Errata for Advanced Modern Algebra, Chapters 8 to 11

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This list of errors will be corrected in the next printing of the book. If you have found any other mistakes – typos, errors in a proof, false statements, unclear exposition, important omission – please write me at

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Page 523, line 4 Should read:

$$\int_0^\infty |f(x)|\,dx = \lim_{t\to\infty}\int_0^t |f(x)|\,dx < \infty.$$

Page 523, line 22 Change "$a + b \in S$" to "$a - b \in S$"

Page 524, line 4 Change "$ab + (ad + bc)i$" to "$ac + (ad + bc)i$"

Page 528, line 8 Should read: "$\sigma\colon R \to \mathrm{End}_{\mathbb{Z}}(M)$."

Page 528, line -8 Change "8.37" to "8.8"

Page 529, lines 16 and 18 Change "μ'" to "μ^{op}"

Page 529, line 21 Change "because R" to "because M"

Page 531 Remove (ii) of Exercise 8.3, and change the wording of (i) because the term *left artinian* has not yet been introduced.

(i) For every sequence of left ideals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$, there exists N so that $L_i = L_{i+1}$ for all $i \geq N$.

Page 532, line -4 Change hint to read:

Hint. Define $R = \mathrm{End}_k(V)$, where V is a vector space over a field k with basis $\{v_n : n \geq 1\}$, and define $a \in R$ by $a(v_n) = v_{n+1}$.

Page 532, line -1 Change formula in hint to read "$u + a^n(1 - au)$"

Page 537, line 14 Should read "$v_j \notin \langle v_{j+1}, \ldots, v_n\rangle$"

Page 538, line 15 Change "the restriction map" to "the map $\psi \mapsto f\psi f^{-1}$"

Page 541, line 8 Change "$\mathrm{End}_k(V) =$" to "$\mathrm{End}_\Delta(V) =$"

Page 542 Rewrite Example 8.27(ii) as follows.

(ii) Every ring R is a $\mathbb{Z}$-algebra, and every ring homomorphism is a $\mathbb{Z}$-algebra map. This example shows why, in the definition of R-algebra, we do not demand that k be isomorphic to a subring of R.

Page 545, lines 15-17 Should read

$\Rightarrow$ (i) If $x(R/I) = \{0\}$, then $x(1+I) = x + I = I$; that is, $x \in I$. Therefore, if $x(R/I) = \{0\}$ for every maximal left ideal I, then $x \in \bigcap_I I = J(R)$.

Page 546, line 11 Rewrite as follows:

Recall that A^m is the set of all sums of the form $a_1 \cdots a_m$, where $a_j \in A$; that is, $A^m = \{\sum_i a_{i1} \cdots a_{im} : a_{ij} \in A\}$.

Page 546, line -2 Change "$1 - b$" to "$1 - x$"

Page 549 Replace Exercise 8.34, as follows.

8.34. If R is a ring and M is a left R-module, prove that $\mathrm{Hom}_R(R, M)$ is a left R-module, and prove that it is isomorphic to M.
Hint. If $f : R \to M$ and $r' \in R$, define $r'f : r \mapsto rr'$. Compare Exercise 7.5 on page 440.

Page 551, line 19 Should read: $\varphi\tau(x) = \sigma(x)\varphi$

Page 552, lines 13, 14, 15 Rewrite:

Since R is left semisimple, it is a direct sum of minimal left ideals: $R = \sum_i L_i$. Let $1 = \sum_i e_i$, where $e_i \in L_i$. If $r = \sum_i r_i \in \sum_i L_i$, then $r = 1r$ and so $r_i = e_i r_i$. Hence, if $e_i = 0$, then $L_i = \{0\}$. We conclude that there are only finitely many nonzero L_i; that is, $R = L_1 \oplus \cdots \oplus L_n$. Now the series

Page 553, line 12 Change "Corollary 7.17" to "Corollary 7.18"

Page 554, line 2 Change "Corollary 7.17" to "Corollary 7.18"

Page 555 Change the first paragraph of proof of Theorem 8.46

Proof. It suffices to prove that R, regarded as a left module over itself, has a composition series, for then Proposition 8.17 applies at once to show that R is left noetherian as a module over itself; that is, R has the ACC on left ideals.

Page 557, lines -7 to -5 Change as follows.

If E is a left R-module, then Proposition 7.64 says that E is injective if every exact sequence $0 \to E \to B \to C \to 0$ splits. By hypothesis, B is a semisimple module, and so Proposition 8.42 implies that the sequence splits; thus, E is injective.

Page 560, line -6 Change "$J(R) = \{0\}$" to "$L^2 \neq \{0\}$"

Page 560, line -2 Change "$bb' \neq \{0\}$" to "$bb' \neq 0$"

Page 561, line 4 Change "(and $J(R) = \{0\}$)" to "and $L^2 \neq \{0\}$"

Page 561 Replace lines 5 through 8 as follows.

Assume now that $L^2 \neq \{0\}$, so there are $x, y \in L$ with $xy \neq 0$. If $g\colon L \to L'$ is an isomorphism, then $0 \neq g(xy) = xg(y) \in LL'$, and so $LL' \neq \{0\}$. •

Note that if $J(R) = \{0\}$, then $L^2 \neq \{0\}$. Otherwise, L is a nilpotent left ideal and Corollary 8.33 gives $L \subseteq J(R) = \{0\}$, a contradiction.

Page 561, line 11 Change "Exercise 7.5 on page 440" to "Exercise 8.34 on page 549"

Page 561, line 12 Change "$S \cong L_j$" to "$\text{Hom}_R(L_j, S) \neq \{0\}$"

Page 564, line -14 Change "$\text{Mat}_n(\Delta)$" to "$\text{Mat}_n(\Delta^{\text{op}})$"

Page 565 lines 12 - 14 should read:

Recall, for each i, that B_i is a direct sum of left ideals L isomorphic to L_i. If $L \cong L_i$ and $L' \cong L_j$, then Lemma 8.53 applies to give $LL' = \{0\}$ if $j \neq i$. Hence, if $j \neq i$,

Page 566 Add the following at end of line 10.

Write $R = B_I \oplus B_J$, where $B_I = \sum_i B_i$ with $B_i \subseteq D$ and $B_J = \sum_j B_j$ with $B_j \not\subseteq D$. By Corollary 7.18 (which holds for modules over noncommutative rings), $D = B_I \oplus (D \cap B_J)$. But $D \cap B_J = \{0\}$; otherwise, it would contain a minimal left ideal $L \cong L_j$ for some $j \in J$ and, as above, this would force $B_j \subseteq D$. Therefore, $D = B_I$.

Page 566 Add following to Corollary 8.62(i).

Moreover, if L is a minimal left ideal in A, then $\Delta^{\text{op}} \cong \text{End}_A(L)$.

Page 566 Add following paragraph to proof.

We may now assume that $A = \text{Mat}_n(\Delta)$ and that $L = \text{Col}(1)$, the minimal left ideal consisting of all the $n \times n$ matrices whose last $n-1$ columns are 0 (see Proposition 8.49). Define $\varphi\colon \Delta \to \text{End}_A(L)$ as follows: if $d \in \Delta$ and $\ell \in L$, then $\varphi_d\colon \ell \mapsto \ell d$. Note that φ_d is an A-map: it is additive and, if $a \in A$ and $\ell \in L$, then $\varphi_d(a\ell) = (a\ell)d = a(\ell d) = a\varphi_d(\ell)$. Next, φ is a ring antihomomorphism: $\varphi_1 = 1_L$, it is additive, and $\varphi_{dd'} = \varphi_{d'}\varphi_d$: if $\ell \in L$, then $\varphi_{d'}\varphi_d(\ell) = \varphi_d(\ell d') = \ell d' d = \varphi_{dd'}(\ell)$; that is, φ is a ring homomorphism $\Delta^{\text{op}} \to \text{End}_A(L)$. To see that φ is injective, note that each $\ell \in L \subseteq \text{Mat}_n(\Delta)$ is a matrix with entries in Δ; hence, $\ell d = 0$ implies $\ell = 0$. Finally, we show that φ is surjective. Let $f \in \text{End}_A(L)$. Now $L = AE_{11}$, where E_{11} is the matrix unit (every simple module is generated by any nonzero element in it). If $u_i \in \Delta$, let $[u_1, \ldots, u_n]$ denote the $n \times n$ matrix in L whose first column is $(u_1, \ldots, u_n)^t$ and whose other entries are all 0. Write $f(E_{11}) = [d_1, \ldots, d_n]$. If $\ell \in L$, then ℓ has the form $[u_1, \ldots, u_n]$, and using only the definition of matrix multiplication, it is easy to see that $[u_1, \ldots, u_n] = [u_1, \ldots, u_n]E_{11}$. Since f is an A-map,

$$\begin{aligned} f([u_1, \ldots, u_n]) &= f([u_1, \ldots, u_n]E_{11}) \\ &= [u_1, \ldots, u_n]f(E_{11}) \\ &= [u_1, \ldots, u_n][d_1, \ldots, d_n] \\ &= [u_1, \ldots, u_n]d_1 = \varphi_{d_1}([u_1, \ldots, u_n]). \end{aligned}$$

Therefore, $f = \varphi_{d_1} \in \operatorname{im}\varphi$, as desired. •

Page 567 Rewrite the last paragraph of the proof of Theorem 8.64

Dropping subscripts, it remains to prove that if $B = \operatorname{Mat}_n(\Delta) \cong \operatorname{Mat}_{n'}(\Delta') = B'$, then $n = n'$ and $\Delta \cong \Delta'$. In Proposition 8.49, we proved that $\operatorname{Col}(\ell)$, consisting of the matrices with jth columns 0 for all $j \neq \ell$, is a minimal left ideal in B, so that $\operatorname{Col}(\ell)$ is a simple B-module. Therefore,

$$\{0\} \subseteq \operatorname{Col}(1) \subseteq \operatorname{Col}(1) \oplus \operatorname{Col}(2) \subseteq \cdots \subseteq \operatorname{Col}(1) \oplus \cdots \oplus \operatorname{Col}(n) = B$$

is a composition series of B as a module over itself. By the Jordan-Hölder theorem (Theorem 8.18), n and the factor modules $\operatorname{Col}(\ell)$ are invariants of B. Now $\operatorname{Col}(\ell) \cong \operatorname{Col}(1)$ for all ℓ, by Corollary 8.63, and so it suffices to prove that Δ can be recaptured from $\operatorname{Col}(1)$. But this has been done in Corollary 8.63(i): $\Delta \cong \operatorname{End}_B(\operatorname{Col}(1))^{\mathrm{op}}$.

Page 567 Rewrite the proof of Corollary 8.65

Proof. By Maschke's theorem, kG is a semisimple ring, and its simple components are isomorphic to matrix rings of the form $\operatorname{Mat}_n(\Delta)$, where Δ arises as $\operatorname{End}_{kG}(L)^{\mathrm{op}}$ for some minimal left ideal L in kG. Therefore, it suffices to show that $\operatorname{End}_{kG}(L)^{\mathrm{op}} = \Delta = k$.

Now $\operatorname{End}_{kG}(L)^{\mathrm{op}} \subseteq \operatorname{End}_k(L)^{\mathrm{op}}$, which is finite-dimensional over k because L is; hence, $\Delta = \operatorname{End}_{kG}(L)^{\mathrm{op}}$ is finite-dimensional over k. Each $f \in \operatorname{End}_{kG}(L)$ is a kG-map and, hence, a k-map; that is, $f(au) = af(u)$ for all $a \in k$ and $u \in L$. Therefore, the map $\varphi_a \colon L \to L$, given by $u \mapsto au$, commutes with f; that is, k (identified with all φ_a) is contained in $Z(\Delta)$, the center of Δ. If $\delta \in \Delta$, then δ commutes with every element in k, and so $k(\delta)$, the subdivision ring generated by k and δ, is a (commutative) field. As Δ is finite-dimensional over k, so is $k(\delta)$; that is, $k(\delta)$ is a finite extension of the field k, and so δ is algebraic over k, by Proposition 3.117. But k is algebraically closed, so that $\delta \in k$ and $\Delta = k$. •

Page 570, line -7 Add a phrase:

Then U is abelian and there are positive integers m_i with

Page 572, line 12 Change "has is of" to "has"

Page 573, line 9 Replace Exercise 8.36 with:

8.36 Let A be an n-dimensional k-algebra over a field k. Prove that A can be imbedded as a k-subalgebra of $\operatorname{Mat}_n(k)$.

Hint. If $a \in A$, define $L_a \colon A \to A$ by $L_a \colon x \mapsto ax$.

Page 575, line 12 Should read:

tensor product of the homology groups of the factors X and Y.

Pages 582-583 Replace the proof of Proposition 8.84 through line -10 on page 583 by the following.

Proof. Define a ***triadditive*** function $f\colon A \times B \times C \to G$, where G is an abelian group, to be a function that is additive in each of the three variables (when we fix the other two),

$$f(ar, b, c) = f(a, rb, c), \quad \text{and} \quad f(a, bs, c) = f(a, b, sc),$$

for all $r \in R$ and $s \in S$. Consider the univeral mapping problem described by the diagram

$$\begin{array}{ccc} A \times B \times C & \xrightarrow{\ h\ } & T(A, B, C), \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \widetilde{f}} & \\ & G & \end{array}$$

where G is an abelian group, f is triadditive, and $\widetilde{f}$ is a $\mathbb{Z}$-homomorphism. As for biadditive functions and tensor products of two modules, define $T(A, B, C) = F/N$, where F is the free abelian group on all ordered triples $(a, b, c) \in A \times B \times C$, and N is the obvious subgroup of relations. Define $h\colon A \times B \times C \to T(A, B, C)$ by

$$h\colon (a, b, c) \mapsto (a, b, c) + N$$

(denote $(a, b, c) + N$ by $a \otimes b \otimes c$). A routine check shows that this construction does give a solution to the universal mapping problem for triadditive functions.

We now show that $A \otimes_R (B \otimes_S C)$ is another solution to this universal problem. Define a triadditive function $\eta\colon A \times B \times C \to A \otimes_R (B \otimes_S C)$ by $\eta\colon (a, b, c) \mapsto a \otimes (b \otimes c)$. For each $a \in A$, the S-biadditive function $f_a\colon B \times C \to G$, defined by $(b, c) \mapsto f(a, b, c)$, gives a unique homomorphism $\widetilde{f}_a\colon B \otimes_S C \to G$ taking $b \otimes c \mapsto f(a, b, c)$. If $a, a' \in A$, then $\widetilde{f}_{a+a'}(b \otimes c) = f(a + a', b, c) = f(a, b, c) + f(a', b, c) = \widetilde{f}_a(b \otimes c) + \widetilde{f}_{a'}(b \otimes c)$. It follows that the function $\varphi\colon A \times (B \otimes_S C) \to G$, defined by $\varphi(a, b \otimes c) = \widetilde{f}_a(b \otimes c)$, is additive in both variables. It is R-biadditive, for if $r \in R$, then $\varphi(ar, b \otimes c) = \widetilde{f}_{ar}(b \otimes c) = f(ar, b, c) = f(a, rb, c) = \widetilde{f}_a(rb \otimes c) = \varphi(a, r(b \otimes c))$. Therefore, there is a unique homomorphism $\widetilde{\varphi}\colon A \otimes_R (B \otimes_S C) \to G$ with $a \otimes (b \otimes c) \mapsto \varphi(a, b \otimes c) = f(a, b, c)$. Uniqueness of solutions to universal mapping problems shows that there is an isomorphism $T(A, B, C) \to A \otimes_R (B \otimes_S C)$ with $a \otimes b \otimes c \mapsto a \otimes (b \otimes c)$. Similarly, $T(A, B, C) \cong (A \otimes_R B) \otimes_S C$ via $a \otimes b \otimes c \mapsto (a \otimes b) \otimes c$, and so $A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C$ via $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$. •

Remark. That the elements $a \otimes b \otimes c \in T(A, B, C)$ have no parentheses will be exploited in the next chapter when we construct tensor algebras. ◂

We now present properties of tensor products that will help us compute them. First, we give a result about Hom, and then we give the analogous result for tensor.

Recall Exercise 8.34 on page 549: For any left R-module M, for any $f \in \mathrm{Hom}_R(R, M)$, and for any $r, s \in R$, define

$$rf\colon s \mapsto f(sr).$$

Using the fact that a ring R is an (R, R)-bimodule, we can check that rf is an R-map and that $\mathrm{Hom}_R(R, M)$ is a left R-module. We incorporate this into the next result.

Page 585, line 16 Change $\dim(V \otimes_k K)$ to "$\dim(V \otimes_k W)$"

Page 587, line -6 Should read

$$1 \otimes p : \sum a_i \otimes b_i \mapsto \sum a_i \otimes pb_i = \sum a_i \otimes b_i''.$$

Page 589, line 2 Change "$m \otimes b \mapsto mb$" to "$m \otimes b \mapsto nmb$"

Page 591, line 12 Should read

$$0 = (1_M \otimes i)u = \sum_{j=1}^{n} x_j \otimes iy_j.$$

Page 591, line -12 Change "$\sum_k (x_j, y_j)$" to "$\sum_k (x_j, iy_j)$"

Page 592 Change bottom 4 lines

C_S is a module, then it is easy to see that $\mathrm{Hom}_S(B, C)$ is a right R-module, where $(fr)(b) = f(rb)$; thus $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$ makes sense, for it consists of R-maps between right R-modules. Finally, if $F \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$, we denote its value on $a \in A$ by F_a, so that $F_a : B \to C$, defined by $F_a : b \mapsto F(a)(b)$, is a one-parameter family of functions.

Page 593 restate the theorem.

Theorem 8.99 (**Adjoint Isomorphism**) *Given modules A_R, ${}_RB_S$, and C_S, where R and S are rings, there is an isomorphism*

$$\tau_{A,B,C} : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C));$$

namely, if $f : A \otimes_R B \to C$ and $a \in A$ and $b \in B$, then

$$\tau_{A,B,C} : f \mapsto f^*, where\ f_a^* : b \mapsto f(a \otimes b).$$

Indeed, fixing any two of A, B, C, the maps $\tau_{A,B,C}$ constitute natural equivalences

$$\mathrm{Hom}_S(\ \otimes_R B, C) \to \mathrm{Hom}_R(\ , \mathrm{Hom}_S(B, C)),$$

$$\mathrm{Hom}_S(A \otimes_R\ , C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(\ , C)),$$

and

$$\mathrm{Hom}_S(A \otimes_R B,\ ) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,\ )).$$

Page 594, line -6 Change "$A \otimes_R M_i$" to "$A \otimes_R B_i$"

Page 594, lines -3, -2, -1   Should read

each pair $i, j \in I$ with $i \preceq j$ in the partially ordered index set I, define B_{ij} to be a module isomorphic to B_i by a map $b_i \mapsto b_{ij}$, where $b_i \in B_i$, and define $\sigma\colon \sum_{ij} B_{ij} \to \sum_i B_i$ by

$$\sigma\colon b_i \mapsto \lambda_j \varphi^i_j b_i - \lambda_i b_i,$$

Page 595, line 14   Should read

$$\widetilde{\sigma}\colon a \otimes b_{ij} \mapsto (1 \otimes \lambda_j)(a \otimes \varphi^i_j b_i) - (1 \otimes \lambda_i)(a' \otimes b_i).$$

Page 597, line 10   Change   "$= rj(m) \in D$;"   to $= j(rm) \in D$;

Page 598, lines -13, -12   Should read

For the converse, it suffices to prove that $\ker\alpha = \operatorname{im}\beta$ without assuming either α^* surjective or β^* injective.

Page 599, line 7   After   "Theorem 8.104,"   add the phrase

with B playing the role of R (so that flatness implies that the map $A' \otimes_R B \to A \otimes_R B$ is injective)

Page 599   Restate Lemma 8.109

Lemma 8.109. *Given modules $({}_R X, {}_R Y_S, Z_S)$, where R and S are rings, there is a natural transformation in X, Y, and Z*

$$\tau_{X,Y,Z}\colon \operatorname{Hom}_S(Y, Z) \otimes_R X \to \operatorname{Hom}_S(\operatorname{Hom}_R(X, Y), Z).$$

Moreover, $\tau_{X,Y,Z}$ an isomorphism whenever X is a finitely generated free left R-module.

Page 600, line 4   Should read:

Theorem 8.110. *A finitely presented right R-module B is flat if ...*

Page 600, line 7   Change   "finitely related"   to   "finitely presented"

Page 600, line -4 Change   "finitely related"   to   "finitely presented"

Page 601, line 4   Change   "finitely related"   to   "finitely presented"

Page 601, lines 11-14   Change all occurences of   "A"   to   "P"

Page 601,   Change all occurences of   "${}_S\mathbf{Mod}$"   to   "$\mathbf{Mod}_S$"

Page 601, line -2   Change   "Corollary 8.80"   to   "Exercise 8.45(ii) on page 603"

Page 602, line -1   Change   "${}_{\mathrm{Mat}_n(R)}\mathbf{Mod}$"   to   "$\mathbf{Mod}_{\mathrm{Mat}_n(R)}$"

Page 603, line -4 Change Exercise 8.45,

8.45 This exercise generalizes Corollary 8.81.

(i) Given a bimodule ${}_R A_S$, prove that $\text{Hom}_R(A, \;\;) \colon {}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$ is a functor, where $\text{Hom}_R(A, B)$ is the left S-module defined by $sf \colon a \mapsto f(as)$.

(ii) Given a bimodule ${}_R A_S$, prove that $\text{Hom}_S(A, \;\;) \colon \mathbf{Mod}_S \to \mathbf{Mod}_R$ is a functor, where $\text{Hom}_S(A, B)$ is the right R-module defined by $fr \colon a \mapsto f(ra)$.

(iii) Given a bimodule ${}_S B_R$, prove that $\text{Hom}_R(\;\;, B) \colon \mathbf{Mod}_R \to {}_S\mathbf{Mod}$ is a functor, where $\text{Hom}_R(A, B)$ is the left S-module defined by $sf \colon a \mapsto s[f(a)]$.

(iv) Given a bimodule ${}_S B_R$, prove that $\text{Hom}_S(A, \;\;) \colon {}_S\mathbf{Mod} \to \mathbf{Mod}_R$ is a functor, where $\text{Hom}_S(A, B)$ is the right R-module defined by $fr \colon a \mapsto f(a)r$.

Page 604. In Exercise 8.46, change "$v_i \otimes w_k$" to "$v_i \otimes w_j$" and change the bottom row of the matrix to

$$a_{m1}B \quad a_{m2}B \quad \cdots \quad a_{mm}B$$

Page 607, line -10 Add "direct"

... ***completely reducible*** if it is a direct sum of irreducible

Page 608, line 4 Should read

where $B_i \cong \text{End}_{\mathbb{C}}(L_i)$ for all i; we will usually abbreviate $\text{End}_{\mathbb{C}}(L_i)$ to $\text{End}(L_i)$. In

Page 608 Change the first paragraph as follows.

Recall the proof of the Wedderburn–Artin theorem: There are pairwise nonisomorphic minimal left ideals $L_1, \ldots, L_r$ in $\mathbb{C}G$ and $\mathbb{C}G = B_1 \oplus \cdots \oplus B_r$, where B_i is generated by all minimal left ideals isomorphic to L_i. Now $B_i \cong \text{Mat}_{n_i}(\mathbb{C})$, by Corollary 8.65. But all minimal left ideals in $\text{Mat}_{n_i}(\mathbb{C})$ are isomorphic, by Lemma 8.61(ii), so that $L_i \cong \text{COL}(1) \cong \mathbb{C}^{n_i}$ (see Example 8.30). Therefore,

$$B_i \cong \text{End}(L_i),$$

where we have abbreviated $\text{End}_{\mathbb{C}}(L_i)$ to $\text{End}(L_i)$.

Page 608, lines $13 - 16$ Should read

(ii) *The representation* λ_i *extends to a* $\mathbb{C}$*-algebra map* $\widetilde{\lambda}_i \colon \mathbb{C}G \to \mathbb{C}G$ *if we define*

$$\widetilde{\lambda}_i(g)u_j = \begin{cases} gu_i & if\ j = i \\ 0 & if\ j \neq i \end{cases} \tag{2}$$

for $g \in G$ *and* $u_j \in B_j$.

Page 609 Add two more parts to Corollary 8.120.

(ii) *Every irreducible representation of a finite abelian group is linear*.

(iii) *If* $\sigma\colon G \to \mathrm{GL}(V)$ *is a representation of a finite group* G*, then* $\sigma(g)$ *is similar to a diagonal matrix for each* $g \in G$.

Here are the proofs of these parts.

(ii) Since G is abelian, $\mathbb{C}G = \sum_i B_i$ is commutative, and so all $n_i = 1$. But $n_i = \mathrm{degree}(\lambda_i)$.

(ii) If $\sigma' = \sigma|\langle g\rangle$, then $\sigma'(g) = \sigma(g)$. Now σ' is a representation of the abelian group $\langle g\rangle$, and so part (ii) implies that the module $V^{\langle g\rangle}$ is a direct sum of one-dimensional submodules. If $V^{\langle g\rangle} = \langle v_1\rangle \oplus \cdots \oplus \langle v_m\rangle$, then the matrix of $\sigma(g)$ with respect to the basis $v_1, \ldots, v_m$ is diagonal. •

Page 621, line -2. After "$\tau(g) = \omega I$," add "by Corollary 8.20(iii)"

Page 622, lines 6 - 8 Should read as follows.

If $g \in \ker\theta$, then $\theta(g) = \theta(1)$. Suppose that $\chi_{j'}(g) \neq \chi_{j'}(1)$ for some j'. Since $\chi_{j'}(g)$ is a sum of roots of unity, Proposition 1.42 applies to force $|\chi_{j'}(g)| < \chi_{j'}(1)$, and so $\theta(g) = \sum_j m_j \chi_j(g) \neq \theta(1)$. Therefore, $g \in \bigcap_j \ker\chi_j$. For the reverse inclusion, if

Page 626, line 10 Should read $h^{-1}(t_i^{-1} g t_i)h$

Page 626, line -5 Change "by substituting" to "by collecting terms involving g_is"

Page 630, line 9 Change "$\chi_3(1) = 3$" to "$\chi_4(1) = 3$"

Page 636, line -3 Change "$= h_j \chi_i(g)$" to "$= h_j \chi_i(g_j)$"

Page 641, line 13

Change "Exercise 2.88 on page 113" to "Exercise 2.99 on page 114"

Page 642, line -10 Change "$\psi^* = \varphi|^G + \chi_1$" to "$\psi^* = \varphi|^G + d\chi_1$"

Page 644, line -14 Change "$\chi_\rho(h) = \sum_i n_i$" to "$\chi_\rho(h) = \sum_i n_i \psi_i(h)$"

Page 645, line 11 Change "$\big(\bigcup_{g\in G}(A^G \cap N)\big)$" to "$\big(\bigcup_{g\in G}(A^g \cap N)\big)$"

Page 647, line 14 Change "$m, m' \in M$" to "$m, m' \in tM$"

Page 648 Rewrite the first paragraph as follows.

(ii)If $\varphi\colon M \to M'$ is an isomorphism, then $\varphi(tM) \subseteq tM'$, for if $rm = 0$ with $r \neq 0$, then $r\varphi(m) = \varphi(rm) = 0$ (this is true for any R-homomorphism); hence, $\varphi|tM\colon tM \to tM'$ is an isomorphism (with inverse $\varphi^{-1}|tM'$). For the second statement, the map $\overline{\varphi}\colon M/tM \to M'/tM'$, defined by $\overline{\varphi}\colon m + tM \mapsto \varphi(m) + tM'$, is easily seen to be an isomorphism. •

Page 648, line -1 Change "$\mathrm{im}\, f$" to "$\mathrm{im}\,\varphi$"

Page 649, line 8 Change "f.g." to "finitely generated"

Page 652, line 17 Add phrase: with the finiteness hypothesis eliminated.

Page 652, line 19 Change "finite" to "torsion"

Page 652, line 21 Delete "finitely generated"

Page 652, line -3 Change "$s_1, \ldots, s_n \in R$" to "$s_1, \ldots, s_n \in \mathbb{Z}$"

Page 653, line 17 Rewrite the proof without the finiteness hypothesis. In particular, change the reference to Proposition 5.4 to Exercise 7.79 on page 519.

Page 655, lines 14 Delete "invariant factors $c_1, \ldots, c_t$ and"

Page 655, lines 16, 17 Should read
Definition. If M is a finitely generated torsion R-module, where R is a PID, then the ***order*** of M is the principal ideal generated by the product of its elementary divisors, namely, $\left(\prod_{ij} p_i^{e_{ij}}\right)$.

Page 659, line 11 Should read: "{complex pth power roots of unity}"

Page 662, line 16 Change "$\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$" to "$\mathbb{Q} \to T$"

Page 663, Change the first two parts of Exercise 9.1:

(i) Prove that $tG = \sum_p \langle a_p \rangle$.

(ii) Prove that G/tG is a vector space over $\mathbb{Q}$.
Hint. Show that G/tG is torsion-free and divisible.

Page 666, line 8 Should read: "of copies of $\mathbb{Z}(p_i^\infty)$"

Page 666, line -10 Change "**Corollary 3.95**" to "**Corollary 3.101**"

Page 667, line 16 Remove parentheses in last term in display. Should read:

$$\ldots = \sum_{i=0}^{m} c_i A^i v.$$

Page 668, line -7 Change "L is the companion matrix of $(g(x) = c_0)/x$" to "$L = xI - C((g(x) = c_0)/x)$"

Page 672, line -12 Change "$\deg(g) = s - 1$." to "$\deg(g) = s$."

Page 683, line 9 Change "Proposition 3.98 on page 175" to "Corollary 3.101 on page 176"

Page 683 Restate Proposition 9.54

If R is a commutative ring, then finite presentations of (finitely presented) R-modules M and M' give exact sequences

$$R^t \xrightarrow{\lambda} R^n \xrightarrow{\pi} M \to 0 \quad and \quad R^{t'} \xrightarrow{\lambda'} R^{n'} \xrightarrow{\pi'} M' \to 0,$$

and choices of bases Y, Y' of R^t and Z, Z' of R^n give matrices $\Gamma = {}_Z[\lambda]_Y$ and $\Gamma' = {}_{Z'}[\lambda]_{Y'}$. If $t' = t$, $n' = n$, and Γ and Γ' are R-equivalent, then $M \cong M'$.

Page 683 bottom: should read

P determines an R-isomorphism $\varphi\colon R^t \to R^t$, and Q determine an R-isomorphism $\theta\colon R^n \to R^n$.

$$\begin{array}{ccccccc} R^t & \xrightarrow{\lambda} & R^n & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \phi\downarrow & & \downarrow\theta & & \downarrow\nu & & \\ R^t & \xrightarrow[\lambda']{} & R^n & \xrightarrow[\pi']{} & M' & \longrightarrow & 0 \end{array}$$

Page 684, line 2 Change "$\pi(u) = n$; set $\nu(m) = \pi'\varphi^{-1}(u)$" to "$\pi(u) = m$; set $\nu(m) = \pi'\theta(u)$"

Page 685 Restate part (ii) of Theorem 9.56

If A is an $n \times n$ matrix over k, then the matrix Γ given by this presentation of $(k^n)^A$ (relative to the standard basis $E = e_1, \ldots, e_n$ of k^n) is $\Gamma = xI - A$..

Page 686 Replace the first 6 lines.

For any $i \geq 1$, we are going to rewrite the ith summand $x^i v_i - T^i v_i$ of u as a telescoping sum, each of whose terms lies in $\operatorname{im}\lambda$; this will suffice to prove that $\ker\pi \subseteq \operatorname{im}\lambda$.

$$\begin{aligned} \sum_{j=0}^{i-1} \lambda(x^{i-1-j}T^j v_i) &= \sum_{j=0}^{i-1}\Big(x^{i-j}T^j v_i - x^{i-1-j}T^{j+1}v_i\Big) \\ &= (x^i v_i - x^{i-1}Tv_i) + (x^{i-1}Tv_i - x^{i-2}T^2 v_i) + \\ &\qquad \cdots + (xT^{i-1}v_i - T^i v_i) \\ &= x^i v_i + \Big[\sum_{j=1}^{i-1}(-x^{i-j}T^j v_i + x^{i-j}T^j v_i)\Big] - T^i v_i \\ &= x^i v_i - T^i v_i. \end{aligned}$$

Page 683, lines -8, -7 Change "r" to "ρ"

Page 688, lines -5, -4, -3 Should read

We claim that σ_1 divides every entry of Δ'. Let a be an entry not in σ_1's row or column; schematically, we have $\begin{pmatrix} a & b \\ c & \sigma_1 \end{pmatrix}$, where $b = u\sigma_1$ and $c = v\sigma_1$. Replace $\text{ROW}(1)$ by $\text{ROW}(1) + (1-u)\text{ROW}(2) = (a + (1-u)c \;\; \sigma_1)$. As above, $\sigma_1 \mid a + (1-u)c$. Since $\sigma_1 \mid c$, we have $\sigma_1 \mid a$.

Page 689, line 1 Change "η_{ij}" to "η_{1j}"

Page 690, line -13 Change "not constant" to "not units"

Page 690, line -5 Change

"$M \cong R/(\sigma_s) \oplus \cdots \oplus R/(\sigma_q)$," to "$M \cong R^{n-q} \oplus R/(\sigma_s) \oplus \cdots \oplus R/(\sigma_q)$,"

Page 692, line -8 Change "not constants." to "not units."

Page 693 In third matrix display, change the 22 entry in both matrices from $x^2 - 4$ to $x^2 - 4x + 1$; in the 2×2 matrix just below, change the 11 entry from $x^2 - 4$ to $x^2 - 4x + 1$.

Page 693 The rational canonical form for A is $\begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & 15 \\ 0 & 1 & 0 \end{pmatrix}$.

Page 697 Replace (ii) of the Proposition by

(ii) *If $B^t AC = B^t A'C$ for all column vectors B and C, then $A = A'$.*

Page 697 Replace the proofs of (ii) and (iii) by

(ii) If $b = \sum_i b_i e_i$ and $c = \sum_i c_i e_i$, then we have seen that $f(b, c) = B^t AC$, where B and C are the column vectors of the coordinates of b and c with respect to E. In particular, if $b = e_i$ and $c = e_j$, then $f(e_i, e_j) = a_{ij}$ is the ij entry of A.

(iii) Let the coordinates of b and c with respect to the basis E' be B' and C', respectively, so that $f'(b, c) = (B')^t A'C'$, where $A' = [f(e'_i, e'_j)]$. If P is the transition matrix ${}_E[1]_{E'}$, then $B = PB'$ and $C = PC'$. Hence, $f(b, c) = B^t AC = (PB')^t A(PC') = (B')^t (P^t AP)C'$. By part (ii), we must have $P^t AP = A'$.

For the converse, the given matrix equation $A' = P^t AP$ yields equations:

$$\begin{aligned} [f'(e'_i, e'_j)] &= A' \\ &= P^t AP \\ &= \Big[\sum_{\ell q} p_{\ell i} f(e_\ell, e_q) p_{qj}\Big] \\ &= \Big[f\Big(\sum_\ell p_{\ell i} e_\ell, \sum_q p_{qj} e_q\Big)\Big] \\ &= [f(e'_i, e'_j)]. \end{aligned}$$

Hence, $f'(e'_i, e'_j) = f(e'_i, e'_j)$ for all i, j, from which it follows that $f'(b, c) = f(b, c)$ for all $b, c \in V$. Therefore, $f = f'$ •.

Page 698 Modify the definition of discriminant:

The ***discriminant*** of a bilinear form f is either 0 or

$$\det(A)(k^\times)^2 \in k^\times/(k^\times)^2,$$

where A is an inner product matrix of f.

Pages 698-699 Replace text from line -6 on page 698 through line 8 on page 699 by following.

Definition. If (V, f) is an inner product space and $W \subseteq V$ is a subspace of V, then the ***left orthogonal complement*** of W is

$$W^{\perp L} = \{b \in V : f(b, w) = 0 \text{ for all } w \in W\};$$

the ***right orthogonal complement*** of W is

$$W^{\perp R} = \{c \in V : f(w, c) = 0 \text{ for all } w \in W\}.$$

It is easy to see that both $W^{\perp L}$ and $W^{\perp R}$ are subspaces of V. Moreover, $W^{\perp L} = W^{\perp R}$ if f is either symmetric or alternating, in which case we write $W^{\perp}$.

Let (V, f) be an inner product space, and let A be the inner product matrix of f relative to a basis $e_1, \ldots, e_n$ of V. We claim that $b \in V^{\perp L}$ if and only if b is a solution of the homogeneous system $A^t x = 0$. If $b \in V^{\perp L}$. then $f(b, e_j) = 0$ for all j. Writing $b = \sum_i b_i e_i$, we see that $0 = f(b, e_j) = f\big(\sum_i b_i e_i, e_j\big) = \sum_j b_i f(e_i, e_j)$. In matrix terms, $b = (b_1, \ldots, b_n)^t$ and $B^t A = 0$; transposing, b is a solution of the homogeneous system $A^t x = 0$. The proof of the converse is left to the reader. A similar argument shows that $c \in V^{\perp R}$ if and only if c is a solution of the homogeneous system $Ax = 0$.

Proposition 9.72. *If (V, f) is an inner product space, then f is nondegenerate if and only if $V^{\perp L} = \{0\} = V^{\perp R}$; that is, if $f(b, c) = 0$ for all $c \in V$, then $b = 0$, and if $f(b, c) = 0$ for all $b \in V$, then $c = 0$.*

Proof. Our remarks above show that $b \in V^{\perp L}$ if and only if b is a solution of the homogeneous system $A^t x = 0$. Therefore, $V^{\perp L} \neq \{0\}$ if and only if there is a nontrivial solution b, and this holds if and only if $\det(A^t) = 0$. Since $\det(A^t) = \det(A)$, we have f degenerate. A similar argument shows that $V^{\perp R} \neq \{0\}$ if and only if there is a nontrivial solution to $Ax = 0$. •

Page 701, lines 10, 11. Should read

We have just seen that every two-dimensional alternating space (V, f) in which f is not identically zero has an inner product matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

Page 706, line 6 Change "are congrent" to "have congruent inner product matrices"

Page 707, line 3 Change "${}_E[T]_E$" to "${}_E[\varphi]_E$"

Page 708, line 9 Change "φ" to "φ_j"

Pages 710-711 Replace line -8 on page 710 through line 21 on page 711 by

The next question is whether $\text{Isom}(V, f)$ depends on the choice of nondegenerate alternating bilinear form f. Observe that $\text{GL}(V)$ acts on $k^{V \times V}$, the set of all functions $V \times V \to k$: If $f : V \times V \to k$ and $\varphi \in \text{GL}(V)$, then define $\varphi f = f^{\varphi}$, where

$$f^{\varphi}(b, c) = f(\varphi^{-1} b, \varphi^{-1} c).$$

This formula does yield an action: If $\theta \in \mathrm{GL}(V)$, then $(\varphi\theta) f = f^{\varphi\theta}$, where

$$\begin{aligned}(\varphi\theta) f(b, c) &= f^{\varphi\theta}(b, c)\\ &= f((\varphi\theta)^{-1}b, (\varphi\theta)^{-1}c)\\ &= f(\theta^{-1}\varphi^{-1}b, \theta^{-1}\varphi^{-1}c).\end{aligned}$$

On the other hand, $\varphi(\theta f)$ is defined by

$$\begin{aligned}(f^{\theta})^{\varphi}(b, c) &= f^{\theta}(\varphi^{-1}b, \varphi^{-1}c)\\ &= f(\theta^{-1}\varphi^{-1}b, \theta^{-1}\varphi^{-1}c),\end{aligned}$$

so that $(\varphi\theta) f = \varphi(\theta f)$.

Definition. Let V and W be finite-dimensional vector spaces over a field k, and let $f \colon V \times V \to k$ and $g \colon W \times W \to k$ be bilinear forms. Then f and g are ***equivalent*** if there is an isometry $\varphi \colon V \to W$.

Theorem 9.91. *If V is a finite-dimensional vector space over a field k and if $f, g \colon V \times V \to k$ are bilinear forms, then the following statements are equivalent.*

(i) *f and g are equivalent.*

(ii) *If $E = e_1, \ldots, e_n$ is a basis of V, then the inner product matrices of f and g with respect to E are congruent.*

(iii) *There is $\varphi \in \mathrm{GL}(V)$ with $g = f^{\varphi}$.*

Proof. (i) $\Rightarrow$ (ii) If $\varphi \colon V \to V$ is an isometry, then $g(\varphi(b), \varphi(c)) = f(b, c)$ for all $b, c \in V$. If $E = e_1, \ldots, e_n$ is a basis of V, then $E' = \varphi(e_1), \ldots, \varphi(e_n)$ is also a basis, because φ is an isomorphism. Hence, $A' = [g(\varphi(e_i), \varphi(e_j))] = [f(e_i, e_j)] = A$ for all i, j; that is, the inner product matrix A' of g with respect to E' is equal to the inner product matrix A of f with respect to E. By Proposition 9.70(iii), the inner product matrix A'' of g with respect to E is congruent to A.

(ii) $\Rightarrow$ (iii) If $A = [f(e_i, e_j)]$ and $A' = [g(e_i, e_j)]$, then there exists a nonsingular matrix $Q = [q_{ij}]$ with $A' = Q^t AQ$. Define $\theta \colon V \to V$ to be the linear transformation with $\theta(e_j) = \sum_\nu q_{\nu j} e_\nu$. Finally, $g = f^{\theta^{-1}}$:

$$\begin{aligned}[g(e_i, e_j)] = A' = Q^t AQ &= \Big[f\Big(\sum_\nu q_{\nu i} e_\nu, \sum_\lambda q_{\lambda j} e_\lambda\Big)\Big]\\ &= [f(\theta(e_i), \theta(e_j))] = [f^{\theta^{-1}}(e_i, e_j)].\end{aligned}$$

(iii) $\Rightarrow$ (i) It is obvious from the definition that $\varphi^{-1} \colon (V, g) \to (V, f)$ is an isometry:

$$g(b, c) = f^{\varphi}(b, c) = f(\varphi^{-1}b, \varphi^{-1}c).$$

Therefore, g is equivalent to f. •

Page 711, line -7 Replace "$f(P^{-1}u, P^{-1}v)$." by "$f(P^{-1}u, P^{-1}v) = f(u, v)$."

Page 717, line 7 Replace "$[M_1' \times \cdots \times M_p']$" by "$M_1' \times \cdots \times M_p'$"

Page 717, line -9 Replace "$[M_1 \times \cdots \times M_p]$" by "$M_1 \times \cdots \times M_p$"

Page 717, line -7 Delete "h"

Page 718 Replace Proposition 9.98.

Proposition 9.98 (**Generalized Associativity**) *Let R be a commutative ring and let $M_1, \ldots, M_p$ be R-modules. If $M_1 \otimes_R \cdots \otimes_R M_p$ is an iterated tensor product in some association, then there is an R-isomorphism $U[M_1, \ldots, M_p] \to M_1 \otimes_R \cdots \otimes_R M_p$ taking $h(m_1, \ldots, m_p) \mapsto m_1 \otimes \cdots \otimes m_p$.*

Remark. We are tempted to quote Theorem 2.20: Associativity for three factors implies associativity for many factors, for we have proved the associative law for three factors in Proposition 8.84. However, we did not prove equality, $A \otimes_R (B \otimes_R C) = (A \otimes_R B) \otimes_R C$; we only constructed an isomorphism. There is an extra condition, due, independently, to Mac Lane and Stasheff: If the associative law holds up to isomorphism and if a certain "pentagonal" diagram commutes, then generalized associativity holds up to isomorphism (see Mac Lane, *Categories for the Working Mathematician*, pages 157–161). ◂

Proof. The proof is by induction on $p \geq 2$. The base step is true, for $U[M_1, M_2] = M_1 \otimes_R M_2$. For the inductive step, let us assume that

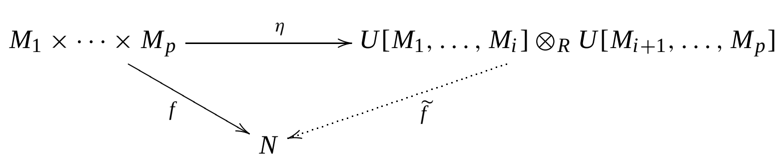
$$M_1 \otimes_R \cdots \otimes_R M_p = U[M_1, \ldots, M_i] \otimes_R U[M_{i+1}, \ldots, M_p].$$

We have indicated the final factors in the association; for example,

$$((M_1 \otimes_R M_2) \otimes_R M_3) \otimes_R (M_4 \otimes_R M_5) = U[M_1, M_2, M_3] \otimes_R U[M_4, M_5].$$

By induction, there are multilinear functions $h' \colon M_1 \times \cdots \times M_i \to M_1 \otimes_R \cdots \otimes_R M_i$ and $h'' \colon M_{i+1} \times \cdots \times M_p \to M_{i+1} \otimes_R \cdots \otimes_R M_p$ with $h'(m_1, \ldots, m_i) = m_1 \otimes \cdots \otimes m_i$ associated as in $M_1 \otimes_R \cdots \otimes_R M_i$, and with $h''(m_{i+1}, \ldots, m_p) = m_{i+1} \otimes \cdots \otimes m_p$ associated as in $M_{i+1} \otimes_R \cdots \otimes_R M_p$. Induction gives isomorphisms $\varphi' \colon U[M_1, \ldots, M_i] \to M_1 \otimes_R \cdots \otimes_R M_i$ and $\varphi'' \colon U[M_{i+1}, \ldots, M_p] \to M_{i+1} \otimes_R \cdots \otimes_R M_p$ with $\varphi' h' = h|(M_1 \times \cdots \times M_i)$ and $\varphi'' h'' = h|(M_{i+1} \times \cdots \times M_p)$. By Corollary 8.78, $\varphi' \otimes \varphi''$ is an isomorphism $U[M_1, \ldots, M_i] \otimes_R U[M_{i+1}, \ldots, M_p] \to M_1 \otimes_R \cdots \otimes_R M_p$.

We now show that $U[M_1, \ldots, M_i] \otimes_R U[M_{i+1}, \ldots, M_p]$ is a solution to the universal problem for multilinear functions. Consider the diagram



where $\eta(m_1, \ldots, m_p) = h'(m_1, \ldots, m_i) \otimes h''(m_{i+1}, \ldots, m_p)$, N is an R-module, and f is multilinear. We must find a homomorphism $\widetilde{f}$ making the diagram commute.

If $(m_1, \ldots, m_i) \in M_1 \times \cdots \times M_i$, the function $f_{(m_1,\ldots,m_i)} \colon M_{i+1} \times \cdots \times M_p \to N$, defined by $(m_{i+1}, \ldots, m_p) \mapsto f(m_1, \ldots, m_i, h''(m_{i+1}, \ldots, m_p))$, is multilinear; hence, there is a unique homomorphism $\widetilde{f}_{(m_1,\ldots,m_i)} \colon U[M_{i+1}, \ldots, M_p] \to N$ with

$$\widetilde{f}_{(m_1,\ldots,m_i)} \colon h''(m_{i+1}, \ldots, m_p) \mapsto f(m_1, \ldots, m_p).$$

If $r \in R$ and $1 \le j \le i$, then

$$\begin{aligned}\widetilde{f}_{(m_1,\ldots,rm_j,\ldots,m_i)}(h''(m_{i+1}, \ldots, m_p)) &= f(m_1, \ldots, rm_j, \ldots, m_p)\\ &= rf(m_1, \ldots, m_j, \ldots, m_i)\\ &= r\,\widetilde{f}_{(m_1,\ldots,m_i)}(h''(m_{i+1}, \ldots, m_p)).\end{aligned}$$

Similarly, if $m_j, m_j' \in M_j$, where $1 \le j \le i$, then

$$\widetilde{f}_{(m_1,\ldots,m_j+m_j',\ldots,m_i)} = \widetilde{f}_{(m_1,\ldots,m_j,\ldots,m_i)} + \widetilde{f}_{(m_1,\ldots,m_j',\ldots,m_i)}.$$

The function of $i+1$ variables $M_1 \times \cdots \times M_i \times U[M_{i+1}, \ldots, M_p] \to N$, defined by $(m_1, \ldots, m_i, u'') \mapsto \widetilde{f}_{(m_1,\ldots,m_i)}(u'')$, is multilinear, and so it gives a bilinear function $U[M_1, \ldots, M_i] \times U[M_{i+1}, \ldots, M_p] \to N$, namely, $(u', u'') \mapsto (h'(u'), h''(u''))$. Thus, there is a unique homomorphism $\widetilde{f} \colon U[M_1, \ldots, M_i] \otimes_R U[M_{i+1}, \ldots, M_p] \to N$ which takes $h'(m_1, \ldots, m_i) \otimes h''(m_{i+1}, \ldots, m_p) \mapsto \widetilde{f}_{(m_1,\ldots,m_i)}(h''(m_{i+1}, \ldots, m_p)) = f(m_1, \ldots, m_p)$; that is, $\widetilde{f}\eta = f$. Therefore, $U[M_1, \ldots, M_i] \otimes_R U[M_{i+1}, \ldots, M_p]$ is a solution to the universal mapping problem. By uniqueness of such solutions, there is an isomorphism $\theta \colon U[M_1, \ldots, M_p] \to U[M_1, \ldots, M_i] \otimes_R U[M_{i+1}, \ldots, M_p]$ with $\theta h(m_1, \ldots, m_p) = h'(m_1, \ldots, m_i) \otimes h''(m_{i+1}, \ldots, m_p) = \eta(m_1, \ldots, m_p)$. Finally, $(\varphi' \otimes \varphi'')\theta$ is the desired isomorphism $U[M_1, \ldots, M_p] \cong M_1 \otimes_R \cdots \otimes_R M_p$. •

Page 720, line -12 Change to read:

If R and S are k-algebras, where k is a commutative ring, then every (R, S)-bimodule M is a left $R \otimes_k S^{\mathrm{op}}$-module, where

Page 720, line -10 Add at beginning of Proof:

The function $R \times_k S^{\mathrm{op}} \times M \to M$, given by $(r, s, m) \mapsto rms$, is k-trilinear, and this can be used to prove that $(r \otimes s) = rms$ is well-defined.

Page 720 Restate first line of Corollary 9.102 so it treats $R \otimes_k S$, where k is a commutative ring, instead of $R \otimes_{\mathbb{Z}} S$.

Page 722 Replace Proposition 9.106 by following.

If R is a commutative ring and A and B are R-modules, define a ***word*** on A and B to be an R-module of the form

$$W(A, B) = T^{e_1}(A) \otimes_R T^{f_1}(B) \otimes_R \cdots \otimes_R T^{e_r}(A) \otimes_R T^{f_r}(B),$$

where all e_i, f_i are integers, $e_1 \geq 0$, $f_r \geq 0$, and all the other exponents are positive. We say that $W(A, B)$ has ***length*** p if $\sum_i (e_i + f_i) = p$.

Proposition 9.106. *If A and B are R-modules, then for all $p \geq 0$,*

$$T^p(A \oplus B) \cong \sum_{j=0}^{p} W(A, B)_j \otimes_R W'(A, B)_{p-j},$$

where $W(A, B)_j$, $W'(A, B)_{p-j}$ range over all words of length j and $p - j$, respectively.

Proof. The proof is by induction on $p \geq 0$. For the base step,

$$T^0(A \oplus B) = R \cong R \otimes_R R \cong T^0(A) \otimes_R T^0(B).$$

For the inductive step,

$$\begin{aligned} T^{p+1}(A \oplus B) &= T^p(A \oplus B) \otimes_R (A \oplus B) \\ &\cong \big(T^p(A \oplus B) \otimes_R A\big) \oplus \big(T^p(A \oplus B) \otimes_R B\big) \\ &\cong \sum_{j=0}^{p} W(A, B)_j \otimes_R W'(A, B)_{p-j} \otimes_R X, \end{aligned}$$

where $X \cong A$ or $X \cong B$. This completes the proof, for every word of length $p - j + 1$ has the form $W'(A, B) \otimes_R X$. •

Page 725, line 12 Replace "$R = T(V)/I$" by "$R = (\sum_p T^p(V))/I$. [This is only to make it easier to recognize, two lines below, that $T^1(V)$ denotes the terms in $T(V)$ of degree 1.]

Page 725, line 22 Should read $\widetilde{\varphi}\colon k\langle X\rangle \to A$,

Page 730, Should read

$$1 \otimes i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \text{and} \qquad 1 \otimes k \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Page 731, line 20 Redo first paragraph of proof.

Proof. Associativity of the multiplication in A shows that A can be viewed as a (B, A)-bimodule. As such, it is a left $(B \otimes_k A^{\mathrm{op}})$-module, where $(b \otimes a)x = bxa$ for all $x \in A$; we denote this module by A^*. But $B \otimes_k A^{\mathrm{op}}$ is a simple k-algebra, by Theorem 9.112, so that Corollary 8.63 gives $B \otimes_k A^{\mathrm{op}} \cong \mathrm{Mat}_s(\Delta)$ for some integer s and some division algebra Δ over k; in fact, $B \otimes_k A^{\mathrm{op}}$ has a unique (to isomorphism) minimal left ideal L, and $\Delta^{\mathrm{op}} \cong \mathrm{End}_{B \otimes_k A^{\mathrm{op}}}(L)$. Therefore, as $(B \otimes_k A^{\mathrm{op}})$-modules, Corollary 8.44 gives $A^* \cong L^r$, the direct sum of r copies of L, and so $\mathrm{End}_{B \otimes_k A^{\mathrm{op}}}(A^*) \cong \mathrm{Mat}_r(\Delta)$.

Page 732, line 3 Change "$= s^2[L : k]$" to "$= s^2[\Delta : k]$"

Page 733, line -1 Should read:

$$[{}_fA : \Delta] = [A : \Delta] = [{}_gA : \Delta],$$

Page 734, line 13 Change "g" to "ψ"

Page 735, line -16 Replace the sentence beginning "If $x^2 > 0$" by

If $x^2 > 0$, then there is $t \in \mathbb{R}$ with $x^2 = t^2$. Now $0 = (x+t)(x-t)$, so that $x = \pm t \in \mathbb{R}$, and this contradicts $-i = xix^{-1}$.

Page 737, lines 9 – 16 Change the proof of (iv).

Define $f : A \times A^{\mathrm{op}} \to \mathrm{End}_k(A)$ by $f(a, c) = \lambda_a \circ \rho_c$, where $\lambda_a : x \mapsto ax$ and $\rho_c : x \mapsto xc$; it is routine to check that λ_a and ρ_c are k-maps (so their composite is also a k-map), and that f is k-biadditive. Hence, there is a k-map $\widehat{f} : A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_k(A)$ with $\widehat{f}(a \otimes c) = \lambda_a \circ \rho_c$. Now associativity $a(xc) = (ax)c$ in A says that $\lambda_a \circ \rho_c = \rho_c \circ \lambda_a$,from which it easily follows that $\widehat{f}$ is a k-algebra map. As $A \otimes_k A^{\mathrm{op}}$ is a simple k-algebra and ker $\widehat{f}$ is a proper two-sided ideal, we have $\widehat{f}$ injective. Now $\dim_k(\mathrm{End}_k(A)) = \dim_k(\mathrm{Hom}_k(A, A)) = n^2$, where $n = [A : k]$. Since $\dim_k(\mathrm{im}\,\widehat{f}) = \dim_k(A \otimes_k A^{\mathrm{op}}) = n^2$, it follows that $\widehat{f}$ is a k-algebra isomorphism: $A \otimes_k A^{\mathrm{op}} \cong \mathrm{End}_k(A)$.

Page 737 line -4 Change "to be the abelian group" to "to be the set"

Page 747 The correct spelling is Grassmann.

Page 750 line 10 Change "*Moreover,*" to "*Thus,*"

Page 751: Rewrite Theorem 9.143 as follows:

Theorem 9.143. *For all $p \geq 0$ and all k-modules A and B, where k is a commutative ring,*

$$\bigwedge^p (A \oplus B) \cong \sum_{i=0}^{p} \Big(\bigwedge^i (A) \otimes_k \bigwedge^{p-i} (B) \Big).$$

Sketch of proof. Let $\mathcal{A}$ be the category of all alternating anticommutative graded k-algebras $R = \sum_{p \geq 0} R^p$ ($r^2 = 0$ for all $r \in R$ of odd degree and $rs = (-1)^{pq} sr$ if $r \in R^p$ and $s \in S^q$); by Theorem 9.136, the exterior algebra $\bigwedge(A) \in \mathrm{obj}(\mathcal{A})$ for every k-module A. If $R, S \in \mathrm{obj}(\mathcal{A})$, then one verifies that $R \otimes_k S = \sum_{p \geq 0} \big(\sum_{i=0}^{p} R^i \otimes_k S^{p-i} \big) \in \mathrm{obj}(\mathcal{A})$; using anticommutativity, a modest generalization of Proposition 9.101 shows that $\mathcal{A}$ has coproducts.

We claim that $(\bigwedge, D)$ is an adjoint pair of functors, where $\bigwedge : {}_k\mathbf{Mod} \to \mathcal{A}$ sends $A \mapsto \bigwedge(A)$, and $D : \mathcal{A} \to {}_k\mathbf{Mod}$ sends $\sum_{p \geq 0} R^p \mapsto R^1$, the terms of degree 1. If $R = \sum_p R^p$, then there is a map $\pi_R : \bigwedge(R^1) \to R$; define $\tau_{A,R} : \mathrm{Hom}_{\mathcal{A}}(\bigwedge(A), R) \to \mathrm{Hom}_k(A, R^1)$ by $\varphi \mapsto \pi_R(\varphi|A)$. It follows from Theorem 7.105 that $\bigwedge$ preserves coproducts; that is, $\bigwedge(A \oplus B) \cong \bigwedge(A) \otimes_k \bigwedge(B)$, and so $\bigwedge^p(A \oplus B) \cong \sum_{i=0}^{p} \Big(\bigwedge^i (A) \otimes_k \bigwedge^{p-i} (B) \Big)$. •

Here is an explicit formula for an isomorphism. In $\bigwedge^3(A \oplus B)$, we have

$$\begin{aligned}(a_1+b_1)\wedge(a_2+b_2)\wedge(a_3+b_3) = {} & a_1\wedge a_2\wedge a_3 + a_1\wedge b_2\wedge a_3 \\ & + b_1\wedge a_2\wedge a_3 + b_1\wedge b_2\wedge a_3 + a_1\wedge a_2\wedge b_3 \\ & + a_1\wedge b_2\wedge b_3 + b_1\wedge a_2\wedge b_3 + b_1\wedge b_2\wedge b_3.\end{aligned}$$

By anticommutativity, this can be rewritten so that each a precedes all the b's:

$$\begin{aligned}(a_1+b_1)\wedge(a_2+b_2)\wedge(a_3+b_3) = {} & a_1\wedge a_2\wedge a_3 - a_1\wedge a_3\wedge b_2 \\ & + a_2\wedge a_3\wedge b_1 + a_3\wedge b_1\wedge b_2 + a_1\wedge a_2\wedge b_3 \\ & + a_1\wedge b_2\wedge b_3 - a_2\wedge b_1\wedge b_3 + b_1\wedge b_2\wedge b_3.\end{aligned}$$

An i-***shuffle*** is a partition of $\{1, 2, \dots, p\}$ into two disjoint subsets $\mu_1 < \dots < \mu_i$ and $\nu_1 < \dots < \nu_{p-i}$; it gives the permutation $\sigma \in S_p$ with $\sigma(j) = \mu_j$ for $j \le i$ and $\sigma(i+\ell) = \nu_\ell$ for $j = i+\ell > i$. Each "mixed" term in $(a_1+b_1)\wedge(a_2+b_2)\wedge(a_3+b_3)$ gives a shuffle, with the a's giving the μ and the b's giving the ν; for example, $a_1\wedge b_2\wedge a_3$ is a 2-shuffle and $b_1\wedge a_2\wedge b_3$ is a 1-shuffle. Now $\operatorname{sgn}(\sigma)$ counts the total number of leftward moves of a's so that they precede all the b's, and the reader may check that the signs in the rewritten expansion are $\operatorname{sgn}(\sigma)$. Define $f\colon \bigwedge^p(A\oplus B) \to \sum_{i=0}^p \left(\bigwedge^i(A) \otimes_k \bigwedge^{p-i}(B)\right)$ by

$$f(a_1+b_1, \dots, a_p+b_p) = \sum_{i=0}^{p}\left(\sum_{i\text{-shuffles }\sigma} \operatorname{sgn}(\sigma) a_{\mu_1}\wedge\cdots\wedge a_{\mu_i} \otimes b_{\nu_1}\wedge\cdots\wedge b_{\nu_{p-i}}\right).$$

Page 758, line 9: Should end: "$du \wedge dv \wedge dw$."

Page 761 Change the proof of Proposition 9.155 as follows:

Proof. If $A = [a_{ij}]$, write the complete expansion of $\det(A)$ more compactly:

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i a_{\sigma(i),i}.$$

For any permutation $\tau \in S_n$, we have $i = \tau(j)$ for all i, and so

$$\prod_i a_{\sigma(i),i} = \prod_j a_{\sigma(\tau(j)),\tau(j)},$$

for this merely rearranges the factors in the product. Choosing $\tau = \sigma^{-1}$ gives

$$\prod_j a_{\sigma(\tau(j)),\tau(j)} = \prod_j a_{j,\sigma^{-1}(j)}.$$

Therefore,

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_j a_{j,\sigma^{-1}(j)}.$$

Now $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$ [if $\sigma = \tau_1 \cdots \tau_q$, where the τ are transpositions, then $\sigma^{-1} = \tau_q \cdots \tau_1$]; moreover, as σ varies over S_n, so does σ^{-1}. Hence, writing $\sigma^{-1} = \rho$ gives

$$\det(A) = \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_j a_{j,\rho(j)}.$$

Now write $A^t = [b_{ij}]$, where $b_{ij} = a_{ji}$. Then

$$\det(A^t) = \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_j b_{\rho(j),j} = \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_j a_{j,\rho(j)} = \det(A). \quad \bullet$$

Page 772 Exercise 9.97(ii) Change "$D_n = F_n$" to "$D_n = F_{n+1}$"

Page 775, lines 7 and 9 Change "$[a,[b,c]]$" to "$[b,[a,c]]$"

Page 779, line 13 Should read: "binary operation $A * B$, where"

Page 779, line 14 Should read: "$A * B = \frac{1}{2}(AB + BA)$"

Page 780 line -6 Should read:

(iii) If L is nilpotent and $L \neq \{0\}$, prove that $Z(L) \neq \{0\}$.

Chapter 10, Change "0" to "$\{0\}$" in many places.

Page 788 Replace (iii) in Proposition 10.5.

(iii) *Let K and Q be subgroups of a group G with $K \lhd G$. Then G is a semidirect product of K by Q if and only if $K \cap Q = \{0\}$, $K + Q = G$, and each $g \in G$ has a unique expression $g = a + x$, where $a \in K$ and $x \in Q$.*

Page 789 Add the following to the proof of $\tilde{(}$iii).

Conversely, each $g \in G$ has a unique factorization $g = ax$ for $a \in K$ and $x \in Q$; define $p \colon G \to Q$ by $p(ax) = x$. It is easy to check that p is a surjective homomorphism with $\ker p = K$.

Page 788, line -4 Change "$pi(x) = 0$" to "$pi(a) = 0$"

Page 789 line -1 Change "$\ker N \cong U(2, \mathbb{C})$" to "$\ker N \cong SU(2, \mathbb{C})$"

Page 793 Change Exercise 10.1(i) to read:

Prove that $\mathrm{SL}(2, \mathbb{F}_5)$ is an extension of $\mathbb{I}_2$ by A_5 which is not a semidirect product.

Page 793 In Exercise 1.2, assume that K is a normal subgroup of order m.

Page 802 line 1 Change "$i'(a) = 2g$" to "$i'(a) = 2pg$"

Page 807, line -11, -12 Should read:

... inner stabilizing automorphism by some $a_0 \in K$ if and only if

Page 807, line -5 Change "$-xa_0$" to "$+xa_0$"

Page 810, line 15 Equation should begin

$$= x(y[z] - [yz] + [y])$$

Page 810, line 15 The subscripts in the displayed and in the diagram below should be d_3^*, d_2^*, d_1^* instead of d_2^*, d_1^*, d_0^*

Page 812. Rewrite first line of Exercise 10.15: Change "Recall that a ***generalized***" to "Recall Example 5.79 on page 307: a ***generalized***"

Page 812. Delete part (iii) of Exercise 10.15.

Page 816, line -10 Should read "Introducing"

Page 823 In definition of the connecting homomorphism, both z_n should be z_n'' (not just one of them).

Page 824, line -12 Replace "$d''pu'd''c''$" to "$d''pu = d''c''$"

Page 827, Exercise 10.24. Change "whenever $m < n$" to "whenever $m \le n$"
Change "is a functor" to "is a contravariant functor"

Page 867, line -9 Replace "V/E" by "V/A"

Page 874, line 1 Delete second $H_0(G, \mathbb{Z}G)$.

Page 881, lines 9, 10 Change "*prove that*" to "*then*"

Page 886, line -1 Change "n" to "$n + 1$"

Page 889, line 9 Change "$\sigma, \tau, \omega inG$:" to "$\sigma, \tau, \omega \in G$:"

Page 891, line 13 Change "throught" to "through"

Page 892, line 6 Change "some $\varphi(x) \in k$" to "some $z \in k^\times$"

Page 893, line 8 Should read: "$R = \{a \in L : v(a) \le 1\}$"

Page 904, lines 11, 12 Replace the sentence beginning "The classification" with:

A. G. Kurosh classified torsion-free abelian groups G of finite rank n with invariants $n = \text{rank}(G)$, $\dim(\mathbb{F}_p \otimes G)$ for all primes p, and an equivalence class of sequences (M_p), where M_p is an $n \times n$ nonsingular matrix over the p-adic numbers $\mathbb{Q}_p$ (this theorem is not easy to use, for it is almost impossible to determine whether two groups have equivalent matrix sequences).

Page 907, line 16 Change definition of S_0:

$$S_0 = \{s_1, \dots, s_n\} \cup \{\text{nonzero coefficients of all } f_i(X)\}$$

Page 912, line -1 Replace "s_2^{-1}" by "$s_2^{-1}m_2$" (two times)

Page 914, line 3 Add: "(see Exercise 6.67 on page 398)."

Page 929, line 10 Replace the sentence beginning "By Lemma 11.50 " by

If $S = R - \mathfrak{p}$, then $S^{-1}R^*$ is an extension of $R_\mathfrak{p}$ (since localization is an exact functor, R contained in R^* implies $R_\mathfrak{p}$ contained in $S^{-1}R^*$); by Lemma 11.50, $S^{-1}R^*$ is integral over $R_\mathfrak{p}$.

Page 934, line 14 Change "R/I" to "$R/\mathfrak{q}$"

Page 936. Replace lines -8 through -1 as follows:

ideals. Thus, u is algebraic over F, and hence u is algebraic over R. Since $R[u]$ is a G-domain, Proposition 11.58 says that R is a G-domain. Now R is a Jacobson ring, and so R is a field, by Exercise 11.34 on page 938. But if R is a field, so is $R[u]$, for u is algebraic over R. Therefore, $R[u] = R[x]/\mathfrak{q}$ is a field, so that $\mathfrak{q}$ is a maximal ideal, and $R[x]$ is a Jacobson ring. •

Page 973, line 3 Change "$\operatorname{im}\eta$" to "$\operatorname{coker}\eta$"

Page 973, line 4 Change "$\operatorname{im} d^{n-1}$" to "$\operatorname{coker} d^{n-1}$"

Page 980, line -4 Change "$\operatorname{Hom}_R(R/cR, A)$" to "$\operatorname{Hom}_R(R/cR, A^*)$"

Page 984, line -6 Change "$\operatorname{Hom}_k(k/ck, A)$" to "$\operatorname{Hom}_k(k/ck, A^*)$"

Page 989. Restate Lemma 11.160 and insert two lines at beginning of proof.

Lemma 11.160. *Let a and b be nonzero elements in a domain R. If there exists $c \in R$ such that $ca^2 \in (b)$ implies $ca \in (b)$, then the series $(a, b) \supseteq (a) \supseteq (a^2)$ and $(a^2, b) \supseteq (a^2, ab) \supseteq (a^2)$ have isomorphic factor modules.*[7]

Proof. Now $(a, b)/(a) \cong (a^2, ab)/(a^2)$, for multiplication by a sends (a, b) onto (a^2, ab) and (a) onto (a^2).

Page 990, line 6 Rewrite the sentence beginning "But Lemma 11.160" as follows:

But Lemma 11.160 implies that both (a, b) and its submodule (a^2, b) have length ℓ.

Page 995, line -5 Change "Lemma 11.171" to "Lemma 11.170"

Page 996, line -13 Change "$pd(A) \le n$" to "$pd(A) = n$"

Page 997, line 13 Change "Lemma 11.164" to "Lemma 11.16"

Pages 997–998 Replace everything under line 14 by the following.

Let R be a noetherian ring, let M be a finitely generated R-module, and let I be an ideal such that $IM \neq M$. By Exercise 11.82 on page 115, I contains a longest M-sequence (such sequences are usually called ***maximal** M**-sequences*** in I). We are going to prove, given an ideal I and a finitely generated R-module M, that all maximal M-sequences in I have the same length.

Definition. If R is a commutative ring, then an ***associated prime ideal*** of a nonzero R-module B is a prime ideal of the form $\operatorname{ann}(b)$ for some nonzero $b \in B$.

Lemma 11.777. *Let B be a nonzero finitely generated module over a noetherian ring R.*
(i) *The maximal elements in* $\mathcal{F}(B) = \{\operatorname{ann}(b) : b \in B \text{ and } b \neq 0\}$ *are associated prime ideals of B.*
(ii) *There are finitely many associated prime ideals of B, say, $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, such that*

$$Z(B) = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s,$$

where $Z(B) = \{r \in R : rb = 0 \text{ for some nonzero } b \in B\}$.

Proof. (i) The set of ideals $\mathcal{F}(B)$ has maximal elements, because R is noetherian. Let $\operatorname{ann}(b)$ be such a maximal element. Suppose that $rs \in \operatorname{ann}(b)$, where $r, s \in R$ and $r \notin \operatorname{ann}(b)$. Now $\operatorname{ann}(b) \subseteq \operatorname{ann}(rb)$, for if $ub = 0$, then $u(rb) = 0$; by maximality, $\operatorname{ann}(b) = \operatorname{ann}(rb)$. Hence, $s \in \operatorname{ann}(rb)$ implies $s \in \operatorname{ann}(b)$, and so $\operatorname{ann}(b)$ is a prime ideal.

(ii) For each $r \in Z(B)$, there is a nonzero $b \in B$ with $rb = 0$; that is, $Z(B) = \bigcup_{\operatorname{ann}(b) \in \mathcal{F}(B)} \operatorname{ann}(b)$. If we denote the set of maximal elements in $\mathcal{F}(B)$ by $\mathfrak{M}$, then $Z(B) = \bigcup_{\mathfrak{p} \in \mathfrak{M}} \mathfrak{p}$, for every $\operatorname{ann}(b) \in \mathcal{F}(B)$ is contained in a maximal element.

It suffices to prove that $\mathfrak{M}$ is finite. Define $B' = \langle b : \operatorname{ann}(b) \in \mathfrak{M} \rangle$. Now B' is finitely generated, for R noetherian implies that every submodule of a finitely generated R-module is itself finitely generated; let $B' = \langle b_1, \ldots, b_n \rangle$, and denote $\operatorname{ann}(b_i)$ by $\mathfrak{p}_i$. Suppose there is $\mathfrak{q} = \operatorname{ann}(b_0) \in \mathfrak{M}$ with $b_0 \neq b_i$ for $i = 1, \ldots, n$. As $b_0 \in B'$, there are $r_i \in R$ with $b_0 = \sum_i r_i b_i$. It follows that if $r \in \bigcap_i \mathfrak{p}_i$, then $rb_0 = 0$; that is, $\bigcap_i \mathfrak{p}_i \subseteq \operatorname{ann}(b_0) = \mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal, Proposition 6.13 gives $\mathfrak{p}_i \subseteq \mathfrak{q}$ for some i. As $\mathfrak{p}_i$ is a maximal element in $\mathcal{F}(B)$, we have $\mathfrak{q} = \mathfrak{p}_i$, as desired. •

Remark. The set $\operatorname{Ass}(B)$ of all associated primes of an R-module B is important in deeper studies [$\mathfrak{M}$ may be a proper subset of $\operatorname{Ass}(B)$]. For example, it is related to primary decompositions (see Matsumura, *Commutative Ring Theory*, pages 39–42). ◂

The next lemma is a generalization of the observation that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{I}_m, \mathbb{I}_n) = \{0\}$ if $(m, n) = 1$.

Lemma 11.178. *Let R be a commutative ring, and let A and B be R-modules.*

(i) *If* $\operatorname{ann}(A)$ *contains a B-regular element, then* $\operatorname{Hom}_R(A, B) = \{0\}$.

(ii) *Let R be noetherian and let A and B be finitely generated. If* $\operatorname{Hom}_R(A, B) = \{0\}$, *then* $\operatorname{ann}(A)$ *contains a B-regular element.*

Proof. If $r \in \operatorname{ann}(A)$, then $ra = 0$ for all $a \in A$. Hence, for all $f \in \operatorname{Hom}_R(A, B)$, we have $0 = f(ra) = rf(a)$. On the other hand, if r is B-regular, then $rf(a) = 0$ implies $f(a) = 0$, and so $f = 0$.

Page 998. Replace lines 1 - 3 by the following.

(ii) Assume, on the contrary, that $\mathrm{ann}(A)$ contains no B-regular elements; that is, $\mathrm{ann}(A) \subseteq Z(B)$. By Lemma 11.177, there are finitely many associated prime ideals of B, say, $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, such that $\mathrm{ann}(A) \subseteq Z(B) = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$, and so Proposition 6.14 says that there is some $\mathfrak{p} = \mathfrak{p}_i$ with $\mathrm{ann}(A) \subseteq \mathfrak{p}$.

Page 998. Rewrite the proof of Lemma 11.178 as follows.

Proof. The proof is by induction on $n \geq 0$. We define $I = \{0\}$ in case $n = 0$, and so the base step holds. Assume now that $x_1, \ldots, x_{n+1}$ is a B-sequence in $\mathrm{ann}(A)$, that $I = (x_1, \ldots, x_{n+1})$, and that $J = (x_1, \ldots, x_n)$. Observe first that there is an exact sequence $0 \to B \to B \to B/x_1B \to 0$, for x_1 is a regular element on B. Consider the portion of the long exact sequence, where x_1 is multiplication by x_1:

$$\mathrm{Ext}^n_R(A, B) \to \mathrm{Ext}^n_R(A, B/x_1B) \xrightarrow{\partial} \mathrm{Ext}^{n+1}_R(A, B) \xrightarrow{x_{1*}} \mathrm{Ext}^{n+1}_R(A, B).$$

Since $x_1 \in \mathrm{ann}(A)$, the induced map x_{1*} is the zero map, and so ∂ is surjective. By induction, $\mathrm{Hom}_R(A, B/JB) \cong \mathrm{Ext}^n_R(A, B)$. Multiplication by $x_{n+1}\colon B/JB \to B/JB$ is an injection, because x_{n+1} is (B/JB)-regular, and left exactness of $\mathrm{Hom}_R(A, \ )$ shows that $(x_{n+1})_*$ is an injection $\mathrm{Hom}_R(A, B/JB) \to \mathrm{Hom}_R(A, B/JB)$. On the other hand, $(x_{n+1})_*$ is the zero map, for $x_{n+1} \in \mathrm{ann}(A)$. Hence, $\mathrm{Hom}_R(A, B/JB) = \{0\}$, and $\mathrm{Ext}^n_R(A, B) = \{0\}$. Therefore, $\partial\colon \mathrm{Ext}^n_R(A, B/x_1B) \to \mathrm{Ext}^{n+1}_R(A, B)$ is an isomorphism.

By induction, if $B' = B/x_1B$, then

$$\mathrm{Hom}_R(A, B'/(x_2, \ldots, x_{n+1})B') \cong \mathrm{Ext}^n_R(A, B') = \mathrm{Ext}^n_R(A, B/x_1B) \cong \mathrm{Ext}^{n+1}_R(A, B).$$

But $B'/(x_2, \ldots, x_{n+1})B' \cong (B/x_1B)/(IB/x_1B) \cong B/IB$, so that $\mathrm{Hom}_R(A, B/IB) \cong \mathrm{Ext}^{n+1}_R(A, B)$, as desired. •

Pages 998–999. Relabel "Proposition 11.179" as "Proposition 11.180". Delete Corollary 11.180 and make its statement a new exercise.

Page 1000, line 13 Change "Theorem 11.128" to "Theorem 11.134"

Page 1000, lines −5 and −4 Delete (ii) and (iii); replace by

(ii) *If, in addition, both A and B are free R-modules, then $\overline{f}$ injective implies that f is a (split) injection.*

Page 1001. Delete lines 5 through 13. Replace by:

(ii) Assume that $\overline{f}$ is injective. Let $x_1, \ldots, x_t$ be a basis of A, and let $b_i = f(x_i)$ for $i = 1, \ldots, t$. Since $\overline{f}$ is injective, the elements $\overline{b}_i = b_i + \mathfrak{m}B$ are linearly independent in $B/\mathfrak{m}B$, and so they extend to a basis: There are $c_1, \ldots, c_s \in B$ with $\overline{b}_1, \ldots, \overline{b}_t, \overline{c}_1, \ldots, \overline{c}_s$ a basis of $B/\mathfrak{m}B$. An application of Nakayama's lemma, as in the proof of Proposition 11.23, shows that $b_1, \ldots, b_t, c_1, \ldots, c_s$ is a basis of B. If we define $h\colon B \to A$ by $h(b_i) = x_i$ and $h(c_j) = 0$, then we see that $hf = 1_A$, and so f is injective. •

Page 1002, line 15 Change "inducted" to "induced"

Page 1007, line 15 Add following before Clearly.

We may assume that $a \notin Rx$. If $a = a_1x$ and $a_1 \notin Rx$, then replace a by a_1, for $R_xa = R_xa_1$. If $a_1 = a_2x$ and $a_2 \notin Rx$, then replace a_1 by a_2, for $R_xa_1 = R_xa_2$. If this process does not stop, there are equations $a_m = a_{m+1}x$ for all $m \geq 1$, which give rise to an ascending sequence $Ra_1 \subseteq Ra_2 \subseteq \cdots$. Since R is noetherian, $Ra_m = Ra_{m+1}$ for some m. Hence, $a_{m+1} = ra_m$ for some $r \in R$, and $a_m = a_{m+1}x = ra_mx$. Since R is a domain, $1 = rx$; thus, x is a unit, contradicting Rx being a prime (hence, proper) ideal.

Page A-3, lines 3 - 5 Change the statement of (iv).

Let X be a partially ordered set in which every two elements are comparable. Assuming the axiom of choice, if every strictly decreasing sequence in X is finite, then X is well-ordered.

Page A-3, line 14 Change this line as follows:

Choose $s_0 \in S$; since s_0 is not smallest, it is not true that $s_0 \preceq s$ for all $s \in S$. Thus, either there exists $s_1 \in S$ with $s_0 \succ s_1$ or there is $s \in S$ with s_0 and s not comparable; the latter cannot occur, by hypothesis.

Page A-5, lines 7 – 10 Change the proof of Lemma A.5.

Proof. Necessity is obvious, for every subset of a well-ordered set is well-ordered. Conversely, let S be a nonempty subset of X. Of course, if S is a singleton, then it contains a smallest element, and so we may assume that S contains at least two elements, say, c' and c. Since X is a chain, we may assume that $c' \prec c$. Hence, $\text{Seg}(c) \cap S \neq \varnothing$; as every nonempty subset of a well-ordered set is well-ordered, there is a smallest element, say, z, in $\text{Seg}(c) \cap S$. Now z is the smallest element in S, for if there is $s' \in S$ with $s' \prec z$, then $s' \in \text{Seg}(c) \cap S$, contradicting z being the smallest element in $\text{Seg}(c) \cap S$. Therefore, X is well-ordered. •

Page A-7, line -5 Change "Choose $c_0 \in X$" to "Define $c_0 = g(\varnothing)$"

Page A-8 Change the first paragraph.

If C and D are g-sets, we claim that either $C \trianglelefteq D$ or $D \trianglelefteq C$. Define W to be the union of all those subsets B with $B \trianglelefteq C$ and $B \trianglelefteq D$. We claim that $W \trianglelefteq C$ and $W \trianglelefteq D$; that is, W is closed in C and in D. Take $w \in W$; this element got into W because it lies in some B, where $B \trianglelefteq C$ and $B \trianglelefteq D$. If $c \in C$ and $c \preceq w$, then $c \in B$ (because B is closed in C). Hence, $c \in B \subseteq W$ (for W is, by definition, the union of all such subsets B). Therefore, W is closed in C. Similarly, W is closed in D. If either $W = C$ or $W = D$, then the claim is true. Hence, we may assume that $W \lhd C$ [so that $W = C \cap \text{Seg}(c')$ for some $c' \in C - W$], and $W \lhd D$ [so that $W = D \cap \text{Seg}(d')$ for some $d' \in D - W$]. Since C and D are g-sets, $c' = g(C \cap \text{Seg}(c')) = g(W)$ and $d' = g(D \cap \text{Seg}(d')) = g(W)$. Therefore, $c' = d'$. But now $W \cup \{c'\} = W \cup \{c'\}$ is closed in C and in D, for it is a closed interval. Thus, $W \cup \{c'\} \subseteq W$, contradicting $c' \notin W$. Therefore, either $W = C$ or $W = D$; that is, either $C \trianglelefteq D$ or $D \trianglelefteq C$, as claimed.