11th Homework Set — Solutions

Chapter 7

Problem 7.75 X is a random variable with moment generating function $M_X(t) = \exp\{2e^t - 2\} = \exp\{2(e^t - 1)\}$, i.e., X is Poisson with parameter $\lambda = 2$.

Y is a random variable with moment generating function $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$, i.e., Y is binomial with parameters $(10, \frac{3}{4})$.

(a)

$$\begin{aligned} P\{X+Y=2\} &= P\{X=0\}P\{Y=2\} + P\{X=1\}P\{Y=1\} \\ &\quad + P\{X=2\}P\{Y=0\} \\ &= e^{-2} \cdot \binom{10}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8 + 2e^{-2} \cdot 10\frac{3}{4}\left(\frac{1}{4}\right)^9 \\ &\quad + 2e^{-2} \cdot \left(\frac{1}{4}\right)^{10} \\ &= e^{-2}\left(\frac{1}{4}\right)^{10}(405 + 60 + 2) = \frac{467}{4^{10}e^2}. \end{aligned}$$

(b)

$$\begin{aligned} P\{XY=0\} &= P\{X=0\} + P\{Y=0\} - P\{X=0\}P\{Y=0\} \\ &= e^{-2} + \frac{1}{4^{10}} - e^{-2}\frac{1}{4^{10}} = \frac{4^{10} + e^2 - 1}{4^{10}e^2}. \end{aligned}$$

(c)

$$\begin{aligned} E[XY] &= E[X] \cdot E[Y] \quad \text{by independence} \\ &= 2 \cdot 7.5 \\ &= 15. \end{aligned}$$

Chapter 8

Problem 8.1 $P(0 \le X \le 40) = 1 - P(|X - 20| > 20) \ge 1 - \frac{20}{400} = \frac{19}{20}$.

Problem 8.2 (a) $P(X \ge 85) \le \frac{75}{85} = \frac{15}{17}$.

(b) $P(65 \le X \le 85) = 1 - P(|X - 75| > 10) \ge 1 - \frac{25}{100} = \frac{3}{4}$.

(c) Since

$$P\left(|\sum_{i=1}^{n}\frac{X_i}{n}-75|>5\right)\leq\frac{25}{25n},$$

we need $n=10$.

Problem 8.4 (a) $P(\sum_{i=1}^{20}X_i>15)\leq\frac{20}{15}$.

(b)

$$\begin{aligned}P(\sum_{i=1}^{20}X_i>15) &= P(\sum_{i=1}^{20}X_i>15.5)\approx P(Z>\frac{15.5-20}{\sqrt{20}}\\ &= P(Z>-1.006)\approx .8428.\end{aligned}$$

Problem 8.5 Let X_i be the i-th round-off error, then

$$E(\sum_{i=1}^{50}X_i)=0,\quad \text{Var}(\sum_{i=1}^{50}X_i)=\frac{50}{12}.$$

Hence by the central limit theorem

$$P\left(|\sum_{i=1}^{50}X_i|>3\right)\approx P(|Z|>\frac{3}{\sqrt{12/50}})=2P(Z>1.47)=.1416.$$

Problem 8.7 If we let X_i be the lifetime of the i-th light bulb, then the desired probability is

$$P\left(\sum_{i=1}^{100}X_i>525\right).$$

It follows from the central limit theorem that $\sum_{i=1}^{100}X_i$ is approximately a normal random variable with mean 500 and variance 2500. Consequently the desired probability is equal to

$$P(Z>\frac{525-500}{50})=P(Z>.05)=.3085.$$

Problem 8.8 If we let X_i be the lifetime of the i-th light bulb and R_i be the time to replace the i-th light bulb, then the desired probability is

$$P\left(\sum_{i=1}^{100}X_i+\sum_{i=1}^{99}R_i\leq 550\right).$$

It follows from the central limit theorem that $\sum_{i=1}^{100} X_i$ is approximately a normal random variable with mean 500 and variance 2500 and that $\sum_{i=1}^{99} R_i$ is approximately a normal random variable with mean 24.75 and variance 99/48, therefore $\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i$ is approximately a normal random variable with mean 524.75 and variance 2502.02. Consequently the desired probability is equal to

$$P(Z \leq \frac{550 - 524.75}{\sqrt{2502.02}}) = P(Z \leq .505) = .693.$$