1. Define each of the first five terms.

(i) (4 points) The list \( v_1, \ldots, v_n \) of vectors is linearly independent.

Whenever there are scalars \( c_1, \ldots, c_n \) with \( c_1 v_1 + \cdots + c_n v_n = 0 \), then all \( c_i = 0 \).

(ii) (4 points) Dimension of a finite-dimensional vector space \( V \).

The dimension of \( V \) is the number of elements in a basis of \( V \).

(iii) (4 points) Orthogonal complement \( S^\perp \) of a subspace \( S \) of a vector space \( V \).

\[ S^\perp = \{ v \in V : (v, s) = 0 \text{ for all } s \in S \} \]

(iv) (4 points) A Hermitian matrix \( A = [a_{jk}] \). (You must explain your notation.)

\[ A^H = A. \text{ Here, } A^H = [a_{kj}]^\top. \]

(v) (4 points) An eigenvalue of a linear operator \( L : V \to V \).

A number \( \lambda \) for which there exists a nonzero vector \( v \) with \( Av = \lambda v \).

(vi) (5 points) What are the columns of the transition matrix \( P \) from a basis \( X \) to a basis \( Y \)?

If \( X = x_1, \ldots, x_n \), then the \( j \)th column of \( P \) consists of the coordinates of \( x_j \) with respect to \( Y \).

2. Consider the homogeneous linear system

\[
\begin{align*}
x_1 + x_2 + x_3 - x_4 &= 0 \\
2x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\
x_1 - x_2 + 2x_3 + 3x_4 + x_5 &= 0
\end{align*}
\]

i) (10 points) Find a basis of the row space of this system.

The row reduced echelon form for the coefficient matrix is

\[
E = \begin{bmatrix}
1 & 0 & 0 & -\frac{3}{4} & \frac{1}{2} \\
0 & 1 & 0 & -\frac{4}{7} & -\frac{1}{7} \\
0 & 0 & 1 & 3 & 0
\end{bmatrix}.
\]

A basis of the row space consists of the rows of \( E \).
ii) (15 points) Find a basis of the solution space of this system.

Using $E$, we see that the free variables are $x_4$ and $x_5$, and

\[
\begin{align*}
  x_1 &= \frac{3}{2}x_4 - \frac{1}{2}x_5 \\
  x_2 &= \frac{1}{2}x_4 + \frac{1}{2}x_5 \\
  x_3 &= -3x_4
\end{align*}
\]

Thus, a basis for the solution space is

\[
\left(\frac{2}{3}, \frac{1}{3}, -3\right)^T, \left(-\frac{1}{2}, \frac{1}{2}, 0\right)^T.
\]

3. Let $A$ be an $n \times n$ matrix, where $n \geq 2$.

(i) (10 points) If $A$ is singular, prove that $\text{adj}(A)$ is singular.

We know that $A \text{adj}(A) = \det(A)I$. Since $A$ is singular, $\det(A) = 0$, and so $A \text{adj}(A) = 0$. If $A = 0$, then $\text{adj}(A) = 0$; if $A \neq 0$, then $\text{adj}(A)$ is singular: otherwise, multiply on the right by $\text{adj}(A)^{-1}$ and get $A = 0$.

(ii) (15 points) If $A$ is nonsingular, find $\det(\text{adj}(A))$; justify your answer.

Since $A \text{adj}(A) = \det(A)I$, we have

\[
\det(\text{adj}(A)) = \det(A) \det(\text{adj}(A)) = \det(A)^n.
\]

Therefore, $\det(\text{adj}(A)) = \det(A)^{n-1}$.

4. Let $V$ and $W$ be vector spaces, let $U = u_1, \ldots, u_n$ be a basis of $V$, and let $w_1, \ldots, w_n$ be a list of (not necessarily distinct) vectors in $W$. Prove that there is a unique linear transformation $L: V \rightarrow W$ with $L(u_i) = w_i$ for all $i$. You must show that the $L$ you construct is single-valued.

Each $v \in V$ has coordinates wrt $U$: there are unique scalars $c_1, \ldots, c_n$ with $v = c_1 u_1 + \cdots + c_n u_n$. Define $L$ by

\[
L(v) = c_1 w_1 + \cdots + c_n w_n.
\]

Note that uniqueness of the coordinates shows that $L$ is a single-valued function. Obviously, $L(u_i) = w_i$ for all $i$, and so it remains to prove that $L$ is a linear transformation.

If $v' = c'_1 u_1 + \cdots + c'_n u_n$, then $v + v' = (c_1 + c'_1) u_1 + \cdots + (c_n + c'_n) u_n$, and so $L(v + v') = (c_1 + c'_1) w_1 + \cdots + (c_n + c'_n) w_n$. On the other hand, $L(v) + L(v') = [c_1 w_1 + \cdots + c_n w_n] + [c'_1 w_1 + \cdots + c'_n w_n] = (c_1 + c'_1) u_1 + \cdots + (c_n + c'_n) u_n$.

Finally, if $\alpha$ is a scalar, then

\[
L(\alpha v) = L(\alpha c_1 u_1 + \cdots + \alpha c_n u_n)
= \alpha c_1 w_1 + \cdots + \alpha c_n w_n
= \alpha(c_1 w_1 + \cdots + c_n w_n)
= \alpha L(v).
\]
To prove uniqueness, suppose that $T: V \rightarrow W$ is a linear transformation with $T(u_i) = w_i$ for all $i$. Since $T$ preserves linear combinations, we have

$$T(v) = T(c_1u_1 + \cdots + c_nu_n)$$
$$= c_1T(u_1) + \cdots + c_nT(u_n)$$
$$= c_1w_1 + \cdots + c_nw_n$$
$$= L(v).$$

Therefore, $T = L$.

**5.** Let $S$ be the set of all $4 \times 4$ skew-symmetric matrices.

(i) (10 points) Prove that $S$ is a subspace of $\mathbb{R}^{4 \times 4}$.

Recall that $A$ is skew-symmetric if $A^\top = -A$. Clearly, the zero matrix is skew-symmetric: $0 \in S$. If $A, B \in S$, then $A^\top = -A$ and $B^\top = -B$, so that $(A + B)^\top = A^\top + B^\top = -A - B = -(A + B)$; thus, $A + B \in S$. Finally, if $\alpha$ is a scalar and $A \in S$, then $(\alpha A)^\top = \alpha A^\top = \alpha(-A) = -(\alpha A)$, and so $\alpha A \in S$. Therefore, $S$ is a subspace.

(ii) (15 points) Find dim($S$); justify your answer.

Let $E_{ij}$ be the $4 \times 4$ matrix having 1 in the $ij$ spot and all other entries 0. Now skew-symmetric matrices $A = [a_{ij}]$ must have 0’s on the diagonal: since $a_{ji} = -a_{ij}$, we have $a_{ii} = -a_{ii}$. Therefore,

$$A = \begin{bmatrix}
0 & a & b & c \\
-a & 0 & d & e \\
-b & -d & 0 & f \\
-c & -e & -f & 0
\end{bmatrix},$$

and a basis for $S$ consists of all $E_{ij} - E_{ji}$ for all $i < j$. Hence, dim($S$) = 6.

**6.** (25 points) Let $p$ be a solution of the system $Ax = b$, where $b \neq 0$. Prove that every solution has a unique expression of the form $u + p$, where $u$ is a solution of the homogeneous system $Ax = 0$.

Let $S$ be all the solutions of $Ax = b$, and let $Z$ be the set of all $u + p$ where $Au = 0$.

We claim $S \subseteq Z$. If $s \in S$, then $As = b$. Hence, $A(s - p) = As - Ap = b - b = 0$; that is, $s - p$ is a solution of $Ax = 0$, and so $s = (s - p) + p \in Z$.

For the reverse inclusion $Z \subseteq S$, take $u + p \in Z$. Then $Au = 0$ and $Ap = b$, so that $A(u + p) = Au + Ap = 0 + b = b$; that is, $u + p \in S$.

For uniqueness, suppose that $u + p = u' + p$, where $Au = 0 = Au'$. Just cancel $p$ to obtain $u = u'$. 
7. Find the inverse of \( A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \).

You can either use \( \text{adj}(A) \) or Gaussian elimination. Either way,

\[ A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -3 & -1 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix}. \]

8. Let \( A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \).

(i) (10 points) Prove that there is no matrix \( P \) with real entries such that \( P^{-1} A P \) is diagonal.

The characteristic polynomial of \( A \) is \( p(\lambda) = \lambda^2 - \sqrt{3}\lambda + 1 \), and the quadratic formula gives the eigenvalues to be \( \frac{1}{2}(\sqrt{3} \pm i) \). Now if \( PAP^{-1} = D \), where \( D \) is diagonal, then \( D \) has real entries and its diagonal entries would have to be the eigenvalues of \( A \), for similar matrices have the same eigenvalues. But the eigenvalues of \( A \) are not real.

(ii) (15 points) Find a matrix \( Q \) with complex entries such that \( Q^{-1} A Q \) is diagonal.

If \( Y = y_1, y_2 \) is a basis of eigenvalues of \( \mathbb{R}^2 \), then the matrix of \( A \) wrt to \( Y \) is a diagonal matrix with the eigenvalues of \( A \) on the diagonal. Thus, a matrix \( Q \) is a transition matrix from \( Y \) to the standard basis.

An eigenvector of \( A \) belonging to \( \sqrt{3} + i \) is \( (i, 1)^\top \), and an eigenvector of \( A \) belonging to \( \sqrt{3} - i \) is \( (-i, 1)^\top \). Thus,

\[ Q = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \]

is such a matrix (since eigenvectors are determined up to nonzero scalar multiples, \( Q \) is not unique: we can multiply the first column by any nonzero complex number \( \alpha \) and the second column by any nonzero complex number \( \beta \)).