1. (20 points) \(\mathbf{a}\) and \(\mathbf{b}\) are two unit vectors in \(\mathbb{R}^3\) forming an angle of 30 degrees (i.e., \(\pi/6\) in radians), but are otherwise arbitrary.

(a) Find the area of the triangle spanned by \(\mathbf{a}\) and \(\mathbf{b}\).

Solution. The area is
\[
\frac{1}{2} \|\mathbf{a} \times \mathbf{b}\| = \frac{1}{2} \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta = \frac{1}{2} \sqrt{3} = \frac{1}{4}.
\]

(b) Evaluate \(\| (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) \|\).

Solution.
\[
\| (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) \| = \| (\mathbf{a} - \mathbf{b}) \times \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{b} \|
\]
\[
= \| \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{b} \|
\]
\[
= \| 2\mathbf{a} \times \mathbf{b} \| = 2 \|\mathbf{a} \times \mathbf{b}\|
\]
\[
= 2 \|\mathbf{a}\| \|\mathbf{b}\| \sin \frac{\pi}{6} = 1.
\]

(c) Evaluate \((\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})\).

Solution.
\[
(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{b}
\]
\[
= \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0.
\]

(d) Evaluate \((\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})\).

Solution.
\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0.
\]

2. (20 points) Let \(L_1\) and \(L_2\) denote the lines with parametric equations given by
\[
L_1:\quad x = 2t, \quad y = t + 5, \quad z = 3t - 1,
\]
\[
L_2:\quad x = t - 1, \quad y = -t + 2, \quad z = 0.
\]

(a) Find the distance between \(L_1\) and \(L_2\).

Solution. The point \(A: (0,5,-1)\) is on the line \(L_1\) and the point \(B: (-1,2,0)\) is on the line \(L_2\). We need to find the projection of \(\mathbf{a} = \overrightarrow{AB} = (-1,-3,1)\) onto a vector \(\mathbf{n}\) perpendicular to both \(L_1\) and \(L_2\). For the vector \(\mathbf{n}\), we could use \(\mathbf{n} = \mathbf{b} \times \mathbf{c}\), where \(\mathbf{b} = (2,1,3)\) is the directional vector of \(L_1\) and \(\mathbf{c} = (1,-1,0)\) is the directional vector of \(L_2\). Thus
\[
\mathbf{n} = \mathbf{b} \times \mathbf{c} = (3,3,-3)
\]
and so
\[
\text{proj}_n \mathbf{a} = -\frac{5}{3} (1,1,-1)
\]
and the desired distance is \(\frac{5}{3} \sqrt{3}\).
(b) Find the equation of the plane containing the line \( L_1 \) and the origin.

**Solution.** The plane must be perpendicular to both the directional vector \( \mathbf{b} = (2, 1, 3) \) of \( L_1 \) and the vector \( \mathbf{d} = (0, 5, -1) \). Thus the vector

\[
\mathbf{n}_1 = \mathbf{b} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 0 & 5 & -1 \end{vmatrix} = (-16, 2, 10)
\]

is a normal vector to the desired plane. Thus the equation of the plane is

\[-16x + 2y + 10z = 0\]

which can be rewritten as

\[8x - y - 5z = 0.\]

3. (12 points) Determine if the following limits exist. If the limit does not exist, give an explanation. If the limit exists, find the limit and prove the limit using the \( \varepsilon-\delta \) definition of the limit.

(a) \( \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \).

**Solution.** Along the \( x \)-axis, the function \( f \) reduces to

\[f(x, 0) = \frac{x^2 - 0}{x^2 + 0} = 1.\]

Thus

\[\lim_{(x,y)\to(0,0), \text{ along } y=0} \frac{x^2 - y^2}{x^2 + y^2} = 1.\]

Along the \( y \)-axis, the function \( f \) reduces to

\[f(0,1) = \frac{0 - y^2}{0 + y^2} = -1.\]

Thus

\[\lim_{(x,y)\to(0,0), \text{ along } x=0} \frac{x^2 - y^2}{x^2 + y^2} = -1.\]

Therefore, the limit \( \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \) does not exist.

(b) \( \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} \).

**Solution.**

\[\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0.\]

Here is a proof of the last equality. For any \( \varepsilon > 0 \), take \( \delta = \sqrt{\varepsilon} \). Whenever \( 0 < \|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta \), we have

\[|x^2 - y^2 - 0| = |x^2 - y^2| \leq x^2 + y^2 = \delta^2 = \varepsilon.\]

Therefore \( \lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0. \)
4. (20 points)
(a) Find the equation of the tangent plane to the surface \( x^2 + y^2 - z^2 - 2xy + 4xz - 4 = 0 \) at the point \((1, 0, 1)\).

**Solution.** Let \( f(x, y, z) = x^2 + y^2 - z^2 - 2xy + 4xz - 4 \). Then
\[
\nabla f(x, y, z) = (2x - 2y + 4z, 2y - 2x, -2z + 4x),
\]
and \( \nabla f(1, 0, 1) = (6, -2, 2) \). Thus the equation of the tangent plane to the surface \( x^2 + y^2 - z^2 - 2xy + 4xz - 4 = 0 \) at the point \((1, 0, 1)\) is
\[
6(x - 1) - 2y + 2(z - 1) = 0.
\]

(b) Given that \( x^2 + y^2 - z^2 - 2xy + 4xz - 4 = 0 \), find the partial derivative \( \frac{\partial z}{\partial x} \) by implicit differentiation.

**Solution.** Take the partial derivative with respect to \( x \) on both side of the equation we get
\[
2x - 2z \frac{\partial z}{\partial x} - 2y + 4z + 4x \frac{\partial z}{\partial x} = 0.
\]
Solving for \( \frac{\partial z}{\partial x} \), we get
\[
\frac{\partial z}{\partial x} = \frac{x - y + 2z}{z - 2x}.
\]

5. (20 points) Given a function \( f(x, y) = x^2 + y^2 + x^2y \).

(a) Find and classify all local minima, maxima, and saddle points of \( f \).

**Solution.** \( \nabla f(x, y) = (2x + 2xy, 2y + x^2) \). Solving \( \nabla f(x, y) = (0, 0) \), we get that the critical points of \( f \) are \((0, 0)\), \((\sqrt{2}, -1)\) and \((-\sqrt{2}, -1)\).

The Hessian of \( f \) is given by
\[
Hf(x, y) = \begin{pmatrix}
2 + 2y & 2x \\
2x & 2
\end{pmatrix}.
\]
At \((0, 0)\),
\[
Hf(x, y) = \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix},
\]
the principal minors are 2 and 4, and so \( f \) has a local minimum at \((0, 0)\).
At \((\sqrt{2}, -1)\),
\[
Hf(x, y) = \begin{pmatrix}
0 & 2\sqrt{2} \\
2\sqrt{2} & 2
\end{pmatrix},
\]
and so \( f \) has a saddle point at \((\sqrt{2}, -1)\).
At \((-\sqrt{2}, -1)\),
\[
Hf(x, y) = \begin{pmatrix}
0 & -2\sqrt{2} \\
-2\sqrt{2} & 2
\end{pmatrix},
\]
and so \( f \) has a saddle point at \((-\sqrt{2}, -1)\).
(b) Find the maximum value and minimum value of $f$ in the disk $D = \{(x, y)|x^2 + y^2 \leq 4\}$ by using the Lagrange multiplier method.

**Solution.** Let $g(x, y) = x^2 + y^2$, then $\nabla g(x, y) = (2x, 2y)$. Solving the system
\[
2x + 2xy = 2\lambda x \\
2y + x^2 = 2\lambda y \\
x^2 + y^2 = 4,
\]
we get the critical points are $(0, 2), (0, -2), (\frac{2\sqrt{2}}{\sqrt{3}}, \frac{2}{\sqrt{3}}), (-\frac{2\sqrt{2}}{\sqrt{3}}, \frac{2}{\sqrt{3}}), (\frac{2\sqrt{2}}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$ and $(-\frac{2\sqrt{2}}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$.

Now $f(0, 0) = 0$, $f(0, 2) = f(0, -2) = 4$, $f(\frac{2\sqrt{2}}{\sqrt{3}}, \frac{2}{\sqrt{3}}) = f(-\frac{2\sqrt{2}}{\sqrt{3}}, \frac{2}{\sqrt{3}}) = 4 + \frac{16}{3\sqrt{3}}$, $f(\frac{2\sqrt{2}}{\sqrt{3}}, -\frac{2}{\sqrt{3}}) = f(-\frac{2\sqrt{2}}{\sqrt{3}}, -\frac{2}{\sqrt{3}}) = 4 - \frac{16}{3\sqrt{3}}$. Thus the maximum value of $f$ in $D$ is $4 + \frac{16}{3\sqrt{3}}$ and the minimum value is 0.

6. (8 points) (a) Let $f$ be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$, $a$ a point in $\mathbb{R}^n$ and $u$ a unit vector in $\mathbb{R}^n$. Complete the following definition of the directional derivative of $f$ at $a$ in the direction of $u$ by filling in the blanks.

$$D_uf(a) = \lim_{h \to 0} \cdots.$$

**Solution.**

$$D_uf(a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}.$$

(b) Let $f$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and $a$ a point in $\mathbb{R}^n$. Complete the following definition of differentiability of $f$ at $a$ and the derivative matrix at that point, by filling in the blanks in (1) and (2).

$f$ is differentiable at $a$ with derivative $Df(a) = A$, if

$$f(a + h) = f(a) + \cdots + R(h), \quad (1)$$

where $R(h)$ satisfies

$$\cdots \quad (2)$$

**Solution.** $f$ is differentiable at $a$ with derivative $Df(a) = A$, if

$$f(a + h) = f(a) + Ah + R(h),$$

where $R(h)$ satisfies

$$\lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0.$$