Week 14: 9.1–9.3

Basics of Finance

**Formula:** The formula for the simple interest $I$ of an account with starting value $P$ (called the **principle**) earning an annual **simple interest** of $r$ (the percent as a decimal) for $t$ years is

$$I = Prt$$

**Formula:** The formula for the future value $F$ of an account with starting value $P$ (called the **principle**) earning an annual **simple interest** of $r$ (the percent as a decimal) for $t$ years is

$$F = P(1 + rt)$$

1. Suppose that you borrow $3000 at 9% simple interest per year. Find the interest and the amount to be paid back if the time period is (a) 6 months, (b) 2 years, and (c) 5 years.

   (a) $I = 3000(0.09) \frac{1}{2} \approx \$135$ \quad \$3,135$

   (b) $I = 3000(0.09) \cdot 2 \approx \$540$ \quad \$3,540$

   (c) $I = 3000(0.09) \cdot 5 \approx \$1350$ \quad \$4,350$

2. Suppose you have $8000 and want to have $8750 in 18 months to go on a luxury cruise. What simple interest rate is needed to raise the money in time?

   Using $F = P(1+rt)$,

   $8750 = 8000(1+1.5r)$

   $\frac{35}{32} = 1+1.5r$

   $r = \frac{3}{32} \cdot \frac{2}{3}$

   $r = \frac{2}{32} = 0.0625$

   **6.25% annual interest rate**

3. Suppose that you open a savings account with an initial balance of $100. If the savings account earns 3% simple interest, how long will it take to double the initial balance?

   Using $F = P(1+rt)$,

   $200 = 100(1 + 0.03t)$

   $2 = 1 + 0.03t$

   $1 = 0.03t$

   $t = \frac{100}{3}$

   **33 years and 4 months**

99
**Definition:** Suppose that an account (or loan, mortgage, etc.) earns a fixed annual interest. When we say that that interest is compounded a certain number of times per year, we apply the interest in equally-divided amounts once every period. For example, if an account earns 8% annual interest compounded quarterly (i.e. four times per year), then 2% interest is applied at four evenly-spaced intervals throughout the year. The formula for the future value $F$ of a principle invested at an interest rate of $r$ for $t$ years compounded $m$ times per year is

$$ F = P \left(1 + \frac{r}{m}\right)^{mt}. $$

This type of interest is called **compound interest**.

4. Suppose that you open a savings account with an initial balance of $100 and an annual interest rate of 3%. Compute the amount in the account after five years when the interest is compounded

a) annually.

$$ F = 100 \left(1 + \frac{0.03}{1}\right)^{5} = 100 \left(1.03\right)^{5} \approx$ \$115.927

b) quarterly.

$$ F = 100 \left(1 + \frac{0.03}{4}\right)^{4\cdot5} \approx$ \$116.118

c) monthly.

$$ F = 100 \left(1 + \frac{0.03}{12}\right)^{12\cdot5} \approx$ \$116.162

d) weekly.

$$ F = 100 \left(1 + \frac{0.03}{52}\right)^{52\cdot5} \approx$ \$116.178

e) daily.

$$ F = 100 \left(1 + \frac{0.03}{365}\right)^{365\cdot5} \approx$ \$116.183

f) What pattern arises in this sequence of numbers?

The more often the interest is compounded, the higher the future value.
5. Suppose that you open a savings account with an initial balance of $100. If the annual interest is compounded monthly, and the account has $1,000 in it after ten years, what is the annual interest rate?

\[ m = 12 \quad t = 10 \]

\[ 1,000 = 100 \left(1 + \frac{r}{12}\right)^{120} \]

\[ 10 = \left(1 + \frac{r}{12}\right)^{120} \]

\[ 10 \frac{120}{12} = 1 + \frac{r}{12} \]

\[ 10 \frac{120}{12} - 1 = \frac{r}{12} \]

\[ r = 12 \left(10 \frac{120}{12} - 1\right) \approx 0.232 \]

\[ 23.2\% \text{ annual interest rate} \]

6. Suppose that you open a savings account with an initial balance of $100, and that the account has an annual interest rate of 7% compounded annually. How long would it take to double your initial balance? (Set up the equation, but do not solve.) Is the doubling time affected by the initial balance?

\[ 200 = 100 \left(1 + \frac{0.07}{1}\right)^t \]

\[ 2 = \left(1 + 0.07\right)^t \]

\[ \ln 2 = t \ln 1.07 \]

\[ t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years} \]

7. A table on page 364 of the textbook shows a table of interest rates and associated doubling times. Use this to estimate the doubling time for the previous problem.

\[ 2 = 1.07^t \]

\[ \ln 2 = t \ln 1.07 \]

\[ t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years} \]

8. Suppose that you open a savings account with an initial balance of $150, and that the account's interest is compounded annually. If there is $1,200 in the account after 18 years, what is the approximate annual interest rate?

\[ m = 1 \quad t = 18 \]

\[ 1200 = 150 \left(1 + \frac{r}{1}\right)^{18} \]

\[ 8 = \left(1 + r\right)^{18} \]

\[ 8^{\frac{1}{18}} = 1 + r \]

\[ r = 8^{\frac{1}{18}} - 1 \approx 0.122 \]

approximate annual interest rate of 12.2%
9. Suppose that you want to invest $2,500. You have two choices: you can either invest at a simple interest rate of 3%, or a compound rate of 0.85% (compounded annually).

a) Which should you choose if you want to invest for 30 years?

Using \( F = P(1+rt) \) for the simple interest, we have
\[
F = 2500 (1 + (0.03)30) = \$ 4750
\]

Using \( F = P\left(1 + \frac{r}{m}\right)^{mt} \) for the compound interest, we have
\[
F = 2500 \left(1 + \frac{0.03}{1}\right)^{30} \approx \$ 4578.40
\]

b) Which should you choose if you want to invest for 60 years?

Using \( F = P(1+rt) \) for the simple interest, we have
\[
F = 2500 (1 + (0.03)60) = \$ 7000
\]

Using \( F = P\left(1 + \frac{r}{m}\right)^{mt} \) for the compound interest, we have
\[
F = 2500 \left(1 + \frac{0.03}{1}\right)^{60} \approx \$ 8202.58
\]

10. How could you explain the answers for problem 9 to someone who does not know the formulas for each type of interest? More generally, how could you explain the difference between simple and compound interest to such a person?

With simple interest, interest is paid on the principal.

With compound interest, interest is paid on both the principal AND previously earned interest.

For the short term, it is wise to choose the higher simple interest rate, but in the long run, the compound interest accumulates enough "interest on the interest" to make it the better choice.
Annuities

All of the preceding examples involved setting up an account and then depositing no more money into it. Now, we analyze one way of adding money over time to an account.

Definition: An annuity is a sequence of equal payments made at equal time periods. The term of an annuity is the time from the beginning of the first period to the end of the last, and the future value $FV$ of an annuity is the amount of the annuity, including interest, at the end of its term. The future value of an annuity of $n$ payments of $PMT$ dollars each made at the end of the period with interest $i$ compounded each period is

$$FV = PMT \times \frac{(1+i)^n - 1}{i}$$

With the same notation, the present value $PV$ of an annuity is

$$PV = PMT \times \frac{1 - (1+i)^{-n}}{i}$$

1. Suppose that you open a savings account earning 6% annual interest compounded monthly. You deposit $100 every month for twenty years starting at the end of the first month.

   a) How much money will you have in the account after twenty years?

   $$n = 12 \times 20 = 240 \quad i = \frac{.06}{12} = 0.005$$

   $$FV = 100 \times \frac{(1+.005)^{240} - 1}{.005} \approx 100 \times (462.041) = $46,204.10$$

   b) How much was deposited into the account?

   $$240 \text{ months} \times $100 / \text{month} = $24,000$$

   c) How much interest was earned?

   $$46,204.10 - 24,000 = 22,204.10 \text{ dollars}$$
2. Suppose that you borrow \$15,000 to buy a car from a bank that charges an annual interest rate of 12\%, and will pay back the loan with 60 equal monthly payments.

a) What should the monthly payments be?

Rearranging the formula on the previous page, we see

\[
PMT = \frac{PV \cdot i}{1 - (1+i)^{-n}} \quad i = \frac{12}{12} = .01
\]

\[
= \frac{15000 \cdot .01}{1 - (1+.01)^{-60}}
\]

\[
\approx 15000 \cdot 0.22244 = \$333.66
\]

b) How much money do you pay to the bank?

\[
60(333.66) = \$20,019.60
\]

c) What is the total amount of interest paid on this loan?

\[
\$5019.60
\]