Surface Integrals, Stokes's Theorem and Gauss's Theorem

Specifying a surface:

- Parametric form: $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$ in some region D in $\mathbb{R}^2$.
- Explicit function: $z = g(x,y)$ where g is a function in some region D in $\mathbb{R}^2$.
- Implicit function: $g(x,y,z) = 0$, for some function g of three variables.

Normal vector and unit normal vector:

Normal vectors and unit normal vectors to a surface play an important role in surface integrals. Here are some ways to obtain these vectors.

- Geometric reasoning. For surfaces of simple shape (e.g., planes or spherical surfaces), the easiest way to get normal vectors is by geometric arguments. For example, if the surface is spherical and centered at the origin, then $\mathbf{x}$ is a normal vector. If the surface is horizontal or vertical, one can take one of the unit basis vectors (or its negative) as normal vectors. Such geometric reasoning requires virtually no calculation, and is usually the quickest way to evaluate a surface integral in situations where it can be applied.
- Surfaces given by an equation $g(x,y,z) = 0$. In this case, the gradient of g, ∇g, is normal to the surface. Normalizing this vector, we get a unit normal vector, $\mathbf{n} = \nabla g / \|\nabla g\|$. Similarly, if the surface is given by an equation $z = g(x,y)$, the vector $(g_x, g_y, 1)$ is normal to the surface.
- Surfaces given by a parametric representation $\mathbf{X}(s,t)$. In this case, the vector $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$ is a normal vector, called the standard normal vector of the given parametrization. Normalizing this vector, we get a unit normal vector, $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$.

Surface area element:

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\begin{aligned}
dS &= \|\mathbf{N}\| dsdt = \|\mathbf{T}_s \times \mathbf{T}_t\| dsdt \quad \text{(general parametrized surface)}, \\
dS &= \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2} \quad \text{(general parametrized surface, explicit form)}, \\
dS &= \sqrt{g_x^2 + g_y^2 + 1} dxdy, \quad (\text{ surface } z = g(x,y)).
\end{aligned}
$$

Surface integral of scalar functions $\iint_S f dS$:

- **Applications**: Surface area (with $f = 1$), mass (with f as the mass density), centroid (with f as the x, y, or z coordinates, multiplied by the density), etc.
- **Evaluation**: Reduce to ordinary double integral by parametrizing S, and using the above formula for the surface area elements dS.

Surface integral of vector fields $\iint_S \mathbf{F} \cdot d\mathbf{S}$:

- **Definition, notation and evaluation**:

  (1) Reducing to scalar surface integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS.$$

  (2) Reducing to double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_s \times \mathbf{T}_t) dsdt.$$

- **Interpretation/application**: Flux of $\mathbf{F}$ across S.
- **Direct evaluation via unit normal vector**:
  - Compute the unit normal $\mathbf{n}$.
  - Compute $\mathbf{F} \cdot \mathbf{n}$.
  - Substitute this expression, along with the appropriate formula for the surface area element into the formula (1) above.
- **Direct evaluation via parametrization**:
  - Compute $\mathbf{T}_s \times \mathbf{T}_t$ for the parametrization.
  - Compute $\mathbf{F} \cdot (\mathbf{T}_s \times \mathbf{T}_t)$.

  - Substitute this expression into formula (2) above.

- **Indirect evaluation via Gauss's theorem**: If the conditions for Gauss's theorem are satisfied, this is usually the quickest way to compute the surface integral.

Stokes's Theorem (Divergence Theorem):

Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$. Suppose that ∂S consists of finitely many piecewise smooth, simple, closed curves each of which is oriented consistently with S. If $\mathbf{F}$ is a C^1 vector field whose domain contains S, then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Gauss's Theorem (Divergence Theorem):

Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by the unit outward normal. If $\mathbf{F}$ is a C^1 vector field whose domain contains S, then

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} dV.$$

Analogies to other vector calculus theorems:

- Stokes's Theorem is analogous the curl form of Green's theorem.
- Gauss's Theorem is analogous the divergence form of Green's

Comparison between Stokes's Theorem and Gauss's Theorem :

Both theorems can be used to evaluate certain surface integrals, but there are some significant differences: Gauss's Theorem applies only to surface integrals over closed surfaces; Stokes's Theorem applies to any surface integrals satisfying the above basic assumptions. Stokes's Theorem also applies to closed surfaces as well, but in this case gives only a trivial result, since the boundary of a closed surface is empty, so both sides of Stokes's Theorem become zero.