Fourth Homework Set — Solutions
Chapter 3

Problem 3.81 Using the gambler's ruin formula, the anwser is

$$\frac{1-(9/11)^{15}}{1-(9/11)^{30}}.$$

Problem 3.83 (a) Conditioning on the coin flip

$$P(\text{throw n is red}) = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}.$$

(b)

$$P(R_3|R_1R_2) = \frac{P(R_1R_2R_3)}{P(R_1R_2)} = \frac{\frac{1}{2}(\frac{2}{3})^3 + \frac{1}{2}(\frac{1}{3})^3}{\frac{1}{2}(\frac{2}{3})^2 + \frac{1}{2}(\frac{1}{3})^2} = \frac{3}{5}.$$

(c)

$$P(A|R_1R_2) = \frac{P(R_1R_2|A)P(A)}{P(R_1R_2)} = \frac{(\frac{2}{3})^2\frac{1}{2}}{(\frac{2}{3})^2\frac{1}{2} + (\frac{1}{3})^2\frac{1}{2}} = \frac{4}{5}.$$

Problem 3.84 (a)

$$\begin{aligned}
P(A \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+1) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i}\frac{1}{3} = \frac{1}{3}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{1}{3}\frac{1}{1-\frac{8}{27}} \\
P(B \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+2) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i+1}\frac{1}{3} = \frac{2}{9}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{2}{9}\frac{1}{1-\frac{8}{27}} \\
P(C \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+3) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i+2}\frac{1}{3} = \frac{4}{27}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{4}{27}\frac{1}{1-\frac{8}{27}}
\end{aligned}$$

(b)

$$\begin{aligned}
P(A \text{ win}) &= \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} \\
P(B \text{ win}) &= \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5} \\
P(C \text{ win}) &= \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}.
\end{aligned}$$

Chapter 4

Problem 4.1 Possible values of X: 0,2,4,-1,-2,1 Probabilities:

$$P\{X=0\} = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$P\{X=2\} = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$

$$P\{X=4\} = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$

$$P\{X=-1\} = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

$$P\{X=-2\} = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

$$P\{X=1\} = \frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}} = \frac{32}{91}$$

Problem 4.4

$$P\{X=1\} = \frac{\binom{5}{1}9!}{10!} = \frac{1}{2}$$
$$P\{X=2\} = \frac{\binom{5}{1}\binom{5}{1}8!}{10!} = \frac{5}{18}$$
$$P\{X=3\} = \frac{\binom{5}{2}2!\binom{5}{1}7!}{10!} = \frac{5}{36}$$
$$P\{X=4\} = \frac{\binom{5}{3}3!\binom{5}{1}6!}{10!} = \frac{5}{84}$$
$$P\{X=5\} = \frac{\binom{5}{4}4!\binom{5}{1}5!}{10!} = \frac{5}{252}$$
$$P\{X=6\} = \frac{\binom{5}{5}5!\binom{5}{1}4!}{10!} = \frac{1}{252}$$
$$P\{X=7\} = P\{X=8\} = P\{X=9\} = P\{X=10\} = 0$$

Problem 4.5 The possible values are $n, n-2, n-4, ..., -n+4, -n+2, -n$.

Problem 4.13 Let X be the total dollar value of all sales. Then X can take the values $0, 500, 1000, 1500, 2000$, and we have

$$P\{X=0\} = 0.7 \cdot 0.4 = 0.28$$
$$P\{X=500\} = \frac{1}{2}(0.3 \cdot 0.4 + 0.7 \cdot 0.6) = 0.27$$
$$P\{X=1000\} = \frac{1}{2}(0.3 \cdot 0.4 + 0.7 \cdot 0.6) + \frac{1}{4}0.3 \cdot 0.6 = 0.315$$
$$P\{X=1500\} = 2\frac{1}{4}0.3 \cdot 0.6 = 0.09$$
$$P\{X=2000\} = \frac{1}{4}0.3 \cdot 0.6 = 0.045$$

Problem 4.14

$$\begin{aligned}
P\{X=0\} &= \frac{0!}{2!} = \frac{1}{2} \\
P\{X=1\} &= \frac{1!}{3!} = \frac{1}{6} \\
P\{X=2\} &= \frac{2!}{4!} = \frac{1}{12} \\
P\{X=3\} &= \frac{3!}{5!} = \frac{1}{20} \\
P\{X=4\} &= \frac{4!}{5!} = \frac{1}{5}
\end{aligned}$$

Problem 4.17 (a)

$$\begin{aligned}
P\{X=1\} &= P\{X \leq 1\} - P\{X < 1\} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
P\{X=2\} &= \frac{11}{12} - \frac{3}{4} = \frac{1}{6} \\
P\{X=3\} &= 1 - \frac{11}{12} = \frac{1}{12}
\end{aligned}$$

(b) $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$.

Problem 4.19

$$\begin{aligned}
P\{X=0\} &= \frac{1}{2} \\
P\{X=1\} &= \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \\
P\{X=2\} &= \frac{4}{5} - \frac{3}{5} = \frac{1}{5} \\
P\{X=3\} &= \frac{9}{10} - \frac{4}{5} = \frac{1}{10} \\
P\{X=3.5\} &= 1 - \frac{9}{10} = \frac{1}{10}
\end{aligned}$$

Problem 4.21 (a) $E[X]$ is larger than $E[Y]$ because the random selection of students favors larger busloads.

(b) $E[X] = \frac{40\cdot 40+33\cdot 33+25\cdot 25+50\cdot 50}{40+33+25+50} = \frac{5814}{148} = 39.3$, $E[Y] = \frac{148}{4} = 37$.

Problem 4.23 (a) Suppose that that you use x dollars to buy $x/2$ ounces of the commodity and keep the rest $1000 - x$ dollars as cash, and then sell your commodity at the end of the week. Then the expected amount of money you have at the end of the week is

$$\frac{1}{2}\frac{x}{2} + \frac{1}{2}2x + 1000 - x = 1000 + \frac{x}{4}$$

which is an increasing function of x. Therefore the best strategy is to use all your money to buy 500 ounces of the commodity and then sell at the end of the week.

(b) Suppose that you use x dollars to buy $x/2$ ounces of the commodity at the beginning of the first week and use the remaining $1000 - x$ dollars to buy the commodity after one week, then the expected ounces of the commodity that you own after one week is

$$\frac{x}{2} + \frac{1}{2}(1000 - x) + \frac{1}{2}\frac{1000 - x}{4} == 625 - \frac{x}{8}$$

which is a decreasing function of x. Therefore the best strategy in this case is that you do not immediately buy anything but use all your money after one week to buy the commodity.

Problem 4.32 Let X be the number of tests needed for a group of ten people. Then $X = 1$ or $X = 11$, and $P\{X = 1\} = 0.9^{10} = 0.3487$ and $P\{X = 11\} = 1 - 0.9^{10} = 0.6513$. Hence $E[X] = 7.5132$.

Problem 4.35 Let X be the win/loss after one game. Then $P\{X = 1.1\} = \frac{2\binom{5}{2}}{\binom{10}{2}} = \frac{20}{45} = \frac{4}{9}$, and $P\{X = -1\} = \frac{5}{9}$.

(a) $E[X] = 1.1 \cdot \frac{4}{9} - \frac{5}{9} = -\frac{1}{15}$.

(b) $\mathrm{Var}(X) = E[X^2] - E[X]^2 = 1.21 \cdot \frac{4}{9} + \frac{5}{9} - \frac{1}{225} = 1.0889$.

Problem 4.37

$$\begin{aligned}\mathrm{Var}(X) &= E\left[X^2\right] - E[X]^2 \\ &= \frac{40^3 + 33^3 + 25^3 + 50^3}{148} - \left(\frac{40^2 + 33^2 + 25^2 + 50^2}{148}\right)^2 = 82.2\end{aligned}$$

$$\mathrm{Var}(Y) = \frac{40^2 + 33^2 + 25^2 + 50^2}{4} - 37^2 = 84.5$$

Problem 4.38 Note that $E\left[X^2\right] = \mathrm{Var}\left(X\right) + E\left[X\right]^2 = 5 + 1 = 6$.

(a) $E\left[(2+X)^2\right] = E\left[4 + 4X + X^2\right] = 4 + 4E\left[X\right] + E\left[X^2\right] = 14$.

(b) $\mathrm{Var}\left(4 + 3X\right) = 9\mathrm{Var}\left(X\right) = 45$.