Solutions to selected problems in HW5

1. Show that the distribution of a bounded random variable $X$ is infinitely divisible if and only $X$ is a constant. (Hint: Show $Var(X) = 0$.)

**Proof.** The “if” part is trivial, let’s prove the “only if” part. Suppose that $X$ is infinitely divisible and bounded, then there exists a constant $M$ such that $|X| \leq M$. For any positive integer $n$, there exist independent and identically random variables $X_{n,1}, \ldots, X_{n,n}$ such that

$$X = X_{n,1} + \cdots + X_{n,n}.$$  

We claim that, for any $j = 1, \ldots, n$, $P(|X_{n,j}| \leq \frac{M}{n}) = 1$. In fact, if $P(X_{n,j} > \frac{M}{n}) > 0$, then

$$P(X > M) \geq P(X_{n,1} > \frac{M}{n}, \ldots, X_{n,n} > \frac{M}{n}) = (P(X_{n,1} > \frac{M}{n}))^n > 0$$

which contradicts the fact that $|X| \leq M$, so we must have $P(X_{n,j} \leq \frac{M}{n}) = 1$ for $j = 1, \ldots, n$. Similarly we can show that $P(X_{n,j} \geq -\frac{M}{n}) = 1$ for $j = 1, \ldots, n$. Now for any $n$,

$$Var(X) = \sum_{j=1}^{n} Var(X_{n,j}) = n Var(X_{n,1}) \leq \frac{M^2}{n},$$

and so $Var(X) = 0$ and thus $X$ must be a constant.

2. Let $X_{n,m}, 1 \leq m \leq n$ be independent nonnegative integer valued random variables with $P(X_{n,m} = 1) = p_{n,m}$, $P(X_{n,m} \geq 2) = \epsilon_{n,m}$. Suppose that (i) $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0, \infty)$, (ii) $\max_{1 \leq m \leq n} p_{n,m} \to 0$, and (iii) $\sum_{m=1}^{n} \epsilon_{n,m} \to 0$. Show that $X_1 + \cdots + X_n$ converges weakly to a Poisson random variable with parameter $\lambda$.

**Proof.** Let $X'_{n,m} = 1$ if $X_{n,m} = 1$, and 0 otherwise. Let $S' = X'_{n,1} + \cdots + X'_{n,n}$. Then by the Poisson convergence theorem we proved in class we know that $S'_n$ converges weakly to a Poisson random variable $Z$ with parameter $\lambda$. Now condition (iii) implies that $P(S'_n \neq S'_n) \to 0$, that is, $S_n - S'_n$ converge to zero in probability. Therefore $S_n = S'_n + (S_n - S'_n)$ converges weakly to $Z$. 

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