10th Homework Set — Solutions
Chapter 7

Problem 7.30 Note that $E[X^2] = E[Y^2] = \text{Var}(X) + E[X]^2 = \sigma^2 + \mu^2$. Now we conclude that

$$E\left[(X-Y)^2\right] = E\left[X^2\right] - 2E[X]E[Y] + E\left[Y^2\right] = 2\sigma^2,$$

using the fact that X and Y are independent.

Problem 7.31 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$. Then $\text{Var}(X_i) = \frac{35}{12}$, so that

$$\text{Var}(X_1 + \cdots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.$$

Problem 7.33 (a)

$$E\left[(2+X)^2\right] = 4 + 4E[X] + E\left[X^2\right] = 8 + \text{Var}(X) + E[X]^2 = 14.$$

(b)

$$\text{Var}(4+3X) = 9\text{Var}(X) = 45.$$

Problem 7.38 We have

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^x 2ye^{-2x}dydx = \int_0^\infty x^2e^{-2x}dx = \frac{1}{4}, \\ E[X] &= \int_0^\infty \int_0^x 2e^{-2x}dydx = \int_0^\infty 2xe^{-2x}dx = \frac{1}{2}, \quad \text{and} \\ E[Y] &= \int_0^\infty \int_0^x \frac{2y}{x}e^{-2x}dydx = \int_0^\infty xe^{-2x}dx = \frac{1}{4}. \end{aligned}$$

Hence,

$$\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

Problem 7.39 We have

$$\begin{aligned}
\mathrm{Cov}\,(Y_n, Y_n) &= \mathrm{Var}\,(Y_n) = 3\sigma^2, \\
\mathrm{Cov}\,(Y_n, Y_{n+1}) &= \mathrm{Cov}\,(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \mathrm{Cov}\,(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \\
&= \mathrm{Var}\,(X_{n+1} + X_{n+2}) = 2\sigma^2, \\
\mathrm{Cov}\,(Y_n, Y_{n+2}) &= \mathrm{Cov}\,(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \\
&= \mathrm{Cov}\,(X_{n+2}, X_{n+2}) = \mathrm{Var}\,(X_{n+2}) = \sigma^2, \quad \text{and} \\
\mathrm{Cov}\,(Y_n, Y_{n+j}) &= 0 \quad \text{if } j \geq 3.
\end{aligned}$$

Problem 7.41 the number of carp is a hypergeometric random variable, so that we have

$$E[X] = \frac{20 \cdot 30}{100} = 6,$$

and

$$\mathrm{Var}\,(X) = \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}.$$

Problem 7.42 (a) Let X_i be one if the i-th pair consists of a man and a women, and zero otherwise. Then the sum $X_1 + \cdots + X_{10}$ is the number of pairs that consist of a man and a woman.

We have $E[X_i] = P\{X_i = 1\} = 2 \cdot \frac{10 \cdot 10}{20 \cdot 19} = \frac{10}{19}$, so that

$$E[X_1 + \cdots + X_{10}] = \frac{100}{19}.$$

Now, we have $\mathrm{Var}\,(X_i) = E[X_i^2] - E[X_i]^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}$, and $\mathrm{Cov}\,(X_i, X_j) = E[X_i X_j] - E[X_i]\,E[X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{10}{6137}$ if $i \neq j$, so that

$$\mathrm{Var}\,(X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.$$

(b) Let Y_i be one if the i-th couple are paired together. $E[Y_i] = P\{Y_i = 1\} = \frac{2 \cdot 10 \cdot 18!}{20!} = \frac{1}{19}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\mathrm{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_iY_j] = \frac{8\binom{10}{2}\cdot 16!}{20!} = \frac{1}{323}$, so that $\mathrm{Cov}(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\mathrm{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_iY_j] = \frac{8\binom{10}{2}\cdot 16!}{20!} = \frac{1}{323}$, so that

$$\mathrm{Var}(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$

Problem 7.50 We have

$$f_Y(y) = \int_0^\infty \frac{e^{-\frac{x}{y}-y}}{y} dx = e^{-y}$$

for $y > 0$, so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{e^{-\frac{x}{y}}}{y} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Now, we have

$$E\left[X^2|Y\right] = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$

Problem 7.51 We have

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$

so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$E\left[X^3|Y = y\right] = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

Problem 7.56 Let Y_i be one if the elevator stops at the i-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_{10}$. Let X be the number of passengers, i.e., X is

Poisson with parameter 10. We have $E\left[Y_i = 1 | X = k\right] = 1 - \left(\frac{N-1}{N}\right)^k$, so that

$$E\left[Y | X = k\right] = N\left(1 - \left(\frac{N-1}{N}\right)^k\right).$$

We have

$$\begin{aligned} E\left[Y\right] &= E\left[E\left[Y|X\right]\right] = E\left[N\left(1 - \left(\frac{N-1}{N}\right)^X\right)\right] \\ &= N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k \frac{10^k}{k!}e^{-10} \\ &= N(1 - e^{-\frac{10}{N}}). \end{aligned}$$

Problem 7.57 By Example 5d in Section 7.5, we have

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[N\right]E\left[X_1\right] = 12.5.$$