A Fine Rediscovery

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Abstract

This article explores the history of the two results in integer partitions known as Stanley's theorem and Elder's theorem. While history has credited Richard Stanley with the discovery of the results, we note that Nathan Fine had established these results among a host of other partition identities over a decade earlier. In tribute to Fine, analogues in the sets of odd partitions and distinct partitions are presented.

1 Introduction.

Rediscoveries of theorems are not uncommon and have added human interest to the results in question.

For example, in 1913, the Indian genius, Ramanujan, sent a number of elegant and surprising formulas to G. H. Hardy. Among the most amazing were the Rogers-Ramanujan identities, of which the first is

$$1+\sum_{n\geq 1}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n\geq 1}\frac{1}{(1-q^{5n-4})(1-q^{5n-1})}.$$

It turned out that Ramanujan had no proof. Hardy communicated these formulas to MacMahon and O. Perron, but none could prove them [16, p. 344]. Then in 1917, Ramanujan discovered that L. J. Rogers had proved them in an 1894 paper [10, p. 91].

In 1937, Pólya proved an enumerative theorem that counts objects subject to an equivalence under the action of a group. He noted its wide applicability to many problems stretching from graph theory to molecular formations. It turned out that an essentially unknown mathematician, J. H. Redfield, had published a more general theorem in 1927 [2, p. 6].

Many of us are familiar with the famous 1939 result of Zeckendorf, which he published in 1972 [24]:

> *Every positive integer is uniquely the sum of nonconsecutive Fibonacci numbers.*

But as Ed Berger, et al. have pointed out, a much more general theorem was proved by Ostrowski in 1921 [3].

This paper is devoted to a further rediscovery. In this instance, we are considering partitions of integers, that is, representations of an integer n by the sum of nonincreasing sequence of positive integers. Thus, 4 has five partitions: $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. We denote this fact by $p(4) = 5$. The theorems we shall consider are as follows.

Theorem 1 (Stanley's Theorem)**.** *The number of $1's$ in the partitions of n is equal to the number of parts that appear at least once in a given partition of n, summed over all partitions of n.*

We see seven $1's$ in the partitions of 4. On the other hand, the number of different parts appearing in the partitions of 4 are $1+2+1+2+1 = 7$. The result has been widely attributed to Richard Stanley. It was generalized as follows.

Theorem 2 (Elder's Theorem)**.** *The number of $j's$ used in the partitions of n is equal to the number of parts that appear at least j times in a given partition of n, summed over all partitions of n.*

There are three $2's$ in the partitions of 4, and the number of times parts appear at least twice is $0+0+1+1+1 = 3$. The latter theorem was attributed to Paul Elder by Ross Honsberger [13, p. 8].

In Section 2, we recount the recent history of these theorems. In Section 3, we look at the older and much more general theorems of Nathan Fine. Section 4 extends these ideas to new, related theorems of partitions into odd parts and partitions into distinct parts. We conclude with a few open questions.

2 The Proofs from the 1970's and 1980's.

During the 1970's and 1980's, a number of proofs were presented for the two identities, and the proofs all took on one of three forms. The first was to establish a bijection between two sets of Ferrers diagrams in which each set counted a different quantity. The most elegant of the three methods, bijections, are often the hardest to find. The second was to show that the two quantities in question may be described by the same generating function. Although cumbersome, generating functions are very reliable. The third was to argue combinatorially that both of the quantities in question are equal to some central sum. For these theorems, the central sum involved the partition function, $p(n)$.

The commonly presented history of these results starts in 1972, when Stanley recognized the relationship known now as Elder's theorem. He submitted a bijective proof to the American Mathematical Monthly's Problems and Solutions section, but it was rejected for being "a bit on the easy side, and using only a standard argument" [22, p. 163]. Hidden in a 1974 article, Louis Solomon combinatorially proved Elder's theorem with a central sum argument [20]. In his 1978 book, Daniel Cohen included an exercise in the chapter on generating functions that asked the reader to show that the number of $1's$ equaled the number of different parts over all partitions of n [5, p. 100]. Cohen attributed this problem to Richard Stanley, and consequently, Stanley's name has been attached to this specific identity ever since.

While teaching at the University of Waterloo, Ross Honsberger learned of the identity for counting the number of $1's$ and had a five page proof from his colleague Ian Goulden that used generating functions. Hoping for a cleaner proof, Honsberger wrote to the Dutch mathematician Edsger W. Dijkstra. Dijkstra and his graduate student, Antonetta Johanna Maria van Gasteren, returned a letter to Honsberger in April 1981 with a proof that combinatorially established the result with a central sum [6].

In 1982, M. S. Kirdar and T. H. R. Skyrme stumbled across Elder's theorem while studying the characterization of characters [15]. Their proof utilized generating functions. The following year, Ian Goulden published *Combinatorial Enumeration* with David M. Jackson. Citing private communication with Richard Stanley as their source, Goulden and Jackson included Stanley's theorem as an exercise with solution in their chapter on generating functions [9, pp. 93, 94, and 370]. One year later, Frank W. Schmidt and Rodica Simion published both a combinatorial proof and one that stemmed from character theory for Elder's theorem [18].

In 1984, as Honsberger was working on his book *Mathematical Gems III*, Paul Elder (at the time an undergraduate at the University of Waterloo) independently rediscovered the variation of Stanley's theorem for counting the number of $j's$ over all partitions of n [13]. Honsberger called the case for $j = 1$ Stanley's theorem and the general case Elder's theorem. This book established the precedent that others followed. Although the combinatorial proof for Stanley's theorem is based on the letter from Dijkstra, the one in Honsberger's book is largely his own [14].

Two final publications appeared in 1986. Richard Stanley finally published his bijective proof for the full generality of Elder's theorem in *Enumerative Combinatorics I* [21, pp. 48, 59] and A. H. M. Hoare unknowingly published the same bijection in The American Mathematical Monthly [11]. Hoare cited the article from Kirdar and Skyrme as his source for the identity.

Since these events, the result has been listed elsewhere (for example, see [12]). The story of independent rediscovery takes on a new twist in the next section.

3 The Work of Nathan Fine.

Fifteen years prior to the earliest published proof mentioned above, Nathan Fine published an article in which both Stanley's theorem and Elder's theorem were established, among a host of other partition results [7]. Carlitz referenced Fine's work in [4, §1]. These methods were highlighted by Riordan in [17, §5.5], and the article was later republished as Section 22 in [8]. Fine's method was one of greater generality than those previously mentioned. Instead of starting with an identity and searching for a proof, Fine reversed this process by starting with the construction of functional relationships and then, by considering specific cases, he gleaned numerous partition identities, including both Stanley's theorem and Elder's theorem. Below, we see a glimpse of this prolific work.

Let π be a partition of n and let $\pi(n)$ represent the set of all partitions of n. (In the now standard notation, this would be expressed $\pi \vdash n$, but we adopt Fine's notation in honor of his work.) Then we let $k_i = k_i(\pi)$ represent the number of times i appears in the partition π. If f is a function on a partition π, then it will have input variables $k_1, \ldots, k_n$ and will be denoted by either $f(\pi)$ or $f(k_1, \ldots, k_n)$.

Theorem 3 (Fine's Theorem 1)**.** *Let*

$$\psi_i(q) = \sum_{k \geq 0} C_i(k) q^k, \ (\textit{for } i = 1, 2, \ldots),$$

where the $C_i(k)$ are arbitrary functions on the nonnegative integers. Then,

$$\prod_{i \geq 1} \psi_i(q^i) = \sum_{n \geq 0} q^n \sum_{\pi(n)} C_1(k_1) C_2(k_2) \cdots.$$

Example 1. The case of $C_i(k) = 1$ for all i and k gives

$$\prod_{i \geq 1} \psi_i(q^i) = \prod_{i \geq 1} \left(\sum_{k_1 \geq 1} C_i(k_i) q^{ik_i} \right) = \prod_{i \geq 1} \frac{1}{1 - q^i} = \sum_{n \geq 0} q^n \sum_{\pi(n)} 1 = \sum_{n \geq 0} q^n p(n),$$

which is the most fundamental example of generating functions in integer partitions.

Theorem 4 (Fine's Theorem 3)**.** *Let $L(\pi) = \sum_{i \geq 1} a_i k_i$. Then*

$$\sum_{\pi(n)} L(\pi) = \sum_{\substack{u+v=n \\ u \geq 0, v \geq 1}} p(u) \sum_{i | v} a_i.$$

Example 2. Let $a_i = 1$ for all i. Then we have

$$\sum_{\pi(n)} \sum_{i \geq 1} k_i = \sum_{\pi(n)} k(\pi) = \sum_{\substack{u+v=n \\ u \geq 0, v \geq 1}} p(u) \sum_{i | v} 1 = \sum_{\substack{u+v=n \\ u \geq 0, v \geq 1}} p(u) d(v),$$

where $k(\pi)$ counts the number of parts in a partition π and $d(v)$ is the divisor function.

Example 3. Let $a_i = i$ for all i. Then we have

$$\sum_{\pi(n)} \sum_{i \geq 1} i k_i = \sum_{\pi(n)} n = np(n) = \sum_{\substack{u+v=n \\ u \geq 0, v \geq 1}} p(u) \sum_{i | v} i = \sum_{\substack{u+v=n \\ u \geq 0, v \geq 1}} p(u) \sigma(v),$$

where $\sigma(v)$ is the sum of divisors function.

Example 4. Let $a_1 = 1$ and $a_i = 0$ for $i \neq 1$. Then we have

$$\sum_{\pi(n)} k_1 = \sum_{\substack{u+v=n \\ u \geq 0, v \geq 1}} p(u) = \sum_{u=0}^{n-1} p(u).$$

As noted, many of the proofs mentioned in the previous section relied on relating two quantities to a central sum. Example 4 establishes half of that method for Stanley's theorem.

With $Q(\pi)$ defined as the number of different parts in the partition π of n, Fine presented the following theorem, which has Stanley's theorem among its examples.

Theorem 5 (Fine's Theorem 5)**.** *Let f be any function on the number of different parts ($f = f(Q)$). Then*

$$\sum_{\pi(n)} f(Q(\pi)) = \sum_{\pi(n)} \sum_{m \geq 0} k_1 \cdots k_m \Delta^m f(0),$$

where $\Delta^m f(0)$ represents the m^{th} forward difference of f.

Example 5. Let f be the identity function ($f(Q) = Q$). Looking at the right-hand sum, we have that $k_1 \cdots k_m = 0$ when $m = 0$, $k_1 \Delta f(0) = k_1((0+1)-0) = k_1$ when $m = 1$, $\Delta^2 f(0) = \Delta(\Delta f(0)) = \Delta(1) = 0$ when $m = 2$, and $\Delta^m f(0) = 0$ when $m > 2$ (see [17, p. 201]). Hence, the right-hand sum equals k_1, which gives Stanley's theorem:

$$\sum_{\pi(n)} \sum_{i,\, k_i \geq 1} 1 = \sum_{\pi(n)} Q(\pi) = \sum_{\pi(n)} k_1.$$

A final theorem resulted in Elder's theorem.

Theorem 6 (Fine's Theorem 8)**.** *Let $w(k_i)$ be a function such that $w(0) = 0$. Then*

$$\sum_{\pi(n)} \sum_{i \geq 1} w(k_i) = \sum_{\pi(n)} \sum_{i \geq 1} (w(i) - w(i-1))k_i.$$

Example 6. For some j, let $w(k) = 0$ (for $k < j$) and $w(k) = 1$ (for $k \geq j$). Then,

$$\sum_{\pi(n)} \sum_{i,\, k_i \geq j} 1 = \sum_{\pi(n)} \sum_{i \geq j} (w(i) - w(i-1))\, k_i = \sum_{\pi(n)} k_j.$$

Here, Fine wrote, "the total number of frequencies $\geq i$ in all partitions of n is equal to the number of times that the part i occurs" [7].

4 New Extensions.

In this section, we reveal analogues to Stanley's theorem and Elder's theorem and other results in the sets of odd partitions and distinct partitions. By odd partitions, we mean those partitions composed solely of odd parts, and in considering distinct partitions we limit ourselves to those partitions in which no part is repeated in any partition. Following previous notation, we will let $\pi_o(n)$ and $\pi_d(n)$ represent the sets of odd partitions of n and distinct partitions of n respectively. Similarly, $p_o(n)$ and $p_d(n)$ will indicate the number of odd partitions and distinct partitions of n. Since Euler, it has been known that $p_o(n) = p_d(n)$ for all integers n [1, p. 5].

On a practical level, the number of times j appears in the odd partitions is zero when j is even and a part will appear at most once in any distinct partition. In contrast, an odd part may appear multiple times in the odd partitions, and values of any parity may appear in the distinct partitions. It therefore makes sense to consider first the number of parts which occur at least j times in the set of odd partitions and the number of times j appears in the set of distinct partitions. We obtain the following result.

Theorem 7. *The number of times j appears in the distinct partitions of $1, \ldots, n$ is equal to the number of parts that appear at least j times in a given partition, summed over all odd partitions of $n, n-1, \ldots, n-j+1$. In the summation notation, this is*

$$\sum_{m=0}^{j-1} \sum_{\pi_o(n-m)} \sum_{i,\, k_i \geq j} 1 = \sum_{m=1}^{n} \sum_{\pi_d(m)} k_j. \tag{1}$$

For example, seven $2's$ are used in the distinct partitions of $1, 2, \ldots, 7$ (2, $1+2$, $2+3$, $2+4$, $1+2+3$, $2+5$, and $1+2+4$), and there are seven instances of parts appearing at least twice in a partition summed over all of the odd partitions of both 6 and 7 ($1+1+1+1+1+1$, $1+1+1+3$, $3+3$, $1+1+1+1+1+1+1$, $1+1+1+1+3$, $1+1+5$, and $1+3+3$).

According to the notes in [19, A025147] and [19, A038348], the case of $j = 1$ for Theorem 7 is known. No further evidence suggests that this theorem has previously been established in greater generality, as presented here.

Proof of Theorem 7. Let $P_o(q)$ and $P_d(q)$ represent the generating functions for odd partitions and distinct partitions, respectively. That is,

$$P_o(q) = \prod_{i \geq 1} \frac{1}{1 - q^{2i-1}} = \prod_{i \geq 1} (1 + q^i) = P_d(q).$$

We establish that the two sides of Equation 1 are the coefficients for the generating function given by

$$\frac{q^j}{(1-q)(1+q^j)} P_o(q).$$

In order to do so, we manipulate the initial generating functions, $P_o(q)$ and $P_d(q)$, by inserting a second variable z appropriately into either $P_o(q)$ or $P_d(q)$, taking the derivative with respect to z, and setting z equal to 1.

Placing a z next to q^j in $P_d(q)$, we note that z counts each time j is used in a partition:

$$(1 + zq^j) \prod_{\substack{i \geq 1 \\ i \neq j}} (1 + q^i).$$

Following the second two steps, we obtain

$$q^j \prod_{\substack{i\geq 1\\ i\neq j}} (1+q^i) = \frac{q^j}{(1+q^j)} P_d(q) = \frac{q^j}{(1+q^j)} P_o(q).$$

This means that $\sum\limits_{\pi_d(n)} k_j$ is the coefficient of q^n in $\frac{q^j}{(1+q^j)} P_o(q)$. What we want, however, is the partial sum of these coefficients. Multiplying by $\frac{1}{1-q}$ gives

$$\sum_{n\geq 0} q^n \left(\sum_{m=0}^{n} \sum_{\pi_d(m)} k_j \right) = \frac{q^j}{(1-q)(1+q^j)} P_o(q).$$

To count the number of parts that appear at least j times, summed over all odd partitions, we first note that $P_o(q)$ may be expressed as a product of (geometric) series. We now place a z at the j^{th} terms and beyond for each geometric series:

$$\begin{aligned}\prod_{i \text{ odd}} \left(1+q^i+q^{2i}+\cdots+q^{i(j-1)}+z(q^{ij}+\cdots)\right) &= \prod_{i \text{ odd}} \left(\frac{1-q^{ij}}{1-q^i} + zq^{ij}\frac{1}{1-q^i}\right)\\ &= \prod_{i \text{ odd}} \frac{1-q^{ij}+zq^{ij}}{1-q^i}\\ &= P_o(q) \prod_{i \text{ odd}} (1-q^{ij}+zq^{ij}).\end{aligned}$$

Taking the derivative and setting $z=1$, we have

$$\sum_{n\geq 0} q^n \left(\sum_{\pi_o(n)} \sum_{i,\, k_i\geq j} 1 \right) = \frac{q^j}{1-q^{2j}} P_o(q).$$

Multiplying by $\frac{1}{1-q}$ gives partial sums for the coefficients:

$$\sum_{n\geq 0} q^n \left(\sum_{m=0}^{n} \sum_{\pi_o(n-m)} \sum_{i,\, k_i\geq j} 1 \right) = \frac{q^j}{(1-q)(1-q^{2j})} P_o(q).$$

These coefficients now contain too much information. We desire to subtract the coefficient of q^{n-j} from the coefficient of q^n, in order to get rid of the first $n-j+1$ terms. With generating functions, this means multiplying by $(1-q^j)$, which gives

$$\sum_{n\geq 0} q^n \left(\sum_{m=0}^{j-1} \sum_{\pi_o(n-m)} \sum_{i,\, k_i\geq j} 1 \right) = \frac{q^j}{(1-q)(1+q^j)} P_o(q).$$

$\square$

Theorem 7 may also be established with a bijection, or by demonstrating that both sides of Equation 1 are equal to the central sum

$$\sum_{i\geq 1}\sum_{m=0}^{j-1} p_o((n-m)-(2i-1)j).$$

Although it is useless to count the number of times even parts appear in the odd partitions, we may compare the number of times odd values of j appear throughout the odd partitions with the number of parts that appear at least j times throughout the odd partitions.

Theorem 8. *The number of times j (odd) appears throughout the odd partitions of n is equal to the number of parts that appear at least j times in a given partition, summed over all odd partitions of n and $n-j$, or*

$$\sum_{\substack{\pi_o(n)\\ j\ odd}} k_j = \sum_{\pi_o(n)}\sum_{i,\,k_i\geq j} 1 + \sum_{\pi_o(n-j)}\sum_{i,\,k_i\geq j} 1. \tag{2}$$

There are four uses of 3 in the odd partitions of 8 ($1+1+1+1+1+3$, $1+1+3+3$, $3+5$). Correspondingly, there are four instances of parts appearing at least three times in the odd partitions of 8 and 5 ($1+1+1+1+1+1+1+1$, $1+1+1+1+1+3$, $1+1+1+5$, and $1+1+1+1+1$).

Theorem 8 may be established by using generating functions, with a bijection, or with combinatorial arguments as presented below, where we establish that both sides of Equation 2 are equal to

$$\sum_{\ell\geq 1} p_o(n-\ell\cdot j).$$

Proof of Theorem 8. Suppose we have an odd partition of n with at least one j. If we remove one j, then we have an odd partition of $n-j$. Hence $p_o(n-j)$ counts the number of odd partitions of n with at least one j. Likewise, $p_o(n-i\cdot j)$ counts the number of odd partitions of n with at least i $j's$. In the sum

$$\sum_{\ell\geq 1} p_o(n-\ell\cdot j),$$

a partition with i $j's$ will be counted exactly i times (once by each $p_o(n-j), p_o(n-2j), \ldots, p_o(n-i\cdot j)$). Thus,

$$\sum_{\substack{\pi_o(n)\\ j\text{ odd}}} k_j = \sum_{\ell\geq 1} p_o(n-\ell\cdot j).$$

Let m be odd and suppose that m appears at least j times in an odd partition of n. Remove the first j $m's$ from that partition, and what remains is an odd partition of $n-m\cdot j$. Thus, the number of odd partitions of n that have at least j $m's$ is equal to $p_o(n-m\cdot j)$. Summing $p_o(n-m\cdot j)$ over all possible values of m will give us the total number of times odd parts appear at least j times in odd partitions of n; that is

$$\sum_{i\geq 1} p_o(n-(2i-1)j) = \sum_{\pi_o(n)} \sum_{i,\, k_i \geq j} 1.$$

Moreover,

$$\begin{aligned}
\sum_{\ell\geq 1} p_o(n-\ell\cdot j) &= \sum_{i\geq 1} p_o(n-(2i-1)j) + \sum_{i\geq 1} p_o(n-(2i)j) \\
&= \sum_{i\geq 1} p_o(n-(2i-1)j) + \sum_{i\geq 1} p_o((n-j)-(2i-1)j) \\
&= \sum_{\pi_o(n)} \sum_{i,\, k_i\geq j} 1 + \sum_{\pi_o(n-j)} \sum_{i,\, k_i\geq j} 1,
\end{aligned}$$

as desired. □

The next three results are of a different flavor.

Theorem 9. *The number of times j appears in the distinct partitions of n is equal to the number of distinct partitions of $n-j$ minus the number of times j appeared in the distinct partitions of $n-j$. In summation notation,*

$$\sum_{\pi_d(n)} k_j = p_d(n-j) - \sum_{\pi_d(n-j)} k_j. \tag{3}$$

For example, of the five distinct partitions of 7 ($1+6$, $2+5$, $3+4$, $1+2+4$, and 7) only two contain a 4. So, $p_d(7) - \sum_{\pi(7)} k_4 = 5-2 = 3$. There are indeed three occurrences of 4 in the distinct partitions of 11 ($1+4+6$, $2+4+5$, $4+7$).

Proof of Theorem 9. If j appears in a distinct partition, it appears just once and the remaining parts sum to $n-j$. Hence, each distinct partition of n that contains a j corresponds to a distinct partition of $n-j$ that did not already have a j. If we subtract $\sum_{\pi_d(n-j)} k_j$ from $p_d(n-j)$, then we are removing those distinct partitions of $n-j$ that already had a j. □

Similar to Theorem 9, we have a corollary of Theorem 7.

Corollary 1. *The number of times j appears in the distinct partitions of n is equal to the number of parts that appear at least j times in the odd partitions of n minus the number of parts that appear at least j times in the odd partitions of $n-j$. That is,*

$$\sum_{\pi_d(n)} k_j = \sum_{\pi_o(n)} \sum_{i,\, k_i\geq j} 1 - \sum_{\pi_o(n-j)} \sum_{i,\, k_i\geq j} 1. \tag{4}$$

We see that there is one part that appears at least twice in the odd partitions of 4 ($1+1+1+1$) and one part that appears at least twice in the odd partitions of 2 ($1+1$), which corresponds to the fact that there are no $2's$ in the distinct partitions of 4.

The next result is parallel to Theorem 9.

Theorem 10. *The number of parts appearing at least j times in an odd partition of n summed over all odd partitions of n equals the number of odd partitions of $n-j$ plus the number of parts appearing at least j times in an odd partition of $n-2j$, summed over all odd partitions of $n-2j$ or*

$$\sum_{\pi_o(n)} \sum_{i,\, k_i \geq j} 1 = p_o(n-j) + \sum_{\pi_o(n-2j)} \sum_{i,\, k_i \geq j} 1. \tag{5}$$

We find three times in which a part appears at least twice in the odd partitions of 6 ($1+1+1+1+1+1$, $1+1+1+3$, and $3+3$). This is equal to the number of odd partitions of 4 ($p_o(4) = 2$) plus the one occurrence of a part appearing at least twice in the odd partitions of 4 (namely, $1+1+1+1$).

Proof of Theorem 10. Rearranging Theorem 9, we have

$$p_o(n) = p_d(n) = \sum_{\pi_d(n+j)} k_j + \sum_{\pi_d(n)} k_j.$$

Applying Corollary 1, we obtain

$$p_o(n) = \sum_{\pi_o(n+j)} \sum_{i,\, k_i \geq j} 1 - \sum_{\pi_o(n-j)} \sum_{i,\, k_i \geq j} 1. \tag{6}$$

Rearranging Equation 6 gives the desired result. □

The value of Theorems 9 and 10 is that they allow for quick calculation of their respective integer sequences, given the well-studied sequence for $p_o(n)$.

5 Concluding Remarks and Acknowledgements.

In this article, we have explored the history of the identities known as Stanley's theorem and Elder's theorem. We then introduced similar results in the sets of odd partitions and distinct partitions.

At present, one may establish both Theorems 7 and 8 with generating functions, bijections, or through a combinatorial argument relating the quantities to a central sum. (We are currently preparing a paper containing the proofs not presented here.) However, in light of the results of Nathan Fine, it would be interesting to explore whether a similar set of prolific theorems exists for the sets of odd partitions and distinct partitions.

Acknowledgments. I would like to gratefully acknowledge the support and guidance I received from my adviser, George E. Andrews, while writing this paper, and for the support of my husband, Ian Gilbert.

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