10th Homework Set — Solutions
Chapter 7

Problem 7.30 Note that \( E[X^2] = E[Y^2] = \text{Var}(X) + E[X]^2 = \sigma^2 + \mu^2 \). Now we conclude that
\[
\]
using the fact that \( X \) and \( Y \) are independent.

Problem 7.31 Let \( X_i \) be the outcome of the \( i \)-th roll of the die, for \( i = 1, \ldots, 10 \). Then \( \text{Var}(X_i) = \frac{35}{12} \), so that
\[
\text{Var}(X_1 + \cdots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.
\]

Problem 7.33 (a)
\[
\]

(b)
\[
\text{Var}(4 + 3X) = 9\text{Var}(X) = 45.
\]

Problem 7.38 We have
\[
E[XY] = \int_0^\infty \int_0^x 2ye^{-2x} dy dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4},
\]
\[
E[X] = \int_0^\infty \int_0^x 2e^{-2x} dy dx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}, \quad \text{and}
\]
\[
E[Y] = \int_0^\infty \int_0^x \frac{2y}{x} e^{-2x} dy dx = \int_0^\infty xe^{-2x} dx = \frac{1}{4}.
\]

Hence,
\[
\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.
\]
Problem 7.39 We have
\begin{align*}
\text{Cov} (Y_n, Y_n) &= \text{Var} (Y_n) = 3\sigma^2, \\
\text{Cov} (Y_n, Y_{n+1}) &= \text{Cov} (X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \text{Cov} (X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \\
&= \text{Var} (X_{n+1} + X_{n+2}) = 2\sigma^2, \\
\text{Cov} (Y_n, Y_{n+2}) &= \text{Cov} (X_n + X_{n+1} + X_{n+2} + X_{n+3} + X_{n+4}) \\
&= \text{Cov} (X_{n+2}, X_{n+2}) = \text{Var} (X_{n+2}) = \sigma^2, \quad \text{and} \\
\text{Cov} (Y_n, Y_{n+j}) &= 0 \quad \text{if } j \geq 3.
\end{align*}

Problem 7.41 the number of carp is a hypergeometric random variable, so that we have
\begin{align*}
E [X] &= \frac{20 \cdot 30}{100} = 6, \\
\text{Var} (X) &= \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}.
\end{align*}

Problem 7.42 (a) Let \(X_i\) be one if the \(i\)-th pair consists of a man and a women, and zero otherwise. Then the sum \(X_1 + \cdots + X_{10}\) is the number of pairs that consist of a man and a woman.

We have \(E [X_i] = P \{X_i = 1\} = 2 \cdot \frac{10}{20} \cdot \frac{19}{19} = \frac{10}{19}\), so that
\[E [X_1 + \cdots + X_{10}] = \frac{100}{19}.\]

Now, we have \(\text{Var} (X_i) = E [X_i^2] - E [X_i]^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}\), and
\(\text{Cov} (X_i, X_j) = E [X_i X_j] - E [X_i] E [X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{90}{6137}\) if \(i \neq j\), so that
\[\text{Var} (X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.\]

(b) Let \(Y_i\) be one if the \(i\)-th couple are paired together. \(E [Y_i] = P \{Y_i = 1\} = \frac{2 \cdot 10 \cdot 18!}{20!} = \frac{1}{19}\), so that
\[E [Y_1 + \cdots + Y_{10}] = \frac{10}{19}.\]
We have $\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and 
$E[Y_i Y_j] = \frac{8(10)^{16t}}{20!} = \frac{1}{323}$, so that $\text{Cov}(Y_i, Y_j) = \frac{10}{19}$.

We have $\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_i Y_j] = \frac{8(10)^{16t}}{20!} = \frac{1}{323}$, so that 
$$\text{Var}(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$ 

Problem 7.50 We have 
$$f_Y(y) = \int_0^\infty \frac{e^{-\frac{x}{y} - y}}{y} dx = e^{-y}$$  
for $y > 0$, so that 
$$f_{X|Y}(x|y) = \begin{cases} \frac{e^{-\frac{x}{y}}}{y} & x > 0 \\ 0 & x \leq 0. \end{cases}$$ 

Now, we have 
$$E[X^2|Y] = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$ 

Problem 7.51 We have 
$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$  
so that 
$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$ 

We conclude that 
$$E[X^3|Y = y] = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$ 

Problem 7.56 Let $Y_i$ be one if the elevator stops at the $i$-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_{10}$. Let $X$ be the number of passengers, i.e., $X$ is
Poisson with parameter 10. We have \( E[Y_i = 1|X = k] = 1 - \left( \frac{N-1}{N} \right)^k \), so that
\[
E[Y|X = k] = N \left( 1 - \left( \frac{N-1}{N} \right)^k \right).
\]

We have
\[
E[Y] = E[E[Y|X]] = E \left[ N \left( 1 - \left( \frac{N-1}{N} \right)^X \right) \right] = N - N \sum_{k=0}^{\infty} \left( \frac{N-1}{N} \right)^k \frac{10^k}{k!} e^{-10} = N(1 - e^{-\frac{10}{N}}).
\]

Problem 7.57 By Example 5d in Section 7.5, we have
\[
E \left[ \sum_{i=1}^{N} X_i \right] = E[N] E[X_1] = 12.5.
\]