Page 98. Ex. 15. Let $A$ and $B$ be $n \times n$ matrices. Prove that if $AB = I$, then $BA = I$. What is the significance of this result?

Since $AB = I$, we have $\det(AB) = \det(I) = 1$. Now $\det(AB) = \det(A) \det(B)$, so that $\det(A) \neq 0$ and $A$ is nonsingular (so is $B$). Hence, there is a matrix $C$ with $CA = I = AC$. But

$$C = CI = C(AB) = (CA)B = IB = B.$$

Therefore, $BA = I$.

First, this says that $A$ and $B$ must commute. More important, it says that to verify whether a matrix $B$ is the inverse of $A$, it suffices to show only that $AB = I$, for it follows automatically that $BA = I$.

Page 116. Ex. 3. Prove that the complex numbers $\mathbb{C}$ is a vector space.

There is no quick and clever proof. If nothing else, you now see the difference between a long problem and a hard problem! We must verify the two closure properties and the 8 axioms. Each of these is routine, essentially following from properties of real numbers. We will verify only two.

1. Given $z = a + ib$, there is $w = c + id$ with $z + w = 0$.

Define $c = -a$ and $d = -b$; then $(a + ib) + (c + id) = (a - a) + i(b - b) = 0$.

2. If $r, s$ are scalars, then $(r + s)(a + ib) = r(a + ib) + s(a + ib)$.

$$(r + s)(a + ib) = (r + s)a + i(r + s)b = (ra + sa) + i(rb + sb) = (ra + irb) + (sa + isb) = r(a + ib) + s(a + ib).$$

Page 126. Ex. 4. Let $x_1, \ldots, x_n$ be a spanning set for $V$.

(i) If we add another vector, say $x_{n+1}$ to the set, will we still have a spanning set?

Yes. Take any $v \in V$. Since the original set spans $V$, there are numbers $c_1, \ldots, c_n$ with $v = c_1x_1 + \cdots + c_nx_n$. Hence, if we define $c_{n+1} = 0$, then

$$v = c_1x_1 + \cdots + c_nx_n + c_{n+1}x_{n+1} \in V.$$

(ii) If we delete one of the vectors, say $x_i$ from the set, will we still have a spanning set?

No. We give a concrete counterexample. A spanning set for $\mathbb{R}^2$ is $x_1, x_2$, where $x_1 = (1, 0)$ and $x_2 = (0, 1)$. If we delete $x_2$, then $x_1$ by itself is not a spanning set. For example, there is no number $c_1$ with $x_2 = (0, 1) = c_1x_1 = (c_1, 0)$. 