Math 461, Solution to Written Homework 5

1. Let $X$ be an absolutely continuous random variable with density function

$$f(x) = \begin{cases} 
  c(1 - x^2) & -1 < x < 1 \\
  0 & \text{otherwise}.
\end{cases}$$

Find the value of $c$ and the distribution function of $X$.

**Solution** Since

$$\int_{-1}^{1} (1 - x^2) \, dx = \frac{4}{3},$$

we know that $c = \frac{3}{4}$. Hence the distribution function of $X$ is

$$F(x) = \begin{cases} 
  0 & x < -1 \\
  \frac{3}{4}(x - \frac{x^3}{3} + \frac{2}{3}) & -1 \leq x < 1 \\
  1 & x \geq 1
\end{cases}$$

2. The lifetime of a certain type of electronic device is $X$ hours, where $X$ is an absolutely continuous random variable with density

$$f(x) = \begin{cases} 
  \frac{10}{x^2} & x > 10 \\
  0 & x \leq 10.
\end{cases}$$

(a) Find $P(X \geq 15)$. (b) Assuming independence, find the probability that of 6 such devices at least 3 will function for at least 15 hours.

**Solution** (a)

$$P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2} \, dx = -\frac{10}{x} \bigg|_{15}^{\infty} = \frac{2}{3}.$$  

(b) The answer is

$$1 - \left( \left( \frac{1}{3} \right)^6 + 6 \left( \frac{2}{3} \right) \left( \frac{1}{3} \right)^5 + \left( \frac{6}{2} \right) \left( \frac{2}{3} \right)^2 \left( \frac{1}{3} \right)^4 \right).$$

3. A bus travels between two cities $A$ and $B$, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city $A$ has a uniform distribution over
(0, 100). There is a bus service station in city $A$, city $B$ and in the center of the route between $A$ and $B$. It is suggested that it would be more efficient to have the 3 service stations located at 25, 50 and 75 miles, respectively, from city $A$. Do you agree? Why? (Hint: consider the expected towing distances in these two scenarios.)

**Solution** Let $X$ be the location of the breakdown, which is uniformly distributed in $(0, 100)$. In the first scenario, the towing distance will be

$$
y = \begin{cases} 
X & X \leq 25 \\
50 - X & X \in (25, 50] \\
X - 50 & X \in (50, 75] \\
100 - X & X \in (75, 100),
\end{cases}
$$

while in the second scenario, the towing distance will be

$$z = \begin{cases} 
25 - X & X \leq 25 \\
X - 25 & X \in (25, 37.5] \\
50 - X & X \in (37.5, 50] \\
X - 50 & X \in (50, 62.5] \\
75 - X & X \in (62.5, 75] \\
X - 75 & X \in (75, 100).
\end{cases}
$$

$$E[Y] = \frac{1}{100} \left( \int_0^{25} x \, dx + \int_{25}^{50} (50 - x) \, dx + \int_{50}^{75} (x - 50) \, dx + \int_{75}^{100} (100 - x) \, dx \right) = 12.5$$

$$E[X] = \frac{1}{100} \left( \int_0^{25} (25 - x) \, dx + \int_{25}^{37.5} (x - 25) \, dx + \int_{37.5}^{50} (50 - x) \, dx \right)$$

$$+ \frac{1}{100} \left( \int_{50}^{62.5} (x - 50) \, dx + \int_{62.5}^{75} (75 - x) \, dx + \int_{75}^{100} (x - 75) \, dx \right) = 9.375.$$

So the second scenario is more efficient.

4. You arrive at a bus stop at 10 am, knowing that the bus will arrive at some time uniformly distributed between 10 am and 10:30 am. (a) Find the probability that you will have to wait longer than 10 minutes. (b) If at 10:15 am the bus has not yet arrived, what is the probability that you will have to wait an additional 10 minutes?

**Solution** (a)

$$P(X > 10) = \frac{2}{3}.$$
5. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 foot 2 inches tall?

**Solution**

\[
P(X > 74) = P\left( \frac{X - 71}{\sqrt{6.25}} > \frac{74 - 71}{\sqrt{6.25}} \right) = P\left( \frac{X - 71}{\sqrt{6.25}} > \frac{6}{\frac{6}{5}} \right) = 1 - \Phi\left( \frac{6}{\frac{6}{5}} \right) \approx .1151
\]

6. Let $X$ be a normal random variable with parameters $\mu = 12$ and $\sigma^2 = 4$. Find the value of $c$ so that $P(X > c) = .10$.

**Solution** Since

\[
P(X > c) = P\left( \frac{X - 12}{2} > \frac{c - 12}{2} \right) = 1 - \Phi\left( \frac{c - 12}{2} \right),
\]

in order for $P(X > c) = .10$, we need to have $\Phi\left( \frac{c - 12}{2} \right) = .90$. From the table, we get

\[
\frac{c - 12}{2} \approx 1.28,
\]

and so $c \approx 14.56$. 

(b) \[
P(X > 25 | X > 15) = \frac{P(X > 25)}{P(X > 15)} = \frac{1}{3}.
\]