Solutions to selected problems in HW4

1. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $0 < X_1 < \infty$. Let $T_n = X_1 + \cdots + X_n$ and $N_t = \sup\{n : T_n \leq t\}$. Show that if $E[X_1] = \mu \leq \infty$, then $\frac{N_t}{t}$ converges to $\frac{1}{\mu}$ almost surely.

**Proof.** It follows from the assumption that $\frac{T_n}{n} \to \mu$ almost surely. From the definition of $N_t$ we can see that

$$T_{N_t} \leq t < T_{N_t+1},$$

and so we have

$$\frac{T_{N_t}}{N_t} < \frac{t}{N_t} < \frac{T_{N_t+1}}{N_t + 1}.$$  

Since $T_n$ is a finite random variable for each $n$, we know that $N_t \uparrow \infty$ as $t \uparrow \infty$. Now we can let $t \uparrow \infty$ in the display above to get the desired conclusion.

2. (Exercise 3.12 in the book) Prove that if a sequence $X_n$ of random variables satisfies

$$\lim_{n,m \to \infty} P(\sup_{m < k \leq n} |X_k - X_m| \geq \delta) = 0, \quad \text{for every } \delta > 0,$$

then there is a limiting random variable $X$ such that

$$P(\lim_{n \to \infty} X_n = X) = 1.$$

**Proof.** It follows from Exercise 3.11 we know that there is a random variable $X$ such that $X_n$ converges to $X$ in probability. Therefore for any $\delta > 0$ there exists $N_1$ such that

$$P(|X_n - X| \geq \delta) \leq \delta \quad n \geq N_1.$$

From our assumption we know that there exists $N_2$ such that

$$P(\sup_{m < k \leq n} |X_k - X_m| \geq \delta) \leq \delta \quad n > m \geq N_2.$$

Take $n = N_1 \lor N_2$, we get that

$$P(\sup_{k > m} |X_k - X| \geq 2\delta) \leq 2\delta \quad m \geq N,$$
thus we have shown that for any $\delta > 0$, 

$$
\lim_{n \to \infty} P\left( \sup_{k > n} |X_k - X| \geq \delta \right) = 0.
$$

Therefore we can find an increasing sequence $n_j$ of integers such that 

$$
P\left( \sup_{k > n_j} |X_k - X| \geq 2^{-j} \right) \leq 2^{-j}.
$$

The Borel-Cantelli lemma implies that 

$$
P\left( \sup_{k > n_j} |X_k - X| \geq 2^{-j} \text{ i. o.} \right) = 0
$$

which immediately implies the desired conclusion.