Week 6: 7.1, 7.4

Binomial Distributions

Suppose we have a binomial experiment with n trials. We can define a random variable X to be the number of successes, so X ranges from 0 to n.

Definition: For a given n and p, let \( b(k) = C(n, k)p^kq^{n-k} \), the probability of getting exactly k successes. We sometimes write \( b(k; n, p) \) when we want to emphasize or clarify the values of n and p. The probability distribution of the associated random variable X is called the binomial distribution.

Recall the coffee problem from the previous activity. There, you were asked to determine the probability that at most three students out of 20 prefer coffee. Getting a numerical answer for such a question is tedious, so we have binomial distribution tables that allow us to look up the numerical answers for many such problems.

Definition: For a given n and p, let \( B(x) = b(0) + b(1) + \cdots + b(x) \), the probability of getting at most x successes. We sometimes write \( B(x; n, p) \) when we want to emphasize or clarify the values of n and p. This is called the cumulative binomial distribution function. The table in Appendix A gives values of B for various values of x, n, and p.

1. Use the table in Appendix A to find the probability of getting at most five successes in a binomial experiment with eleven trials and a probability of success \( p = 0.25 \).

\[
B(5; 11, 0.25) = 0.9657
\]

2. Use the table to find the probability of getting exactly five successes in a binomial experiment with eleven trials and a probability of success \( p = 0.25 \). (Hint: think of how B is defined in terms of b.)

\[
b(5) = B(5) - B(4) = 0.9657 - 0.8854 = 0.0803
\]

3. Write \( b(x) \) in terms of the function B.

\[
b(x) = B(x) - B(x - 1)
\]

If \( x > 0 \), \( b(x) = B(x) - B(x - 1) \)
4. The table gives us probabilities of getting at most a certain number of successes. How can we use it to find the probability of getting at least a specified number of successes, for example at least three successes in seven trials with \( p = 0.35 \)?

\[
Pr(\text{At least 3}) = 1 - Pr(\text{At most 2})
= 1 - B(2; 7, 0.35) = 1 - 0.5323
= 0.46770
\]

5. The values of \( p \) in the table are no greater than 0.5. How can we use the table to find, for example, \( B(3; 9, 0.6) \)?

\[
B(3; 9, 0.6) = Pr(\text{At most 3 successes})
= Pr(\text{At least 6 failures})
= 1 - Pr(\text{At most 5 failures})
= 1 - B(5; 9, 0.4) = 1 - 0.9006 = 0.0994
\]

6. Find the probability of getting at least five successes in a binomial experiment with eleven trials and a probability of success \( p = 0.25 \).

\[
Pr(\text{At least 5 successes}) = 1 - Pr(\text{At most 4 successes})
= 1 - B(4; 11, 0.25) = 1 - 0.8854
= 0.11460
\]

7. Find the probability of getting exactly four successes in a binomial experiment with nine trials and a probability of success \( p = 0.4 \).

\[
Pr(\text{Exactly 4 successes}) = B(4; 9, 0.4) - B(3; 9, 0.4)
= 0.7334 - 0.4826
= 0.2508
\]

8. Find the probability of getting at least five successes in a binomial experiment with eleven trials and a probability of success \( p = 0.7 \).

\[
Pr(\text{At least 5 successes}) = Pr(\text{At most 6 failures}) = B(6; 11, 0.5)
= 0.9784
\]
Normal Distributions

**Definition:** A continuous random variable is one that can take on any value on some interval of real numbers.

Recall that a histogram is a visual representation of a random variable, and that the total area under all of the bars is 1. With a continuous random variable, instead of bars we use a probability density function, the area under whose graph is 1.

1. In the following graph of a probability density function $f$ of a random variable $X$, shade the area corresponding to $X$ being between $-2$ and $1$.

2. Use the table in Appendix B to find the following probabilities.

   a) $\Pr(Z \leq 2.1)$
   
   $\boxed{0.9821}$

   b) $\Pr(Z \leq 2.14)$
   
   $0.9838$

   c) $\Pr(-2 \leq Z \leq 1)$
   
   $= \Pr(Z \leq 1) - \Pr(Z \leq -2) = 0.8413 - 0.0228$
   
   $= 0.8185$    

   d) $\Pr(Z \geq 0.5)$
   
   $= 1 - \Pr(Z < 0.5) = 1 - 0.6915$
   
   $= 0.3085$
Definition: A normal distribution is determined by a curve like the one that determines the standard normal distribution, but we are now free to change the mean $\mu$ and standard deviation $\sigma$.

3. Suppose that the following graph is a normal distribution determined by a random variable $X$. Label the mean, and shade the area corresponding to $X$ being between $-2$ and $1$.

![Normal Distribution Graph]

Remark: We can use the same table in Appendix B to find values like $\Pr(X \leq 0)$ by using the following facts. If $X$ has mean $\mu$ and standard variation $\sigma$, then

$$Z = \frac{X - \mu}{\sigma},$$

which means that

$$\Pr(a \leq X \leq b) = \Pr\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right)$$

4. Suppose that $X$ is a normally distributed random variable with $\mu = 2$ and $\sigma = 0.5$. Use the table in Appendix B to find the following probabilities.

a) $\Pr(X \leq 2.1) = \Pr\left(\frac{X - 2}{0.5} \leq \frac{2.1 - 2}{0.5}\right) = \Pr(Z \leq 0.2) = 0.5793$

b) $\Pr(X \leq 2.14) = \Pr\left(\frac{X - 2}{0.5} \leq \frac{2.14 - 2}{0.5}\right) = \Pr(Z \leq 0.28) = 0.6105$

c) $\Pr(1 \leq X \leq 2.1) = \Pr\left(\frac{1 - 2}{0.5} \leq \frac{X - 2}{0.5} \leq \frac{2.1 - 2}{0.5}\right) = \Pr(-2 \leq Z \leq 0.2)$

$$= \Pr(Z \leq 0.2) - \Pr(Z \leq -2) = 0.5793 - 0.0228 = 0.55650$$

d) $\Pr(X \geq 0.5)$

$$= 1 - \Pr(X < 0.5) = 1 - \Pr\left(\frac{X - 2}{0.5} < \frac{0.5 - 2}{0.5}\right)$$

$$= 1 - \Pr(Z < -3) = 1 - 0.0013 = 0.9987$$