Solutions to selected problems in HW2

2.4. For any nonnegative integer $n$, if $E[|X|^n] < \infty$, then the characteristic function $\phi$ of $X$ is $n$ times continuously differentiable and

$$\phi^{(n)}(t) = i^n E[X^n e^{itX}], \quad \forall t \in \mathbb{R}.$$ 

In particular, $\phi^{(n)}(0) = i^n E[X^n]$. Conversely, if $\phi^{(n)}(0)$ exists for some nonnegative even integer $n$, then $E[X^n]$ exists as a finite number.

**Proof.** Suppose that $E[|X|^n] < \infty$. For $k = 0, 1, \ldots, n$, define the functions

$$f_k(t) = E[x^n e^{itX}], \quad t \in \mathbb{R}.$$ 

By using the dominated convergence theorem we can easily see that the functions $f_k$, $k = 0, 1, \ldots, n$ are continuous. For any $k \leq n - 1$, we have by the dominated convergence theorem that

$$\lim_{h \to 0} \frac{f_k(t + h) - f_k(t)}{h} = \lim_{h \to 0} E[X^k e^{itX} e^{ihX} - 1]$$

$$= E[X^k e^{itX} \lim_{h \to 0} e^{ihX} - 1] = if_{k+1}(t).$$

So the first part of the assertion are proved.

Now we prove the second part by induction on $n$. When $n = 0$, the second part of the assertion is trivially true. Suppose the result is true for $n = k - 2$ for some even integer $k \geq 2$, and suppose that $\phi^{(k)}(0)$ exists as a finite number. The inductive hypothesis implies that $E[X^{k-2}]$ is finite. We need to show that $E[X^k]$ is finite. We can easily check that

$$\lim_{t \to 0} \frac{2 - 2 \cos(tx)}{t^2} = x^2.$$

Thus, we have by Fatou’s Lemma that

$$E[X^k] \leq \liminf_{t \to 0} E \left( X^{k-2} \frac{2 - 2 \cos(tx)}{t^2} \right)$$

$$= \liminf_{t \to 0} \left( -E \left( X^{k-2} \frac{e^{itX} - 2 + e^{-itX}}{t^2} \right) \right)$$

$$= \liminf_{t \to 0} \left( -\frac{f_{k-2}(t) - 2f_{k-2}(0) + f_{k-2}(-t)}{t^2} \right). \quad (0.1)$$
Since we know that $E[X^{k-2}]$ is finite, we have $\phi^{(k-2)}(t) = i^{k-2} f_{k-2}(t)$ for all real $t$. By assumption we know that $\phi^{(k)}(0)$ exits and is finite, thus $f_{k-2}''(t)$ exists in a neighborhood of the origin and $f_{k-2}''(0)$ also exists and is finite. Consequently we can apply the l'Hospital rule in (0.1) and get

$$
\liminf_{t \to 0} \left( -\frac{f_{k-2}(t) - 2f_{k-2}(0) + f_{k-2}(-t)}{t^2} \right) \\
= -\lim_{t \to 0} \frac{f_{k-2}'(t) - f_{k-2}'(-t)}{2t} \\
= -\lim_{t \to 0} \frac{[f_{k-2}'(t) - f_{k-2}'(0)] + [f_{k-2}'(0) - f_{k-2}'(-t)]}{2t} \\
= -f_{k-2}''(0).
$$

Thus we have concluded that $E[X^k]$ is finite.

2. Suppose that $X_n = X$ and $Y_n = c$, where $c$ is a constant then $X_n + Y_n = X + c$.

**Proof.** For any real number $x$ and $\epsilon > 0$, we have

$$
P(X_n + Y_n \leq x) \geq P(X_n \leq x - c - \epsilon) - P(Y_n > c + \epsilon).
$$

The second term on the right side goes to zero as $n \to \infty$. If $x - c - \epsilon$ is a continuity point of the distribution function of $X$ then the first term on the right hand side goes to $P(X \leq x - c - \epsilon)$. Letting $\epsilon \downarrow 0$ it follows that if $x$ is a continuity point of the distribution of $X + c$

$$
\liminf_{n \to \infty} P(X_n + Y_n \leq x) \geq P(X + c \leq x).
$$

For any real number $x$ and $\epsilon > 0$, we also have

$$
P(X_n + Y_n \leq x) \leq P(X_n \leq x - c + \epsilon) + P(Y_n < c - \epsilon).
$$

The second term on the right side goes to zero as $n \to \infty$. If $x - c + \epsilon$ is a continuity point of the distribution function of $X$ then the first term on the right hand side goes to $P(X \leq x - c + \epsilon)$. Letting $\epsilon \downarrow 0$ we get that

$$
\limsup_{n \to \infty} P(X_n + Y_n \leq x) \leq P(X + c \leq x).
$$

3. Suppose that $X_n = X$ and $Y_n \geq 0$, and $Y_n \Rightarrow c$, where $c > 0$ is a constant then $X_n Y_n \Rightarrow cX$.

**Proof.** For any $x \geq 0$ and $\epsilon > 0$, we have

$$
P(X_n Y_n \leq x) \geq P(X_n \leq \frac{x}{c + \epsilon}) - P(Y_n > c + \epsilon).
$$
The second term on the right side goes to zero as \( n \to \infty \). If \( \frac{x}{c+\epsilon} \) is a continuity point of the distribution function of \( X \) then the first term on the right hand side goes to \( P(X \leq \frac{x}{c+\epsilon}) \). Letting \( \epsilon \downarrow 0 \) it follows that if \( x \) is a continuity point of the distribution of \( cX \)

\[
\liminf_{n \to \infty} P(X_n Y_n \leq x) \geq P(cX \leq x).
\]

For any \( x \geq 0 \) and \( \epsilon \in (0, c) \), we have

\[
P(X_n Y_n \leq x) \leq P(X_n \leq \frac{x}{c-\epsilon}) + P(Y_n < c - \epsilon).
\]

The second term on the right side goes to zero as \( n \to \infty \). If \( \frac{x}{c-\epsilon} \) is a continuity point of the distribution function of \( X \) then the first term on the right hand side goes to \( P(X \leq \frac{x}{c-\epsilon}) \). Letting \( \epsilon \downarrow 0 \) it follows that

\[
\limsup_{n \to \infty} P(X_n Y_n \leq x) \leq P(cX \leq x).
\]

The proof for \( x < 0 \) is similar.

4. Show that

\[
\rho(F, G) = \inf \{ \epsilon : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x \}
\]

defines a metric on the space of distribution functions and \( \rho(F_n, F) \to 0 \) if and only if \( F_n \Rightarrow F \).

**Proof.** It is clear that \( \rho(F, G) = 0 \) if and only if \( F = G \). It is also easy to see that \( \rho(F, G) = \rho(G, F) \). So to prove that \( \rho \) is a metric we only need to check the triangle inequality. Suppose \( F, G, H \) are three distribution functions. If \( \epsilon \) and \( \eta \) are such that

\[
F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon, \quad \forall x
\]

\[
G(x - \eta) - \eta \leq H(x) \leq F(x + \eta) + \eta, \quad \forall x,
\]

Then for all \( x \) we have

\[
F(x - \epsilon - \eta) - \epsilon - \eta \leq H(x) \leq F(x + \epsilon + \eta) + \epsilon + \eta,
\]

thus we have \( \rho(F, H) \leq \rho(F, G) + \rho(G, H) \), and consequently \( \rho \) is a metric.

If \( \epsilon = \rho(F_n, F) \to 0 \), then for any \( \text{epsilon} > 0 \) there exists \( N \) such that when \( n \geq N \) we have

\[
F(x - \epsilon) - \epsilon \leq F_n(x) \leq F(x + \epsilon) + \epsilon.
\]

Thus we have \( F_n(x) \to F(x) \) at any continuity point \( x \) of \( F \).

Now suppose \( F_n \Rightarrow F \). For any \( \text{epsilon} > 0 \), let \( x_1 < x_2 < \cdots < x_k \) be continuity points of \( F \) such that \( F(x_1) < \epsilon, F(x_k) > 1 - \epsilon \) and \( x_j - x_{j-1} < \epsilon \) for \( j = 2, \ldots, k \). There exists \( N \)
such that when \( n \geq N \), \( |F_n(x_j) - F(x_j)| \leq \epsilon \) for \( j = 1, \ldots, k \). If \( x \in (x_j, x_{j+1}) \), then when \( n \geq N \) we have

\[
F_n(x) \leq F_n(x_{j+1}) \leq F_n(x_{j+1}) + \epsilon \leq F(x + \epsilon) + \epsilon
\]

\[
F_n(x) \geq F_n(x_j) \geq F_n(x_j) - \epsilon \leq F(x - \epsilon) - \epsilon.
\]

If \( x < x_1 \), then when \( n \geq N \) we have

\[
F_n(x) \leq F_n(x_1) \leq F(x_1) + \epsilon \leq 2\epsilon \leq F(x + 2\epsilon) + 2\epsilon
\]

\[
F_n(x) \geq 0 \geq F(x - \epsilon) - \epsilon.
\]

We have a similar situation when \( x > x_k \). Thus we have shown that \( \rho(F_n, F) \leq 2\epsilon \) when \( n \geq N \).