of empty boxes. For $i = 1, \ldots, 5$,

$$E[X_i] = P(X_i = 1) = (1 - p_i)^{10}.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_5] = \sum_{i=1}^{5} (1 - p_i)^{10}.$$

(b) For $i = 1, \ldots, 5$, let $Y_i = 1$ if the i-th box has exactly 1 ball and $Y_i = 0$ otherwise. Then $Y = Y_1 + \cdots + Y_5$ is the number of boxes that have exactly 1 ball. For $i = 1, \ldots, 5$,

$$E[Y_i] = P(Y_i = 1) = 10 p_i (1 - p_i)^{9}.$$

Thus

$$E[Y] = E[Y_1] + \cdots + E[Y_5] = \sum_{i=1}^{5} 10 p_i (1 - p_i)^{9}.$$

Problem 4.85 For $i = 1, \ldots, k$, let $X_i = 1$ if the i-th type appear at least once in the set of n coupons. Then $X = X_1 + \cdots + X_k$ is the number of distinct types that appear in this set. For $i = 1, \ldots, k$,

$$E[X_i] = P(X_i = 1) = 1 - P(X_i = 0) = 1 - (1 - p_i)^n.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_k] = k - \sum_{i=1}^{k} (1 - p_i)^n.$$

Chapter 5

Problem 5.1 (a) We have $1 = \int_{-1}^{1} c(1 - x^2)dx = cx\left(1 - \frac{x^2}{3}\right)\Big|_{-1}^{1} = \frac{4}{3}c$, so that $c = \frac{3}{4}$.

(b) We have $\int_{-1}^{x} f(y)dy = \frac{3}{4}y\left(1-\frac{y^2}{3}\right)|_{-1}^{x} = \frac{1}{2}+\frac{3}{4}x\left(1-\frac{x^2}{3}\right)$ if $-1 \leq x \leq 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{1}{2}+\frac{3}{4}x\left(1-\frac{x^2}{3}\right) & -1 \leq x \leq 1, \\ 1 & x > 1. \end{cases}$$

Problem 5.2 Determine C:

$$\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}}|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x-4)e^{-\frac{x}{2}}|_0^\infty = 4,$$

so that $C = \frac{1}{4}$.

Now, we have $P\{X \geq 5\} = \int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -(\frac{x}{2}+1)e^{-\frac{x}{2}}|_5^\infty = \frac{7}{2}e^{-\frac{5}{2}}$

Problem 5.4 (a) $P\{X > 20\} = \int_{20}^\infty \frac{10}{x^2}dx = -\frac{10}{x}|_{20}^\infty = \frac{1}{2}$.

(b)

$$F(x) = \begin{cases} 0 & x < 10 \\ 1-\frac{10}{x} & x \geq 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is

$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.$$

Problem 5.5 We want to find C such that $F(C) \geq 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5|_0^C = 1-(1-C)^5$. We want $1-(1-C)^5 \geq 0.99$, i.e., $(1-C)^5 \leq 0.01$, hence $C \geq 1-(0.01)^{0.2}$.

Problem 5.6 (a)

$$\begin{aligned} E[X] &= \int_{-\infty}^\infty xf(x)dx = \frac{1}{4}\int_0^\infty x^2e^{-\frac{x}{2}}dx \\ &= \frac{1}{4}\left(-2x^2-8x-16\right)e^{-\frac{x}{2}}|_0^\infty = 4 \end{aligned}$$

(b) $E[X] = \int_{-1}^{1} c(1-x^2)x dx = 0$ by symmetry

(c) $E[X] = \int_5^\infty x\frac{5}{x^2}dx = \int_5^\infty \frac{5}{x} = \infty$

Problem 5.10 (a) Let X be uniform on $[0, 60]$. Then

$$\begin{aligned}
&P\left(\text{passenger goes to } A\right) \\
&= P\{5 \le X < 15\} + P\{20 \le X < 30\} P\{35 \le X < 45\} \\
&\quad + P\{50 \le X < 60\} \\
&= \frac{2}{3}.
\end{aligned}$$

(b) Same as above.

Problem 5.12 If service stations are located in A, B, and the center, then the distance between two service stations is 50 miles, so that the expected distance from a service station at the time of a breakdown is

$$\begin{aligned}
&\frac{1}{50}\left(\int_0^{25} x dx + \int_{25}^{50} (50 - x) dx\right) \\
&= \frac{1}{50}\left(\frac{25^2}{2} + 25 \cdot 50 - \frac{50^2}{2} + \frac{25^2}{2}\right) = 12.5.
\end{aligned}$$

If the service stations are located at mile 25, 50, and 75, then the expected distance from a station at the time of a breakdown is

$$\begin{aligned}
&\frac{1}{50}\left(\int_0^{25} x dx + \int_{25}^{37.5} (x - 25) x dx + \int_{37.5}^{50} (50 - x) dx\right) \\
&= \frac{1}{50}\left(\frac{25^2}{2} + 2\frac{12.5^2}{2}\right) = 9.375.
\end{aligned}$$

The second strategy is more efficient.

Problem 5.13 (a) $P\{X > 10\} = \frac{2}{3}$

(b) $P\{X > 25 | X > 15\} = \frac{P\{X>25\}}{P\{X>15\}} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$.

Problem 5.15 (a) $P\{X > 5\} = P\left\{\frac{X-10}{6} > \frac{5-10}{6}\right\} = 1 - \Phi\left(-\frac{5}{6}\right) = \Phi\left(\frac{5}{6}\right) = 0.7977$

(b)

$$\begin{aligned}
&P\{4 < X < 16\} = P\left\{-1 < \frac{X - 10}{6} < 1\right\} = \Phi(1) - \Phi(-1) \\
&= 2\Phi(1) - 1 = 0.6827
\end{aligned}$$

(c)

$$P\{X<8\} = P\left\{\frac{X-10}{6} < -\frac{1}{3}\right\}$$
$$= \Phi\left(-\frac{1}{3}\right) = 1-\Phi\left(\frac{1}{3}\right) = 0.3695$$

(d) $P\{X<20\} = P\left\{\frac{X-10}{6} < -\frac{10}{6}\right\} = \Phi\left(\frac{5}{3}\right) = 0.9522$

(e) $P\{X>16\} = P\left\{\frac{X-10}{6} > 1\right\} = 1-\Phi(1) = 0.1587$

Problem 5.18 We have $P\{X>9\} = P\left\{\frac{X-5}{\sigma} > \frac{4}{\sigma}\right\} = 1-\Phi\left(\frac{4}{\sigma}\right) = 0.2$, so that $\Phi\left(\frac{4}{\sigma}\right) = 0.8$, hence $\frac{4}{\sigma} = 0.85$. This implies that $\sigma = 4.7059$, so that the variance is $\sigma^2 = 22.145$.

Problem 5.21 Let X be a normal random variable with $\mu = 71$ and $\sigma^2 = 6.25$. Then $P\{X>74\} = P\left\{\frac{X-71}{2.5} > \frac{3}{2.5}\right\} = 1-\Phi\left(\frac{6}{5}\right) = 0.1151$. Moreover, $P\{X>77|X\geq 72\} = \frac{P\left\{\frac{X-71}{2.5} > \frac{6}{2.5}\right\}}{P\left\{\frac{X-71}{2.5} \geq \frac{1}{2.5}\right\}} = \frac{1-\Phi\left(\frac{12}{5}\right)}{1-\Phi\left(\frac{2}{5}\right)} = 0.024$.

Problem 5.22 Let X be normal with $\mu = 0.9$ and $\sigma = 0.003$.

(a) $P\{|X-0.9| > 0.005\} = P\left\{\frac{|X-0.9|}{0.003} > \frac{5}{3}\right\} = 2-2\Phi\left(\frac{5}{3}\right) = 0.095$.

(b) We want $P\left\{\frac{|X-0.9|}{\sigma} > 0.005\right\} = 2-2\Phi\left(\frac{0.005}{\sigma}\right) \leq 0.01$, hence $\Phi\left(\frac{0.005}{\sigma}\right) \geq 0.995$, so that $\frac{0.005}{\sigma} \geq 2.58$, hence $\sigma = 0.0019$.

Problem 5.23 Let X be the number of times the number six appears.

$$P\{149.5 < X < 200.5\}$$
$$= P\left\{\frac{149.5-\frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{X-\frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{200.5-\frac{5000}{36}}{\sqrt{\frac{5000}{36}}}\right\}$$
$$= \Phi(2.87) + \Phi(1.46) - 1 = 0.9258.$$

$$P\{X<149.5\} = P\left\{\frac{X-\frac{800}{5}}{\sqrt{\frac{3200}{25}}} < \frac{149.5-\frac{800}{5}}{\sqrt{320025}}\right\} = 1-\Phi(0.92) = 0.1762.$$

Problem 5.25 Let X be a binomial random variable with $p = 0.05$ and $n = 150$. Then $P\{X\leq 10\} = P\{X\leq 10.5\} = P\left\{\frac{X-7.5}{\sqrt{7.125}} \leq \frac{10.5-7.5}{\sqrt{7.125}}\right\} = \Phi(1.1239) = 0.8695$, using DeMoivre-Laplace.

Problem 5.28 Let X be the number of lefthanders. Then X is binomial with $p = 0.12$ and $n = 200$. Then

$$
\begin{aligned}
&P\{X \geq 20\} = P\{X > 19.5\} \\
&= P\left\{\frac{X - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}} > \frac{19.5 - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}}\right\} \\
&= 1 - \Phi(-0.9792) = \Phi(0.9792) = 0.8363.
\end{aligned}
$$