Solutions to Math 461 Test 1, Fall 2017

1. (14 points) (a) Suppose that $E_1, E_2, E_3, E_4$ are independent and that \( P(E_1) = P(E_2) = \frac{1}{2}, \) \( P(E_3) = P(E_4) = \frac{1}{3}. \) Find \( P((E_1 \cap E_2) \cup E_3) \cap E_4). \)

   Solution. (a) 
   
   \[
P((E_1 \cap E_2) \cup E_3) \cap E_4) = P((E_1 \cap E_2) \cup E_3)P(E_4) \]
   
   \[
   = (1 - P((E_1 \cap E_2)^c \cap E_3^c)) \frac{1}{3} = (1 - P((E_1 \cap E_2)^c)P(E_3^c)) \frac{1}{3} \]
   
   \[
   = (1 - (1 - \frac{1}{4}) \frac{2}{3}) \frac{1}{3} = \frac{1}{6} \]
   
   (b) 
   
   \[
   E[(1 + 2X)^2] = E[1 + 2X + X^2] = 1 + 4E[X] + 4E[X^2] \]
   
   \[
   = 1 + 4 + 4(\text{Var}(X) + (E[X])^2) = 25. \]

2. (12 points) Let $X$ be a random variable whose distribution function $F$ is given by

   \[
   F(x) = \begin{cases} 
   0, & x < 0, \\
   x/4, & 0 \leq x < 1, \\
   x/3, & 1 \leq x < 2, \\
   1, & 2 \leq x. 
   \end{cases} 
   \]

   Find (a) \( P(\frac{1}{2} \leq X \leq 1); \) (b) \( P(1 \leq X < 2); \) (c) \( P(X > 3/2); \) (d) \( P(X = 1). \)

   Solution. (a) \( P(\frac{1}{2} \leq X \leq 1) = F(1) - F(\frac{1}{2}) = \frac{1}{3} - \frac{1}{8} \).

   (b) \( P(1 \leq X < 2) = F(2) - F(1) = \frac{3}{4} - \frac{1}{2}. \)

   (c) \( P(X > 3/2) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{2}. \)

   (d) \( P(X = 1) = F(1) - F(1^-) = \frac{1}{3} - \frac{1}{4}. \)

3. (10 points) 10 people, including John and Mike, are randomly seated at a round table. Find the probability that John and Mike are seated next to each other.

   Solution. The answer is

   \[
   \frac{8!2}{9!} = \frac{2}{9}. \]

4. (14 points) Box A contains 2 white and 4 red balls, whereas box B contains 3 white and 3 red balls. A ball is randomly chosen from box A and put into box B, and a ball is then randomly selected from box B. (a) Find the probability that the ball selected from box B is white; (b) find the conditional probability that the transferred ball was white given that a white ball is selected from box B.

   Solution. Let $E$ be the event that the transferred ball is white and let $F$ be the event that the ball selected box B is white.
\[ P(F) = P(E \cap F) + P(E^c \cap F) = P(E)P(F|E) + P(E^c)P(F|E^c) \]
\[ = \frac{14}{37} + \frac{23}{37} = \frac{10}{21}. \]

(b)
\[ P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F|E)}{P(F)} = \frac{\frac{14}{37}}{\frac{10}{21}} = \frac{2}{5}. \]

5. (15 points) A certain typing agency has 2 typists. The number of errors per paper is a Poisson random variable with parameter 3 when typed by the first typist and the number of errors per paper is a Poisson random variable with parameter 4 when typed by the second typist. If your paper is equally likely to be typed by either typist, find the probability that it will have no more than 1 error.

**Solution.** Let \( E_1 \) be the event that your paper is typed by the first typist and \( E_2 \) the event that your paper is typed by the second typist. Let \( F \) be the event that your paper has no more than 1 error. Then
\[ P(F) = P(E_1 \cap F) + P(E_2 \cap F) = P(E_1)P(F|E_1) + P(E_2)P(F|E_2) \]
\[ \frac{1}{2}(e^{-3} + 3e^{-3} + e^{-4} + 4e^{-4}) = 2e^{-3} + \frac{5}{2}e^{-4}. \]

6. (15 points) Two teams play a series of games. The series is finished as soon as one of the teams wins 4 games. Suppose that the two teams are evenly matched and each has probability \( \frac{1}{2} \) of winning each game. Let \( X \) be the number of games played. (a) Find the mass function of \( X \); (b) find \( E[X] \).

**Solution.** (a)
\[ P(X = 4) = 2\left(\frac{1}{2}\right)^4 = \frac{1}{8}, \]
\[ P(X = 5) = 2\binom{4}{3}\left(\frac{1}{2}\right)^5 = \frac{1}{4}, \]
\[ P(X = 6) = 2\binom{5}{3}\left(\frac{1}{2}\right)^6 = \frac{5}{16}, \]
\[ P(X = 7) = 2\binom{6}{3}\left(\frac{1}{2}\right)^7 = \frac{5}{16}. \]

(b) \( E[X] = 4 \cdot \frac{1}{8} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{5}{16} + 7 \cdot \frac{5}{16} \).

7. (20 points) 13 cards are randomly selected from an ordinary deck of 52 cards without replacement. Find the probability that the 13-card hand (consisting of the selected cards) (a) contains all 4 aces; (b) contains all 4 aces and all 4 kings; (c) contains all 4 aces, all 4 kings and all 4 queens; (d) contains all 4 cards of at least 1 of the 13 denominations.

**Solution.** (a)
\[ \frac{\binom{48}{9}}{\binom{52}{13}}. \]
(b) \( \left( \frac{\binom{44}{5}}{\binom{52}{13}} \right) \).

(c) \( \left( \frac{\binom{40}{1}}{\binom{52}{13}} \right) \).

(d) Let \( E_1 \) be the event that the 13-card hand contains all 4 aces, \( E_2 \) the event that the 13-card hand contains all 4 2’s, \ldots, \( E_{11} \) the event that the 13-card hand contains all 4 jacks, \( E_{12} \) the event that the 13-card hand contains all 4 queens and \( E_{13} \) the event that the 13-card hand contains all 4 kings. Then by the inclusion-exclusion formula,

\[
P(\bigcup_{i=1}^{13} E_i) = \sum_{i=1}^{13} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k)
\]

\[
= 13 \cdot \left( \frac{\binom{48}{9}}{\binom{52}{13}} \right) - \binom{13}{2} \left( \frac{\binom{44}{5}}{\binom{52}{13}} \right) + \binom{13}{3} \left( \frac{\binom{40}{1}}{\binom{52}{13}} \right).
\]