1. A total of 4 buses carrying 148 students from a school arrives at a football stadium. The buses carries, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let $X$ be the number of students that were on the bus carrying this randomly selected student. One of 4 the bus drivers is also randomly selected. Let $Y$ be the number of students on her bus. Find, $E[X]$, $E[Y]$, $\text{Var}(X)$ and $\text{Var}(Y)$.

**Solution.** We know that $P(X = 40) = \frac{40}{148}$, $P(X = 33) = \frac{33}{148}$, $P(X = 25) = \frac{25}{148}$, $P(X = 50) = \frac{50}{148}$, and so

$$E(X) = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148},$$

$$E(X^2) = 40^2 \cdot \frac{40}{148} + 33^2 \cdot \frac{33}{148} + 25^5 \cdot \frac{25}{148} + 50^5 \frac{50}{148}.$$

From these you can get $\text{Var}(X) = E(X^2) - (E(X))^2$ easily.

We know that $P(Y = 40) = P(Y = 33) = P(Y = 25) = P(Y = 50) = \frac{1}{4}$, and so

$$E(Y) = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = \frac{148}{4},$$

$$E(Y^2) = 40^2 \cdot \frac{1}{4} + 33^2 \cdot \frac{1}{4} + 25^5 \cdot \frac{1}{4} + 50^5 \frac{1}{4}.$$

From these you can get $\text{Var}(Y) = E(Y^2) - (E(Y))^2$ easily.

2. If $E[X] = 1$ and $\text{Var}(X) = 5$, find (a) $E[(2 + X)^2]$, (b) $\text{Var}(4 + 3X)$.

**Solution.** (a)


(b)

$$\text{Var}(4 + 3X) = 9\text{Var}(X) = 45.$$
3. Suppose that the distribution function of a random variable is given by

\[ F(x) = \begin{cases} 
0, & x < 0 \\
\frac{x}{4}, & 0 \leq x < 1 \\
\frac{1}{2} + \frac{x-1}{4}, & 1 \leq x < 2 \\
\frac{11}{12}, & 2 \leq x < 3 \\
1, & 3 \leq x.
\end{cases} \]

(a) Find \( P(X = i) \), \( i = 1, 2, 3 \).  (b) Find \( P(1 \leq X < 3) \).

**Solution.** (a) \( P(X = 1) = F(1) - F(1-) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \).  \( P(X = 2) = F(2) - F(2-) = \frac{11}{12} - (\frac{1}{2} + \frac{1}{4}) = \frac{1}{6} \).  \( P(X = 3) = F(3) - F(3-) = 1 - \frac{11}{12} = \frac{1}{12} \).

(b) \( P(1 \leq X < 3) = F(3-) - F(1-) = \frac{11}{12} - \frac{1}{4} = \frac{2}{3} \).

4. A box initially contains 5 white and 7 black balls. Each time a ball is selected, its color noted and it is replaced in the box along with 2 other balls of the same color. Find the probability that the first 2 balls selected are black and the next 2 white.

**Solution.** For \( i = 1, 2, 3, 4 \), let \( B_i \) be the event that the \( i \)-th selected ball is black and \( W_i \) the event that the \( i \)-th selected ball is white.

\[ P(B_1 \cap B_2 \cap W_3 \cap W_4) = P(B_1)P(B_2|B_1)P(W_3|B_1 \cap B_2)P(W_4|B_1 \cap B_2 \cap W_3) = \frac{7}{12} \frac{9}{14} \frac{5}{16} \frac{7}{18}. \]

5. Box I contains 2 white and 4 red balls, whereas box II contains 1 white and 1 red ball. A ball is randomly selected from box I and put into box II, and a ball is then randomly selected from box II. Find (a) the probability that the ball selected box II is white; (b) the conditional probability that the transferred ball was white, given that a white ball is selected from box II.

**Solution.** Let \( W_1 \) be the event that the transferred ball is white, and \( R_1 \) the event that the transferred ball is red. Let \( W_2 \) be the event that the ball selected box II is white.

(a) \( P(W_2) = P(W_1 \cap W_2) + P(R_1 \cap W_2) = P(W_1)P(W_2|W_1) + P(R_1)P(W_2|R_1) = \frac{2}{3} \frac{2}{18} + \frac{1}{3} \frac{1}{3} = \frac{1}{3}. \)

(b) \( P(W_1|W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)} = \frac{2}{2} \frac{2}{18} + \frac{1}{3} \frac{1}{3} = \frac{1}{2}. \)