Name: __________________________________________

“The only way to learn mathematics is to do mathematics.” - Paul Halmos

You were assigned yesterday in Lecture the following: Read section 1.3. In 1.3 do 3, 8, 9, 10, 12, 14, 18, 19, 31, 32, 33, 38, 41, 43. Concepts you should understand after today: function, domain, range, independent variable, dependent variable, vertical line test, even function, odd function, increasing and decreasing as well as transformations of functions.

1. What are the domains and ranges of the following functions? Remember that the domain is all the real valued inputs for which the function makes sense and that the range is the set of all outputs of the function.

   (a) \( f(x) = \frac{1}{x^3 + 27} \)

   Solution. We know we can’t divide by zero. So, all the allowed input will be all real numbers except those who make the denominator zero. Hence, to find those values we must solve, \( x^3 + 27 = 0 \) \( \Rightarrow (x + 3)(x^2 - 3x + 9) = 0 \) so \( x \neq -3 \) and note that \( (x^2 - 3x + 9) \neq 0 \) for all \( x \) in the real numbers. Thus, the domain of \( f \) is \((-\infty, -3) \cup (-3, \infty)\).

   (b) \( g(x) = \sqrt{\frac{1}{1 - |5 - x|}} \)

   Solution. What do we know about the root function? It can have negative inputs. So, we know that \( \frac{1}{1 - |5 - x|} \geq 0 \) When does this happen? When \( 1 - |5 - x| \geq 0 \) However, since this is in the denominator it can’t be zero. Solving for \( x \) in \( 1 - |5 - x| > 0 \), we get

   \(-|5 - x| > -1 \Rightarrow |5 - x| < 1 \Rightarrow -1 < 5 - x < 1 \Rightarrow -6 < -x < -4 \Rightarrow 6 > x > 4.\)

   So, the domain of \( g \) is \((4,6)\). The range is \((0, \infty)\) since there is no value of \( x \) that makes the input 0.

   (c) \( h(x) = \sqrt{(1-x)(-4-x)} \) Solution. Similar to the above exercise, root functions can’t have negative inputs so we must solve \((1-x)(-4-x) \geq 0\) Note we can factor a \(-1\) from the second term and get \(-(1-x)(4+x) \geq 0 \Rightarrow (1-x)(4+x) \leq 0\) One way, to solve this inequality is by using a table. First, set each factor to zero to obtain \( x = 1 \) and \( x = -4 \). Range: The real numbers.

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   \text{Factor/Interval} & (-\infty, -4) & (-4, 1) & (1, \infty) \\
   \hline
   (1 -x) & + & + & - \\
   (4+x) & - & + & + \\
   \text{sign of function} & - & + & - \\
   \hline
   \end{array}
   \]

   So, the domain of \( h \) is where we see negative values: \((-\infty, -4] \cup [1, \infty)\)
2. Starting with a piece of wire 100 inches in length, you use $x$ inches to make a circle and the remaining wire to make a square. Express the total area of the two shapes as a function of $x$.

Solution. We know that the length of the circle is given by its circumference so $2\pi r = x \Rightarrow r = \frac{x}{2\pi}$, and that to make the square we have $100 - x$ amount of string so a side of the square uses $\frac{100-x}{4}$, then $A = \pi r^2 + (100-x)^2$ so we get that the area is $A = \pi \left(\frac{x}{2\pi}\right)^2 - \left(\frac{100-x}{4}\right)^2$.

3. Let $L_1 : 3x + 2y = 8$ and line $L_2 : 2x - 5y = 15$ be two lines. Find the slope and y-intercept of each line. Are the lines increasing or decreasing? Using only the y-intercept and slope, graph each line on the same coordinate plane. Suppose these equations represented the relationship between two physical quantities. What does the slope tell you about that relationship?

Solution. $L_1$ has slope $\frac{3}{2}$ and y-intercept 4, $L_2$ has slope $\frac{2}{5}$ and y-intercept $-3$.

4. Given the function $f(x) = x^3$, sketch graphs of the following functions. Write down a sentence explaining why the graph changes the way it does.

   (a) $f(x + 2)$ shifts 2 left
   (b) $f(x) - 3$ shifts 3 down
   (c) $f(-x)$ flips over x-axis
   (d) $-f(x)$ flips over y-axis
   (e) $f(x - 4) + 3$ shifts right 4 and up 3
   (f) $|f(x)|$ flips everything positive

5. Let $f(x) = \sqrt{3 - 2x}$ and $g(x) = x^2 - 5$. Find the following functions and their domains:

   (a) $f(x) + g(x) = \sqrt{3 - 2x} + x^2 - 5$ Domain: $\frac{3}{2} \geq x$
   (b) $f(x) - g(x) = \sqrt{3 - 2x} - x^2 + 5$ Domain: $\frac{3}{2} \geq x$
   (c) $\frac{f(x)}{g(x)} = \frac{\sqrt{3 - 2x}}{x^2 - 5}$ Domain: $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$
   (d) $\frac{g(x)}{f(x)} = \frac{x^2 - 5}{\sqrt{3 - 2x}}$ Domain: $\frac{3}{2} > x$

6. What does it mean for a function to be even? What does this mean graphically? What does it mean for a function to be odd? What does this mean graphically? Think of a function that is neither even nor odd. Is there a function that is both even and odd?

Solution. even: $f(-x) = f(x)$, symmetric about y-axis, odd: $f(-x) = -f(x)$, symmetric about origin, $f(x) = x^2 + x$ is not even nor odd, $g(x) = 0$ is both even and odd (only function this is true for).

7. Below there are four cases for f and g. Decide if $f + g$, $fg$, and $g(f(x))$ are even, odd, or neither.
(a) f is even, g is even
Solution: \((f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)\)
\((fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)\)
\(f(g(-x)) = f(g(x))\) All 3 are even

(b) f is odd, g is odd
Solution: \(f + g\) odd, \(fg\) is even and \(g \circ f\) is odd
\((f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x)\)
\((fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)\)
\(f(g(-x)) = f(-g(x)) = -f(g(x))\)

(c) f is even, g is odd
Solution: \(f + g\) is neither, \(fg\) is odd and \(g \circ f\) is even \((f + g)(-x) = f(-x) + g(-x) = f(x) - g(x)\)
\((fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -(fg)(x)\)
\(f(g(-x)) = f(-g(x)) = f(g(x))\)

(d) f is odd, g is even
Solution: \(f + g\) is neither, \(fg\) is odd and \(g \circ f\) is even \((f + g)(-x) = f(-x) + g(-x) = -f(x) + g(x)\)
\((fg)(-x) = f(-x)g(-x) = -f(x)(g(x)) = -(fg)(x)\)
\(f(g(-x)) = f(g(x))\)
This part is to discuss with your group if you finish the exercises. In the next discussion, session I’ll discuss this briefly. If you didn’t get to discuss it with your group. Think about it at home, this will be useful in the future.

**BONUS:** When comparing two functions, it’s often useful to talk about which one grows faster. One way to define this is to say that $f(x)$ grows faster than $g(x)$ if eventually (i.e., as $x$ gets really big), $f(x) > g(x)$.

(a) According to this definition, which grows faster, $x$ or $2x$? (Sketch a graph if you’re not sure).

(b) How about $x$ or $2^x$? Use this to figure out which grows faster, $x^2$ or $2^x$. How about $x^3$ and $2^x$?

(c) Is there any positive number $a$ for which $x$ grows faster than $2^x$? How about for which $x^a$ grows faster than $2^x$?

(d) People often talk about polynomial growth versus exponential growth. Which is faster? Does it make sense to group these two concepts?