# Solutions to Math 461 Test 1, Fall 2017

1. (14 points) (a) Suppose that $E_1, E_2, E_2, E_4$ are independent and that $\mathbb{P}(E_1) = \mathbb{P}(E_2) = \frac{1}{2}$, $\mathbb{P}(E_3) = \mathbb{P}(E_4) = \frac{1}{3}$. Find $P(((E_1 \cap E_2) \cup E_3) \cap E_4)$.
(b) Given $E[X] = 1$ and $\text{Var}(X) = 4$. Find $E[(1 + 2X)^2]$.

**Solution**. (a)

$$P(((E_1 \cap E_2) \cup E_3) \cap E_4) = P((E_1 \cap E_2) \cup E_3)P(E_4)$$
$$= (1 - P((E_1 \cap E_2)^c \cap E_3^c))\frac{1}{3} = (1 - P((E_1 \cap E_2)^c)P(E_3^c))\frac{1}{3}$$
$$= (1 - (1 - \frac{1}{4})\frac{2}{3})\frac{1}{3} = \frac{1}{6}$$

(b)

$$E[(1 + 2X)^2] = E[1 + 2X + X^2] = 1 + 4E[X] + 4E[X^2]$$
$$= 1 + 4 + 4(\text{Var}(X) + (E[X])^2) = 25.$$

2. (12 points) Let $X$ be a random variable whose distribution function $F$ is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ x/3 & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

Find (a) $\mathbb{P}(\frac{1}{2} \le X \le 1)$; (b) $\mathbb{P}(1 \le X < 2)$; (c) $\mathbb{P}(X > 3/2)$; (d) $\mathbb{P}(X = 1)$.

**Solution**. (a) $\mathbb{P}(\frac{1}{2} \le X \le 1) = F(1) - F(\frac{1}{2}-) = \frac{1}{3} - \frac{1}{8}$.
(b) $\mathbb{P}(1 \le X < 2) = F(2-) - F(1-) = \frac{2}{3} - \frac{1}{4}$.
(c). $\mathbb{P}(X > 3/2) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{2}$.
(d) $\mathbb{P}(X = 1) = F(1) - F(1-) = \frac{1}{3} - \frac{1}{4}$.

3. (10 points) 10 people, including John and Mike, are randomly seated at a round table. Find the probability that John and Mike are seated next to each other.

**Solution**. The answer is

$$\frac{8!2}{9!} = \frac{2}{9}.$$

4. (14 points) Box A contains 2 white and 4 red balls, whereas box B contains 3 white and 3 red balls. A ball is random chosen from box A and put into box B, and a ball is then randomly selected from box B. (a) Find the probability that the ball selected from box B is white; (b) find the conditional probability that the transferred ball was white given that a white ball is selected from box B.

**Solution**. Let $E$ be the event that the transferred ball is white and let $F$ be the event that the ball selected box B is white.

(a)

$$P(F) = P(E \cap F) + P(E^c \cap F) = P(E)P(F|E) + P(E^c)P(F|E^c)$$
$$= \frac{1}{3}\frac{4}{7} + \frac{2}{3}\frac{3}{7} = \frac{10}{21}.$$

(b)

$$P(E|F) = \frac{P(E \cap F}{P(F)} = \frac{P(E)P(F|E)}{P(F)} = \frac{\frac{1}{3}\frac{4}{7}}{\frac{10}{21}} = \frac{2}{5}.$$

5. (15 points) A certain typing agency has 2 typists. The number of errors per paper is a Poisson random variable with parameter 3 when typed by the first typist and the number of errors per paper is a Poisson random variable with parameter 4 when typed by the second typist. If your paper is equally likely to be typed by either typist, find the probability that it will have no more than 1 error.

**Solution**. Let $E - 1$ be the event that your paper is typed by the first typist and $E_2$ the event that your paper is typed by the second typist. Let $F$ be the event that your paper has no more than 1 error. Then

$$P(F) = P(E_1 \cap F) + P(E_2 \cap F) = P(E_1)P(F|E_1) + P(E_2)P(F|E_2)$$
$$\frac{1}{2}(e^{-3} + 3e^{-3} + e^{-4} + 4e^{-4}) = 2e^{-3} + \frac{5}{2}e^{-4}.$$

6. (15 points) Two teams play a series of games. The series is finished as soon as one of the teams wins 4 games. Suppose that the two teams are evenly matched and each has probability 1/2 of winning each game. Let $X$ be the number of games played. (a) Find the mass function of $X$; (b) find $E[X]$.

**Solution**. (a)

$$P(X = 4) = 2(\frac{1}{2})^4 = \frac{1}{8},$$
$$P(X = 5) = 2\binom{4}{3}(\frac{1}{2})^5 = \frac{1}{4},$$
$$P(X = 6) = 2\binom{5}{3}(\frac{1}{2})^6 = \frac{5}{16},$$
$$P(X = 7) = 2\binom{6}{3}(\frac{1}{2})^7 = \frac{5}{16}.$$

(b) $E[X] = 4 \cdot \frac{1}{8} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{5}{16} + 7 \cdot \frac{5}{16}$.

7. (20 points) 13 cards are randomly selected from an ordinary deck of 52 cards without replacement. Find the probability that the 13-card hand (consisting of the selected cards) (a) contains all 4 aces; (b) contains all 4 aces and all 4 kings; (c) contains all 4 aces, all 4 kings and all 4 queens; (d) contains all 4 cards of at least 1 of the 13 denominations.

**Solution**. (a)

$$\frac{\binom{48}{9}}{\binom{52}{13}}.$$

(b)

$$\frac{\binom{44}{5}}{\binom{52}{13}}.$$

(c)

$$\frac{\binom{40}{1}}{\binom{52}{13}}.$$

(d) Let $E_1$ be the event that the 13-card hand contains all 4 aces, $E_2$ the event that the 13-card hand contains all 4 2's, $\ldots$, $E_{11}$ the event that the 13-card hand contains all 4 jacks, $E_{12}$ the event that the 13-card hand contains all 4 queens and $E_{13}$ the event that the 13-card hand contains all 4 kings. Then by the inclusion-exclusion formula,

$$\begin{aligned} P(\cup_{i=1}^{13} E_i) &= \sum_{i=1}^{13} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) \\ &= 13 \cdot \frac{\binom{48}{9}}{\binom{52}{13}} - \binom{13}{2} \frac{\binom{44}{5}}{\binom{52}{13}} + \binom{13}{3} \frac{\binom{40}{1}}{\binom{52}{13}}. \end{aligned}$$