Soltions to Math 461 Test 2, Fall 2017

1. Suppose that X is uniformly distributed over the interval $(-2, 2)$. Find the density of $Y = X^2$.

Solution. For $y \in (0, 4)$,

$$P(Y \le y) = P(X^2 \le y) = P(-y^{1/2} \le X \le y^{1/2}) = \frac{1}{2}y^{1/2}.$$

Taking derivatives, we get that the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{4}y^{-1/2}, & 0 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

2. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. For $k = 0, 1, \ldots, 50$, find $P(X = k|X + Y = 50)$.

Solution. $X + Y$ is a Poisson random variable with parameter $\lambda + \mu$. Thus, for $k = 0, 1, \ldots, 50$,

$$P(X = k|X + Y = 50) = \frac{P(X = k, X + Y = 50)}{P(X + Y = 50)}$$

$$= \frac{P(X = k, Y = 50 - k)}{P(X + Y = 50)} = \frac{P(X = k)P(Y = 50 - k)}{P(X + Y = 50)}$$

$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{50-k}}{(50-k)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^{50}}{50!}} = \binom{50}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{50-k}.$$

3. Suppose that X and Y are independent exponential random variables with common parameter $\lambda = 1$. Find the density function of $Z = X/Y$.

Solution. For $z > 0$, we have

$$P(Z \le z) = P(X/Y \le z) = P(X \le zY) = \int_0^\infty \int_{x/z}^\infty e^{-x}e^{-y}dydx$$

$$= \int_0^\infty e^{-(1+\frac{1}{z})x}dx = \frac{z}{1+z}.$$

Thus

$$f_Z(z) = \begin{cases} (1+z)^{-2}, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

4. Suppose that U and V are independent random variables and both are uniformly distributed in $(0, 1)$. Define $X = \min(U, V)$ and $Y = \max(U, V)$. (a) Find (or write down) the joint density of X and Y. (b) For $y \in (0, 1)$, find the conditional density $f_{X|Y}(x|y)$.

Solution. (a) The joint density of X and Y is

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) For $y \in (0,1)$,

$$f_Y(y) = \int_0^y 2dx = 2y.$$

Thus

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & x \in (0,y) \\ 0, & \text{otherwise} \end{cases}$$

5. Both players A and B try 100 free throws. It is known that, on average, A will make 80 percent of his free throw attempts and B will make 90 percent of his free throw attempts. Use the normal approximation to find the probability that at least 175 of these 200 free throw attempts will be successful. (For this problem, you can use the following values of the distribution function Φ of the standard normal distribution: $\Phi(.7) = 0.7580$, $\Phi(0.8) = 0.7881$, $\Phi(0.9) = 0.8159$, $\Phi(1) = 0.8413$.)

Solution. By the central limit theorem, the number of free throws X made by A is approximately normal with mean 80 and variance 16, and the number of free throws Y made by A is approximately normal with mean 90 and variance 9. So the total number of free throws $X+Y$ is approximately normal with mean 170 and variance 25. Thus

$$P(X+Y \geq 175) = P(X+Y \geq 174.5) = P(\frac{X+Y-170}{5} \geq \frac{4.5}{5}) = 1 - \Phi(.9) = .1841.$$

6. Suppose that X is a geometric random variable with parameter $p_1 = 1/3$, Y is a geometric random variable with parameter $p_2 = 2/3$, and that X and Y are independent. Find $P(Y < X)$.

Solution.

$$\begin{aligned} P(X > Y) &= \sum_{k=1}^{\infty} P(X > Y, Y = k) = \sum_{k=1}^{\infty} P(X > k, Y = k) \\ &= \sum_{k=1}^{\infty} P(X > k) P(Y = k) = \sum_{k=1}^{\infty} (\frac{2}{3})^k \frac{2}{3} (\frac{1}{3})^{k-1} \\ &= \frac{4}{9} \sum_{n=0}^{\infty} (\frac{2}{9})^n = \frac{4}{7}. \end{aligned}$$

7. 20 balls are randomly distributed into 10 boxes. Let X be the number of boxes that have exactly one ball. Find $E[X]$.

Solution. For $i = 1, \ldots, 10$, let $X_i = 1$ if the i-th box has exactly 1 ball and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_{10}$ is the number of boxes that have exactly 1 ball. For $i = 1, \ldots, 5$,

$$E[X_i] = P(X_i = 1) = 20 \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^{19} = 2 \cdot \left(\frac{9}{10}\right)^{19}.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_{10}] = \sum_{i=1}^{10} 2 \cdot \left(\frac{9}{10}\right)^{19} = 20 \cdot \left(\frac{9}{10}\right)^{19}.$$