Page 43. 8(c). If \( A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \), \( B = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} \), and \( C = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \end{bmatrix} \), verify that
\[
A(B + C) = AB + AC.
\]
We calculate. First, left hand side.
\[
A(B + C) = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \left( \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \end{bmatrix} \right) = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \end{bmatrix} = [10 \ 17]
\]
Now the right hand side.
\[
[10 \ 17] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \end{bmatrix} = [10 \ 17].
\]

Page 44. 15. A matrix \( A \) is said to be skew symmetric if \( A^T = -A \). Show that if a matrix is skew symmetric, then its diagonal entries must all be 0.

If \( A = [a_{ij}] \), then \( A^T = [a_{ji}] \) for all \( i, j \). Hence, if \( A \) is skew symmetric, then \( a_{ji} = -a_{ij} \) for all \( i, j \). In particular, \( a_{ii} = -a_{ii} \) and \( a_{ii} = 0 \).

Page 56. 8. Let
\[
A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.
\]
Compute \( A^2 \) and \( A^3 \). Just using the definition of matrix multiplication, along with some patience, we obtain \( A^2 = I \) and \( A^3 = A \).

Here is a less computational way to do this. If \( J \) is the 4 \times 4 matrix all of whose entries are 1, then \( J^2 = 4J \) (why?); hence, \( \left( \frac{1}{2} J \right)^2 = \frac{1}{4} J^2 = \frac{1}{4} 4J = J \).

Now \( A = \frac{1}{2} (2I - J) \), where \( I \) is the 4 \times 4 identity matrix. But \( 2I \) commutes with \( J \) (in fact, for any scalar \( t \), the matrix \( tI \) commutes with every matrix), and so we can use the Binomial Theorem:
\[
A^2 = \frac{1}{4} (4I - 4J + J^2) = \frac{1}{4} (4J - 4J + 4I) = I.
\]
And it is now easy to compute \( A^3 = AA^2 = AI = A \).

The last part of the problem can be done by induction. We prove that \( A^{2n+1} = A \) by induction on \( n \geq 0 \). The base step \( n = 0 \) says that \( A^1 = A \); sure it is. For the inductive step, the inductive hypothesis \( A^{2n+1} = A \) gives
\[
A^{2(n+1)+1} = A^{2(n+1)+2} = AA^2 = A^3 = A,
\]
for we have already verified that $A^3 = A$.

We prove that $A^{2n} = I$ by induction on $n \geq 0$. The base step $n = 0$ says that $A^0 = I$; this is the definition of exponent 0: $M^0 = I$ for every matrix $M$ (if you don’t like this, start your induction at $n = 1$, so that the base step is $A^2 = I$, which we have already verified). For the inductive step, the inductive hypothesis $A^{2n} = I$ gives

$$A^{2(n+1)} = A^{2n+2} = A^{2n}A^2 = IA^2 = I,$$

for $A^2 = I$.

Here is a proof without induction. If the exponent is even, say $2n$, then $A^{2n} = (A^2)^n = I^n = I$, while if the exponent is odd, say $2n + 1$, then $A^{2n+1} = A^{2n}A = IA = A$ (for we just saw that $A^{2n} = I$).