

Second Homework Set — Solutions

Chapter 2

Problem 8 Suppose that A, B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$.

(a) $P(\text{either } A \text{ or } B) = P(A) + P(B) = 0.8$ because A and B are mutually exclusive.

(b) $P(A \text{ occurs but } B \text{ does not}) = 0.3$.

(c) $P(A \cap B) = 0$.

Problem 9 Let A be the event that a randomly chosen customer carries an Amex card, and let V be the event that a randomly chosen customer carries a Visa card. Then $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(AB) = 0.24 + 0.61 - 0.11 = 0.74$. Hence, 74 percent of all customers carry an acceptable credit card.

Problem 12 Let E be the event that a randomly chosen student is taking Spanish, let F be the event that a randomly chosen student is taking French, and let D be the event that a randomly chosen student is taking German. Note that

$$P(E) = \frac{28}{100} = \frac{7}{25}$$

$$P(F) = \frac{26}{100} = \frac{13}{50}$$

$$P(D) = \frac{16}{100} = \frac{4}{25}$$

$$P(EF) = \frac{12}{100} = \frac{3}{25}$$

$$P(ED) = \frac{4}{100} = \frac{1}{25}$$

$$P(FD) = \frac{6}{100} = \frac{3}{50}$$

$$P(EFD) = \frac{2}{100} = \frac{1}{50}$$

(a) The probability of a student not being in any of these classes is

$$\begin{aligned} 1 - P(E \cup F \cup D) &= 1 - (P(E) + P(F) + P(D) \\ &\quad - P(EF) - P(ED) - P(FD) + P(EFD)) \\ &= 1 - \frac{28 + 26 + 16 - 12 - 4 - 6 + 2}{100} = 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

(b) We have

$$P(\text{only } E) = P(E) - P(EF) - P(ED) + P(EFD) = 0.14$$

$$P(\text{only } F) = P(F) - P(EF) - P(FD) + P(EFD) = 0.1$$

$$P(\text{only } D) = P(D) - P(ED) - P(FD) + P(EFD) = 0.08$$

Hence, $P(\text{exactly one language class}) = 0.14 + 0.1 + 0.08 = 0.32$.

(c) Since fifty students are not taking any of the courses, the probability that neither of two randomly picked students is $\frac{\binom{50}{2}}{\binom{100}{2}} = \frac{49}{198}$, so that the probability of at least one of them taking a language class is $1 - \frac{49}{198} = \frac{149}{198}$.

Problem 17 There are $64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57$ ways of arranging 8 castles on a chess board. Of these, there are $64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1 = \prod_{i=1}^8 i^2$ in which none of the rooks can capture any of the others. So the answer is

$$\frac{\prod_{i=1}^8 i^2}{64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57}.$$

Problem 18

$$\frac{2 \cdot 4 \cdot 16}{52 \cdot 51}.$$

Problem 20 Let A be the event that you are dealt a blackjack, and let B be the event that the dealer is dealt a blackjack.

Then

$$P(A) = P(B) = \frac{2 \cdot 4 \cdot 16}{52 \cdot 51}$$

$$P(AB) = \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49}$$

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.0948.$$

Hence, then probability that neither you nor the dealer is dealt a black-jack is $1 - P(A \cup B) = 0.9052$.

Problem 21 (a) $P(1) = \frac{4}{20} = \frac{1}{5}$, $P(2) = \frac{8}{20} = \frac{2}{5}$, $P(3) = \frac{5}{20} = \frac{1}{4}$, $P(4) = \frac{2}{20} = \frac{1}{10}$, and $P(5) = \frac{1}{20}$.

(b) There are 48 children altogether, so that $P(1) = \frac{4}{48} = \frac{1}{12}$, $P(2) = \frac{2 \cdot 8}{48} = \frac{1}{3}$, $P(3) = \frac{3 \cdot 5}{48} = \frac{5}{16}$, $P(4) = \frac{4 \cdot 2}{48} = \frac{1}{6}$, and $P(5) = \frac{5}{48}$.

Problem 25 Let E_n be the event that a sum of 5 occurs on the n th roll, and no sum of 5 or 7 occurs on the first $n - 1$ rolls. There are 36 outcomes of a single roll, and four of them give a sum of 5, while 6 of them give a sum of 7. Hence,

$$P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{4}{36} = \left(\frac{13}{18}\right)^{n-1} \frac{1}{9}.$$

A sum of 5 occurs before a sum of 7 precisely if the events E_n occurs for some n . Since E_n and E_m are disjoint if $n \neq m$, the desired probability is

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9} = \frac{1}{9} \cdot \frac{1}{1 - \frac{13}{18}} = \frac{1}{9} \cdot \frac{18}{5} = \frac{2}{5}.$$

Problem 27

$$\begin{aligned} P(A \text{ wins in one move}) &= \frac{3}{10} \\ P(A \text{ wins in three moves}) &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{7}{40} \\ P(A \text{ wins in five moves}) &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{1}{12} \\ P(A \text{ wins in seven moves}) &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{1}{40} \\ P(A \text{ wins}) &= \frac{3}{10} + \frac{7}{40} + \frac{1}{12} + \frac{1}{40} = \frac{7}{12} \end{aligned}$$

Problem 28 (a) Without replacement:

$$P(\text{all three balls are the same color}) = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

With replacement:

$$P(\text{all three balls are the same color}) = \left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3$$

(b) Without replacement:

$$P(\text{all three balls are of different colors}) = \frac{\binom{5}{1} \cdot \binom{6}{1} \cdot \binom{8}{1}}{\binom{19}{3}}$$

With replacement:

$$P(\text{all three balls are of different colors}) = 3! \cdot \frac{5}{19} \cdot \frac{6}{19} \cdot \frac{8}{19}$$

Problem 32 There are $(b+g)!$ ways to line up the children. There are $g \cdot (b+g-1)!$ arrangements with a girl in the i th position. The desired probability is $\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$.

Problem 37 (a) There are $\binom{10}{5}$ selections for the final exam. The number of selections that allow the student to solve all problems is $\binom{7}{5}$, so that the desired probability is $\frac{\binom{7}{5}}{\binom{10}{5}} = 0.08333$.

(b) There are $\binom{7}{4} \cdot \binom{3}{1}$ selections that'll let the student solve exactly four problems, so that the probability of solving at least four problems is $\frac{\binom{7}{5} + \binom{7}{4} \cdot \binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}$.

Problem 43 (a) There are $n!$ ways to arrange n people in a line. There are $2(n-1)!$ ways to arrange them such that A and B are next to each other. Hence, the probability of A and B being next to each other is $\frac{2(n-1)!}{n!} = \frac{2}{n}$.

(b) If $n = 2$, then A and B will always be next to each other. Now, assume that $n > 3$. After A picks a seat, there are $n-1$ seats left, two of which are next to A , so that the desired probability is $\frac{2}{n-1}$.

Problem 50 The probability that you have five spades and your partner has the remaining eight spades is

$$\frac{\binom{13}{5} \binom{39}{8} \binom{8}{8} \binom{31}{5}}{\binom{52}{13,13,26}} = 2.6084 \cdot 10^{-6}.$$

Problem 53 Let E_i be the event that the i -th couple sit together, for $j = 1, \dots, 4$. Then $P(E_i) = \frac{2}{8} = \frac{1}{4}$ (Problem 43(a)). Moreover, if $i < j$, then $P(E_i E_j) = \frac{2^2 \cdot 6!}{8!}$. Similarly, if $i < j < k$, then $P(E_i E_j E_k) = \frac{2^3 \cdot 5!}{8!}$. Finally, we have $P(E_1 E_2 E_3 E_4) = \frac{2^4 \cdot 4!}{8!}$. Using inclusion-exclusion, we obtain

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^4 P(E_i) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) - P(E_1 E_2 E_3 E_4) \\ &= 4 \cdot \frac{1}{4} - \binom{4}{2} \frac{2^2 \cdot 6!}{8!} + \binom{4}{3} \frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!} \\ &= 1 - \frac{3}{7} + \frac{2}{21} - \frac{1}{105} = \frac{23}{35}. \end{aligned}$$

Hence, the probability that no husband sits next to his wife is $1 - \frac{23}{35} = \frac{12}{35}$.

Problem 54 Let S , H , C , and D be the event that spades are missing, hearts are missing, etc. Then

$$\begin{aligned} P(S \cup H \cup C \cup D) &= P(S) + P(H) + P(C) + P(D) \\ &\quad - P(SH) - P(SC) - P(SD) - P(HC) - P(HD) - P(CD) \\ &\quad + P(SHC) + P(SHD) + P(SCD) + P(HCD) \\ &\quad - P(SHCD) \\ &= 4 \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - 6 \cdot \frac{\binom{26}{13}}{\binom{52}{13}} + 4 \cdot \frac{1}{\binom{52}{13}} - 0 \\ &= 0.0511. \end{aligned}$$