Third Homework Set — Solutions
Chapter 3

Problem 3.1 Let E be the event that at least one die lands on six, and let F be the event that the dice land of different numbers. Then $P(EF) = 2 \cdot \frac{1}{6} \cdot 56 = \frac{5}{18}$ and $P(F) = \frac{30}{36} = \frac{5}{6}$. Hence,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{1}{3}.$$

Problem 3.5

$$\frac{6 \cdot 5 \cdot 9 \cdot 8}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{6}{91}$$

Problem 3.6 Let A be the event that the sample drawn contains exactly three white balls. Let B be the event that the first and third ball drawn are white.

Without replacement

$P(A) = 4\frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}$ and $P(AB) = 2\frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}$, hence $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}$.

With replacement

$P(A) = \binom{4}{3}\left(\frac{2}{3}\right)^3 \frac{1}{3}$ and $P(AB) = 2\left(\frac{2}{3}\right)^3 \frac{1}{3}$, hence $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}$.

Problem 3.9 Let E_i be the event that the ball drawn from the i-th urn is white, for $i = 1, 2, 3$. Let F be the event that exactly two white balls were drawn. Then

$$\begin{aligned}
P(E_1|F) &= \frac{E_1 F}{P(F)} \\
&= \frac{P(E_1 E_2 E_3^c) + P(E_1 E_2^c E_3)}{P(E_1 E_2 E_3^c) + P(E_1 E_2^c E_3) + P(E_1^c E_2 E_3)} \\
&= \frac{\frac{2 \cdot 8 \cdot 3}{6 \cdot 12 \cdot 4} + \frac{2 \cdot 4 \cdot 1}{6 \cdot 12 \cdot 4}}{\frac{2 \cdot 8 \cdot 3}{6 \cdot 12 \cdot 4} + \frac{2 \cdot 4 \cdot 1}{6 \cdot 12 \cdot 4} + \frac{4 \cdot 8 \cdot 1}{6 \cdot 12 \cdot 4}} \\
&= \frac{7}{11}.
\end{aligned}$$

Problem 3.10 For $i = 1, 2, 3$, let E_i be the event that the i-th card is a spade. Then

$$P(E_1 E_2 E_3) = \frac{13}{52}\frac{12}{51}\frac{11}{50}$$

and

$$P(E_2E_3) = P(E_1E_2E_3) + P(E_1^cE_2E_3) = \frac{13}{52}\frac{12}{51}\frac{11}{50} + \frac{39}{52}\frac{13}{51}\frac{12}{50}.$$

Thus

$$P(E_1|E_2E_3) = \frac{P(E_1E_2E_3)}{P(E_2E_3)} = \frac{11}{50}.$$

Problem 3.20 (a) $P(F|C) = \frac{P(FC)}{P(C)} = \frac{.02}{.05} = \frac{2}{5}$.

(b) $P(C|F) = \frac{P(FC)}{P(F)} = \frac{.02}{.52} = \frac{1}{26}$.

Problem 3.23 (a) Let W be the event that the ball selected urn II is white, E be the event that the transfered ball is white and F be the event that the transfered ball is red, then

$$P(W) = P(E)P(W|E) + P(F)P(W|F) = \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}.$$

(b)

$$P(E|W) = \frac{P(EW)}{P(W)} = \frac{P(E)P(W|E)}{P(W)} = \frac{1}{2}.$$

Problem 3.30 Let B and W be the events that the marble is black and white respectively, and let B_i be the event that box i is chosen. Then

$$P(B) = P(B_1)P(B|B_1) + P(B_2)P(B|B_2) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{2}{3} = \frac{2}{12},$$

$$P(B_1|W) = \frac{P(B_1W)}{P(W)} = \frac{P(B_1)P(W|B_1)}{P(W)} = \frac{\frac{1}{2}\frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}.$$

Problem 3.47 (a)

$$P(\text{all white}) = \frac{1}{6}\left(\frac{5}{15} + \frac{5}{15}\frac{4}{14} + \frac{5}{15}\frac{4}{14}\frac{3}{13} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right).$$

(b)

$$P(3|\text{all white}) = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P(\text{all white})}.$$

Problem 3.51 Let R be the event that she receives a job offer, S be the event that event of a strong recommendation, M the event of a moderate recommendation and W the event of a weak recommendation.

(a)

$$P(R) = P(S)P(R|S) + P(M)P(R|M) + P(W)P(R|W)$$
$$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65.$$

(b)

$$P(S|R) = \frac{P(SR)}{P(R)} = \frac{P(S)P(R|S)}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}.$$

Similarly

$$P(M|R) = \frac{8}{65}, \quad P(W|R) = \frac{1}{65}.$$

Problem 3.56

$$P(\text{new}) = \sum_{i=1}^{m} p_i P(\text{new}|\text{the } n\text{-th coupon is of type i}) = \sum_{i=1}^{m} p_i (1-p_i)^{n-1}.$$

Problem 3.57 (a) $P\,(\text{original price after two days}) = \binom{2}{1} p(1-p) = 2p(1-p)$

(b) $P\,(\text{increase by one after three days}) = \binom{3}{2} p^2(1-p) = 3p^2(1-p)$

(c) $P\,(\text{increase on first day}|\text{increase by one after three days}) = \frac{p\cdot 2p(1-p)}{3p^2(1-p)} = \frac{2}{3}$

Problem 3.59 (a) $P\,(HHHH) = p^4$

(b) $P\,(THHH) = p^3(1-p)$

(c) The pattern $HHHH$ can *only* occur before $THHH$ if the first four coin flips come up heads. Hence, $P\,(THHH \text{ occurs before } HHHH) = 1 - p^4$.

Problem 3.64 Let E be the event that the wife answers correctly, and let F be the event that the husband answers correctly.

(a) If only one of them answers, then the probability of a correct answer is $P\,(E) = P\,(F) = p$.

(b) $P\left(\text{correct answer}\right) = P\left(EF\right) + \frac{1}{2} \cdot 2 \cdot p(1-p) = p^2 + p - p^2 = p$

Problem 3.66 Let E_i be the event that the i-th switch is on.

(a)

$$\begin{aligned}
&P\left(\text{current flows from } A \text{ to } B\right) \\
&= \left(P\left(E_1E_2\right) + P\left(E_3E_4\right) - P\left(E_1E_2E_3E_4\right)\right) P\left(E_5\right) \\
&= (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5
\end{aligned}$$

(b)

$$\begin{aligned}
&P\left(\text{current flows from } A \text{ to } B\right) \\
&= P\left(E_1E_4 \cup E_1E_3E_5 \cup E_2E_5 \cup E_2E_3E_4\right) \\
&= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4 \\
&- p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 \\
&- p_1p_2p_3p_5 - p_1p_2p_3p_4p_5 - p_2p_3p_4p_5 \\
&+ 4p_1p_2p_3p_4p_5 - p_1p_2p_3p_4p_5 \\
&= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4 \\
&- p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 \\
&- p_1p_2p_3p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5
\end{aligned}$$

Problem 3.78 (a) $P\left(\text{exactly four games are played}\right) = P\left(ABAA\right) + P\left(BAAA\right) + P\left(ABBB\right) + P\left(BABB\right) = 2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)\left(p^2 + (1-p)^2\right) = 2p(1-p)(1-2p+2p^2)$

(b) Let E be the event that A wins the match. Conditioning on the first two games of the match, we get $P\left(E\right) = P\left(E|A,A\right)p^2 + P\left(E|A,B\right)p(1-p) + P\left(E|B,A\right)(1-p)p + P\left(E|B,B\right)(1-p)^2 = p^2 + 2P\left(E\right)p(1-p)$ because $P\left(E|A,B\right) = P\left(E|B,A\right) = P\left(E\right)$. Hence, $P\left(E\right) = \frac{p^2}{1-2p(1-p)}$.