Solutions to selected problems in HW3

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $P(X_1 > x) = e^{-x}$ for $x \geq 0$, let $M_n = \max_{1 \leq m \leq n} X_m$. Show that (i)

$$\limsup_{n \to \infty} \frac{X_n}{\log n} = 1$$

and (ii)

$$\frac{M_n}{\log n} \to 1$$

almost surely.

**Proof.** (i). For any $n \geq 1$,

$$P(X_n \geq \log n) = \frac{1}{n},$$

and so we have

$$\sum_{n=1}^{\infty} P(X_n \geq \log n) = \infty.$$ 

Therefore by the second Borel-Cantelli lemma we get that

$$P(X_n \geq \log n \text{ i. o.}) = 1,$$

and consequently

$$\limsup_{n \to \infty} \frac{X_n}{\log n} \geq 1$$

almost surely. On the other hand, for any $\epsilon > 0$,

$$P(X_n \geq (1 + \epsilon) \log n) = \frac{1}{n^{1+\epsilon}},$$

and so the first Borel-Cantelli lemma implies that

$$P(X_n \geq (1 + \epsilon) \log n \text{ i. o.}) = 0.$$ 

Thus we have

$$\limsup_{n \to \infty} \frac{X_n}{\log n} \leq 1$$
almost surely.

(ii). (i) implies that for almost every \( \omega \) and \( \epsilon > 0 \), there exists \( N(\omega) \) such that \( X_n \leq (1 + \epsilon) \log n \). From this one can easily see that

\[
\limsup_{n \to \infty} \frac{M_n}{\log n} \leq 1.
\]

On the other hand, for any \( \epsilon > 0 \),

\[
P(M_n < (1 - \epsilon) \log n) = (1 - n^{(1-\epsilon)})^n \leq e^{-n^\epsilon}.
\]

So we can use the first Borel-Cantelli lemma to get

\[
P(M_n < (1 - \epsilon) \log n \text{ i. o.}) = 0.
\]

From this we immediately get that

\[
\liminf_{n \to \infty} \frac{M_n}{\log n} \geq 1.
\]