Math 461, Solution Written Homework 9

1. Suppose that \( n \) balls are distributed at random into \( r \) boxes so that all \( r^n \) possible arrangements are equally likely. Let \( X \) be the number of empty boxes. Find the expectation and variance of \( X \).

**Solution** For \( i = 1, \ldots, r \), let \( X_i = 1 \) if the \( i \)-th box is empty and \( X_i = 0 \) otherwise. Then for any \( i \),

\[
E[X_i] = \mathbb{P}(X_i = 1) = \left( \frac{r - 1}{r} \right)^n,
\]

and so

\[
E[X] = E\left[ \sum_{i=1}^{r} X_i \right] = r \left( \frac{r - 1}{r} \right)^n.
\]

For \( i \neq j \), we have

\[
E[X_i X_j] = \mathbb{P}(X_i = 1, X_j = 1) = \left( \frac{r - 2}{r} \right)^n
\]

and so

\[
\text{Cov}(X_i, X_j) = \left( \frac{r - 2}{r} \right)^n - \left( \frac{r - 1}{r} \right)^{2n}.
\]

Therefore

\[
\text{Var}(X) = \text{Var}\left( \sum_{i=1}^{r} X_i \right) = \sum_{i=1}^{r} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)
\]

\[
= r \left( \frac{r - 1}{r} \right)^n \left( 1 - \left( \frac{r - 1}{r} \right)^n \right) + (r^2 - r) \left( \left( \frac{r - 2}{r} \right)^n - \left( \frac{r - 1}{r} \right)^{2n} \right).
\]

2. Consider \( n \) independent flips of a coin having probability \( p \) of coming up heads. We say that a changeover occurs whenever an outcome differs from the one preceding it. Find the expected number of changeovers.

**Solution** For \( i = 2, \ldots, n \), let \( X_i = 1 \) if a changeover occurs at \( i \) and \( X_i = 0 \) otherwise. Then \( E[X_i] = \mathbb{P}(X_i = 1) = 2p(1 - p) \). Thus

\[
E[X] = E\left[ \sum_{i=2}^{n} X_i \right] = 2(n - 1)p(1 - p).
\]
3. A certain region is inhabited by \( r \) distinct types of insect species, and each insect caught will, independently of the types of the previous catches, be of type \( i \) with probability \( p_i, i = 1, 2, \ldots, r, \sum_{i=1}^{r} p_i = 1 \). (a) Find the expected number of insects that are caught before the first type 1 catch. (b) Find the expected number of types of insects that are caught before the first type 1 catch.

**Solution** (a) The number of insects that are caught before the first type 1 catch is \( X - 1 \), where \( X \) is a geometric random variable with parameter \( p_1 \). Thus the expected number of insects that are caught before the first type 1 catch is equal to

\[
\frac{1}{p_1} - 1.
\]

(b) For \( i = 2, \ldots, r \), let \( Y_i = 1 \) if a there is a type \( i \) catch before the first type 1 catch and \( Y_i = 0 \) otherwise. Then \( \sum_{i=2}^{r} Y_i \) is the total number of types of insects that are caught before the first type 1 catch. For any \( i = 2, \ldots, r \),

\[
E[Y_i] = P(Y_i = 1) = \frac{p_i}{p_1 + p_i}.
\]

Thus

\[
E[\sum_{i=2}^{r} Y_i] = \sum_{i=2}^{r} \frac{p_i}{p_1 + p_i}.
\]

4. The joint density function of \( X \) and \( Y \) is given by

\[
f(x, y) = \begin{cases} 
  ye^{-y(x+1)} & x > 0, y > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

For any \( y > 0 \), find (a) the conditional density \( f_{X|Y}(x|y) \); (b) \( P(X > 2|Y = y) \); (c) \( E[X|Y = y] \).

**Solution** (a) For any \( y > 0 \),

\[
f_Y(y) = \int_0^\infty ye^{-y(x+1)} dx = e^{-y},
\]

thus

\[
f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} 
  ye^{-yx} & x > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

Therefore, \( f_{X|Y}(x|y) \) is an exponential density with parameter \( y \).

(b) It follows from (a) that \( P(X > 2|Y = y) = e^{-2y} \).

(c) It follows from (a) that \( E[X|Y = y] = \frac{1}{y} \).

5. Suppose that \( X \) and \( Y \) are independent Poisson random variables with parameters \( \lambda_1 \) and \( \lambda_2 \) respectively. For any \( n > 2 \). Find \( E[X|X + Y = n] \).
Solution Since $X$ and $Y$ are independent Poisson random variables with parameters $\lambda_1$ and $\lambda_2$ respectively, $X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. Thus, for $x = 0, 1, \ldots, n$,

$$p_{X|X+Y}(x|n) = \frac{\mathbb{P}(X = x, X + Y = n)}{\mathbb{P}(X + Y = n)} = \frac{\mathbb{P}(X = x, x + Y = n)}{\mathbb{P}(X + Y = n)} = \frac{\mathbb{P}(X = x)\mathbb{P}(Y = n - x)}{\mathbb{P}(X + Y = n)}$$

$$= \frac{e^{-\lambda_1} \frac{\lambda_1^x}{x!} e^{-\lambda_2} \frac{\lambda_2^{n-x}}{(n-x)!}}{e^{-(\lambda_1 + \lambda_2)} \frac{\lambda_1^x \lambda_2^{n-x}}{(\lambda_1 + \lambda_2)^n}} = \binom{n}{x} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^x \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-x}.$$

That is, $p_{X|X+Y}(x|n)$ is binomial mass function with parameters $(n, \frac{\lambda_1}{\lambda_1 + \lambda_2})$. Therefore

$$E[X|X + Y = n] = \frac{n\lambda_1}{\lambda_1 + \lambda_2}.$$