Solutions to selected problems in HW6

Suppose that \((S, \mathcal{S})\) is a standard measurable space, \(\mu\) is a probability measure on \((S, \mathcal{S})\) and that \(p : S \times \mathcal{S} \rightarrow \mathbb{R}\) is a transition probability. Define a probability measure \(P_\mu\) on \((\mathbb{S}^n, \mathcal{S}^n)\) by setting

\[
P_\mu(\omega \in \mathbb{S}^n : \omega_i \in B_i, 0 \leq i \leq n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n)
\]

for all \(n\) and all \(B_i \in \mathcal{S}\). Show that, under \(P_\mu\), the coordinate map \(X_n(\omega) = \omega_n\) is a Markov chain with transition matrix \(p\).

**Proof.** Using the definition of \(P_\mu\) and standard measure theory techniques we can easily get that for any positive integer \(n\) and \(\mathcal{S}\)-measurable functions on \(S\) \(f_i, i = 0, 1, \ldots, n\) we have

\[
E_\mu \left[ \prod_{i=0}^{n} f_i(X_i) \right] = \int_{B_0} f_0(x_0) \mu(dx_0) \int_{B_1} f_1(x_1)p(x_0, dx_1) \cdots \int_{B_n} f_n(x_n)p(x_{n-1}, dx_n).
\]

For any sets \(B_i \in \mathcal{S}, i = 0, 1, \ldots, n\) and any \(B \in \mathcal{S}\), from the equation above we get

\[
E_\mu \left[ 1_{\{X_{n+1} \in B\}} 1_{\{X_0 \in B_0, X_1 \in B_1, \ldots, X_n \in B_n\}} \right] = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n) \int_{B} p(x_n, dx_{n+1})
\]

and

\[
E_\mu \left[ p(X_n, B) 1_{\{X_0 \in B_0, X_1 \in B_1, \ldots, X_n \in B_n\}} \right] = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_n, B)p(x_{n-1}, dx_n).
\]

Thus we have

\[
E_\mu \left[ 1_{\{X_{n+1} \in B\}} 1_{\{X_0 \in B_0, X_1 \in B_1, \ldots, X_n \in B_n\}} \right] = E_\mu \left[ p(X_n, B) 1_{\{X_0 \in B_0, X_1 \in B_1, \ldots, X_n \in B_n\}} \right].
\]

Again using standard measure theory techniques we can get from the above that for any \(A \in \mathcal{F}_n\),

\[
E_\mu \left[ 1_{\{X_{n+1} \in B\}} 1_A \right] = E_\mu \left[ p(X_n, B) 1_A \right],
\]

consequently

\[
P_\mu(X_{n+1} \in B | \mathcal{F}_n) = p(X_n, B).
\]

The proof is finished.