Analytical Calculation of Risk Measures for Variable Annuity Guaranteed Benefits

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http://dx.doi.org/10.1016/j.insmatheco.2012.09.007

Abstract

With the increasing complexity of investment options in life insurance, more and more life insurers have adopted stochastic modeling methods for the assessment and management of insurance and financial risks. The most prevalent approach in market practice, Monte Carlo simulation, has been observed to be time consuming and sometimes extremely costly. In this paper we propose alternative analytical methods for the calculation of risk measures for variable annuity guaranteed benefits on a stand-alone basis. The techniques for analytical calculations are based on the study of geometric Brownian motion and its integral. Another novelty of the paper is to propose a quantitative model which assesses both market risk on the liability side and revenue risk on the asset side in the same framework from the viewpoint of risk management. As we demonstrate by numerous examples on quantile risk measure and conditional tail expectation, the methods and numerical algorithms developed in this paper appear to be both accurate and computationally efficient.

Key Words. Variable annuity guaranteed benefit, Asian option, risk measures, value at risk, conditional tail expectation, geometric Brownian motion, integral of geometric Brownian motion, Hartman-Watson density, modified Bessel functions.

1 Introduction

In recent developments within both insurance and banking industries, risk metrics such as value at risk and conditional tail expectation have been employed in nearly all major capital requirements and risk measurement models. Hence the accuracy and efficiency in the implementation of these risk metrics become increasingly important for stakeholders in the industries.

The current market practice of implementing capital requirements includes factor-based formulas, stress and scenario testing, and stochastic approach, etc. According to Farr et al. [11], the most prevalent method is the stochastic approach, of which various forms have been adopted by the

majority (57%) of insurance companies that responded to the Tillinghast Enterprise Risk Management survey and that calculate economic capital. As shown in their report, the stochastic approach used in the insurance industry generally refers to the implementation of Monte Carlo simulations. Although the approach is often applied to different financial measures for various purposes of risk assessment, we can nevertheless summarize the main procedure as follows.

- A set of future scenarios on a certain time frame is generated to reflect a company or regulator's view on the variability of economic outcomes.
- Under each scenario the economic outcomes are assessed according to certain accounting conventions to produce relevant economic measures. Combining all scenarios, the economic measures are then ranked to form an empirical distribution.
- A capital requirement is then determined by an estimation of a chosen risk metric, such as value at risk or conditional tail expectation, applied to the distribution of economic measures.

Despite its engineering flexibility, the simulation approach is often resource demanding and time consuming. The results may sometimes be unreliable due to the sampling variability even with the aid of variance reduction techniques. Besides the potential problems with inaccuracy and inefficiency, the simulation approach may also be undesirable from the cost and benefit perspective. According to Farr et al. [11], "76% of the respondents in the survey whose companies have more than $10 billion of annual revenue use a form of stochastic approach. In contrast, only 27% of the respondents whose companies have less than $1 billion use a form of stochastic approach. ... These statistics suggest that a fully robust economic capital model [using stochastic approach] may present prohibitive cost and implementation challenges to small and medium sized insurers."

There are many papers in recent actuarial literature regarding the pricing and valuations of variable annuity guarantees such as Bacinello et al. [3], Ballotta and Haberman [4], Coleman et al. [6], Van Haastrecht et al. [14], Wang [24], etc. under various equity price and interest rate models. However, there is scarce literature on risk assessment and management of variable annuity guarantees. Simulation appears to be the only available stochastic modeling tool in practice for risk capital calculation and sensitivity test of investment-linked products, due to the lack of development on alternative techniques and methodologies. No previous work has been done on the analytical calculation of risk measures, despite the widely acknowledged issues with Monte Carlo simulation. As the research on analytical methods for risk management of variable annuities is in the early stage, this work is by no means an attempt to address the complex issue of model efficiency all at once. Much more work needs to be done for more general and practical applications. Like many other analytical approaches, the application of methods in this work hinges on the tractability of underlying models. Hence, we shall investigate an example of calculating risk metrics for two types of variable annuity guaranteed benefits under the geometric Brownian motion asset pricing model, which is often the base model for simulation approach as well.

In Section 2, we first introduce the basics of variable annuity guaranteed benefits and formulate the gross and net liabilities of these product designs in a probabilistic setting from an insurer's

perspective. Two most widely used risk measures in practice are discussed in Section 3. Due to the technical advances on exponential functionals of geometric Brownian motions in the past two decades, it becomes possible for us to derive analytic solutions to these measures for two most basic types of guaranteed benefits. More details on the development of these techniques in both actuarial and finance literature can be found in Section 3. For reasons to be seen, we provide two alternative sets of solutions to facilitate practical implementation. The pros and cons of both approaches will become self-evident in Section 4, in which we enter discussions with a few numerical examples.

2 Variable Annuity Guaranteed Benefits

Variable annuity is a type of accumulation annuity that offers participation in the profit sharing of investment funds. Purchase payments net of fees and charges are deposited in various subaccounts of policyholders' choosing. Each subaccount is invested in a particular fund with a distinct investment objective. To protect investors from downside risk of fund participation, insurers also sell guarantee riders such as guaranteed minimum maturity benefit (GMMB), guaranteed minimum death benefit (GMDB), guaranteed minimum accumulation benefit (GMAB), etc. Under a guarantee rider, an insurer receives the proceeds from fees and charges, and is responsible for covering financial losses to policyholders in adverse economic scenarios. The risk management of these benefits often hinges on the insurer's ability to accurately quantify and assess its liabilities. Hence effective and efficient computational methods and models for risk management are crucial to the development and maintenance of variable annuity products. Readers are referred to Hardy [16] for detailed accounts of variable annuity guarantees.

A variety of equity return models have been used in the actuarial and financial literature for variable annuity guarantee products, the most popular being the lognormal model or also known as the geometric Brownian motion model. There are many benefits from adopting this model in the current context. (1) One can use the celebrated Black-Scholes formula to explicitly price variable annuity guaranteed benefits as well as to use dynamic strategies to hedge these option-like benefits. (2) It is also the basis of further extension to the regime-switching lognormal model with multiple regimes which are also well accepted in the industry for economic scenario generation. (See Hardy [15].) (3) The lognormal model is one of the recommended models in the American Academy of Actuaries report and has been tested and used to generate the pre-packaged economic scenarios. (See AAA [1], Appendix 2.) (4) The lognormal model in which we compute risk measures for risk management purpose will be the same as the typical model in which the volatility smile/surface is estimated for pricing. Hence practitioners can assess the impact of equity volatility for both pricing and risk assessment in a consistent manner and hence reduces model risks. For these reasons, We shall assume in this paper that the dynamics of underlying asset price is driven by a geometric Brownian motion. However, readers should be reminded that, as every model has its own limitation, the lognormal distribution is known to have lighter tails than the probability distributions of equity prices from empirical data. A comparative study of the lognormal and

alternative time series models can be seen in the aforementioned references. An analysis of risk measures under other models should be conducted in future research.

We now illustrate how the liability distribution is determined by the actuarial approach, which assumes no dynamic hedging on investment guarantees. The use of stochastic simulation for modeling the guarantee liability was extensively studied in Bacinello et al. [3], Bauer et al. [5] and Chapter 6 of Hardy [16]. To facilitate the comparison with the stochastic models in Hardy [16], we follow their notation for cash flow variables with slight modifications.

- G, the guarantee level, typically ranging from 75% to 100% of the purchase payments under the GMMB rider. The guarantee may also accrue compound interest up to an advanced age. This is referred to as a roll-up option often associated with the GMDB rider.

- F_t, the market value of the separate account at $t \geq 0$. F_0 is considered to be the initial purchase payment at the beginning of the rider. For simplicity, we assume that no additional purchase payments or withdrawal is allowed.

- S_t, the market value of the underlying equity fund at t. For simplicity, we assume the account is invested entirely in one fund. The asset price process in this fund is defined, on a probability space denoted by $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, by

$$S_t = S_0 e^{\mu t + \sigma B_t}, \qquad t > 0, \tag{2.1}$$

  where B is a standard Brownian motion.

- m, the annualized rate at which fees and charges are deducted from the separate account. Except for purchase payments based withdrawal charges, all contract fees and expenses are typically calculated and accrued on daily basis. Since no withdrawal is considered on policy anniversaries, it is reasonable for us to model all fees and charges as being taken continuously. The portion available for funding the guarantee cost is called margin offset or rider charge and is usually split by benefit. In this paper, we denote the annualized rate of charges allocated to the GMMB by m_e and that of the charges allocated to the GMDB by m_d. Note that in general $m > m_e + m_d$ to allow for other expenses.

- T, the target value date (or called maturity date), typically a rider anniversary on which the insurer is liable for guarantee payments.

- L_0, the net present value of future liabilities, discounted at a constant risk-free force of interest of r per year. The rate r generally is selected to reflect the overall yield on bonds backing the general account of the variable annuity writer.

At the end of each trading day, the account value is adjusted according to the performance of funds in which it invests and deducted by mortality and expenses (M&E) fees and rider charges. Hence, without the effect of investment guarantees the account value at time t is given by

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad 0 \leq t \leq T, \tag{2.2}$$

and the margin offset income at time t is given by

$$M_t = m_x F_t, \qquad 0 \le t \le T,$$

where m_x is replaced with m_e for the GMMB or m_d for the GMDB.

In this paper, we investigate two basic designs of investment guarantee - guaranteed minimum maturity benefit and guaranteed minimum death benefit in their plain vanilla forms. Neither annual ratchet features nor policyholder behavior is considered in this model.

2.1 GMMB

A variable annuity writer's incomes and liabilities are generally dependent on the survivorship of policyholders. We set out the actuarial notation for modeling mortality.

- τ_x, the future-lifetime of a policyholder at age x;
- ${}_tp_x$, the probability that a policyholder at age x will survive t periods;
- μ_{x+t}, the force of mortality for a policyholder at age $x+t$;

The GMMB rider provides a minimum guarantee on the balance of policyholder's separate account at the time of maturity. From an insurer's point of view, the present value of the added cost of a GMMB rider, which will be called **gross liability** in this paper, is given by the discounted payoff

$$e^{-rT}(G - F_T)_+ I(\tau_x > T),$$

with $(x)_+ = \max(x, 0)$, and $I(A) = 1$ if the statement A is true or $I(A) = 0$ otherwise.

It is usually assumed that policy exits due to deaths are independent of fund performance. In Hardy [16, Chapter 6], the exits are treated deterministically so that the only source of randomness is the equity price process. The rationale behind this may be explained as follows. According to the strong law of large numbers, for a pool of N contracts of the same size with policyholders of age x at time 0 where N is large enough, there are approximately ${}_tp_x\, N$ contracts still in force at time t. We might as well average out the remaining liabilities at time t over all original contracts and treat them as if every contract reduces to the portion ${}_tp_x$ at time t of its original size. Hence, it is argued that the cash flow at time t for $0 \le t \le T$ is generated as follows

$$C_t = \begin{cases} -{}_tp_x M_t, & 0 \le t < T; \\ {}_Tp_x (G - F_T)_+, & t = T. \end{cases}$$

Therefore the present value of net loss is given by

$$L_0^* = \int_0^T e^{-rs} C_s \, \mathrm{d}s + e^{-rT} C_T. \tag{2.3}$$

Even though ${}_tp_x$ has its probabilistic interpretation, it declines deterministically over time.

However, it is questionable whether the strong law of large numbers is applicable when modeling variable annuity contracts. In most contracts, there is no standardized purchase payment and the size of payment varies largely by contract ranging from $\$10,000$ to $\$1$ million. When risk measures are calculated with the cash flow (2.3), the assumption is implicitly made under the strong law of large numbers that all contracts are of equal size, say for instance, $\$10,000$. A single contract with a purchase payment of $\$1$ million then should be treated as 100 units. Hence the length of all 100 contracts depends on the future lifetime of a single policyholder, which clearly violates the independence assumption of the strong law of large number.

Since we do not assume all contract units to be mutually independent in this paper, the calculations of risk measures are done on an individual contract basis. However, as one shall see later, it does not mean that we have to compute risk measures repeatedly for each contract in practice. Risk measures are in fact proportional to the size of contract (see Remark 3.2) and hence one can quickly determine the risk measures for each contract by multiplying risk measures per unit by appropriate units of contract size. This linearity has nothing to do with whether or not the contracts are independent.

The randomness of insurer's liability to each contract arises from two independent sources – the equity price process and the future-lifetime of contract. Therefore the present value of insurer's liability net of margin offsets from the GMMB, which will be called **net liability** in this paper, is formulated as

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T\wedge\tau_x} e^{-rs} M_s \,\mathrm{d}s. \tag{2.4}$$

This concept essentially corresponds to the term **present value of accumulated deficiencies** in the AAA report for each individual stand-alone contract. We shall leave the technical details and their distinctions in another paper.

The fundamental difference between (2.3) and (2.4) is that the mortality is modeled by a deterministic function of time in the former and by a random variable in the latter. One should note that the net liability formulated in Hardy [16] denoted by L_0^* here is in fact the conditional expectation of net liability in this paper with respect to the natural filtration of asset price process.

$$\begin{aligned}
\mathbb{E}[L_0|\mathcal{F}_T] &= \mathbb{E}\left[e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T\wedge\tau_x} e^{-rs} M_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_T\right] \\
&= {}_Tp_x e^{-rT}(G - F_T)_+ - \int_0^T {}_tp_x \mu_{x+t} \int_0^t e^{-rs} M_s \,\mathrm{d}s \,\mathrm{d}t - \int_T^\infty {}_tp_x \mu_{x+t} \int_0^T e^{-rs} M_s \,\mathrm{d}s \,\mathrm{d}t \\
&= {}_Tp_x e^{-rT}(G - F_T)_+ - \int_0^T {}_sp_x e^{-rs} M_s \,\mathrm{d}s = L_0^*.
\end{aligned}$$

Under the circumstances where the strong law of large numbers does apply to contract length, i.e. all contract lifetimes are independent and of similar size, the average across all policies of aggregate liabilities converges to the expectation of individual liability and then our formulation reduces to that of Hardy [16].

2.2 GMDB

The death benefit is often determined by the greater of total purchases payments with roll-up and the account value at death. We assume that the death benefit is payable immediately at the time of death and that the guarantee increases at a roll-up rate δ available up to the maturity. Note that the roll-up rate is usually offered at a modest rate less than the risk-free rate which reflects insurer's own rate of return on bonds investment, i.e. $0 \leq \delta \leq r$. From an insurer's point of view, the present value of the added cost of a GMDB rider, or called **gross liability**, is given by

$$e^{-r\tau_x}(e^{\delta\tau_x}G - F_{\tau_x})_+ I(\tau_x \leq T).$$

Similarly, the present value of the GMDB gross liability less margin offsets, or called **net liability**, is determined by

$$L_0 = e^{-r\tau_x}(e^{\delta\tau_x}G - F_{\tau_x})_+ I(\tau_x \leq T) - \int_0^{T\wedge\tau_x} e^{-rs} M_s \, \mathrm{d}s. \tag{2.5}$$

For practical applications, we consider that death benefits are payable on discrete time basis. Introduce the curtate future lifetime $\kappa_x^{(n)}$ in years rounded to the upper one n-th of a year.

$$\kappa_x^{(n)} := \frac{1}{n}\lceil n\tau_x \rceil,$$

where $\lceil x \rceil$ is the integer ceiling of x. Due to the time lag often resulted from claims and investigation, we assume the investment account is accumulated and rider charges are deducted up until the end of the one n-th year of the policyholder's death and the death benefit is payable at the end of the one n-th year. Then the net liability is given by

$$L_0^{(n)} = e^{-r\kappa_x^{(n)}}(e^{\delta\kappa_x^{(n)}}G - F_{\kappa_x^{(n)}})_+ I(\kappa_x^{(n)} \leq T) - \int_0^{T\wedge\kappa_x^{(n)}} e^{-rs} M_s \, \mathrm{d}s. \tag{2.6}$$

When n is taken to be large, $L_0^{(n)}$ can be used as a close estimate of L_0.

In this paper we only consider the risk measures of the GMMB and the GMDB separately. However, the work can be easily extended with the combination of both riders.

3 Analytical Solutions to Risk Measures

The two most common risk measures used by practitioners are quantile risk measure, also known as value at risk, and conditional tail expectation risk measure. The quantile risk measure for L_0 is defined for α $(0 \leq \alpha \leq 1)$ as

$$V_\alpha := \inf\{y : \mathbb{P}[L_0 \leq y] \geq \alpha\}.$$

Alternatively, since L_0 is modeled by a continuous random variable in this paper, we shall compute the quantile risk measure by V_α such that

$$\mathbb{P}[L_0 > V_\alpha] = 1 - \alpha. \tag{3.1}$$

The quantile risk measure V_α is interpreted as the minimum capital required to ensure that there is sufficient fund to cover future liability with the probability of at least α. The conditional tail expectation risk measure for L_0 is also defined for $\alpha\,(0 \le \alpha \le 1)$ as

$$\mathrm{CTE}_\alpha := \mathbb{E}[L_0 | L_0 > V_\alpha].$$

It is the capital required to cover the average amount of liabilities when they exceed the quantile measure with the probability of at most $1-\alpha$. There is extensive literature on the pros and cons for each of the two measures. Since the calculation of conditional tail expectation depends on that of quantile risk measure, we develop computational algorithms of both risk measures for GMMB and GMDB.

Readers should be reminded that these risk metrics are typically used as measures of extreme events. The two formulations of liabilities L_0 and L_0^* generally result in different values of the risk measures, since the randomness of the policyholder's future lifetime also contributes to the occurrence of extreme events.

We also remark that the net liabilities (gross liabilities) are expected to be negative (zero) in most cases as such products are designed to be profitable. However, from the perspective of risk management, risk capitals are determined to ensure that sufficient funds are available to cover unexpected losses in adverse scenarios and hence we are only concerned with the levels of risk measures corresponding to positive liabilities in worst cases, which form the basis of calculation for risk capitals. Thus in most parts of the paper we consider α to be greater than the probability of non-positive liabilities, which shall be denoted by ξ_e for GMMB and ξ_d for GMDB.

3.1 GMMB

The formulation of gross liabilities provides a quantitative model for assessing market risk from the liability side of investment guarantees whereas the formulation of net liabilities offers the quantification of combined effects of market risk on both liability and asset sides. Hence we treat them separately in two subsections.

3.1.1 Gross Liability

We first determine the probability that no guarantee payment will be made at maturity.

$$\xi_e = 1 - \mathbb{P}[G \ge F_T, \tau_x > T] = 1 - {}_Tp_x \mathbb{P}\left[e^{-mT}\frac{S_T}{S_0} \le \frac{G}{F_0}\right] = 1 - {}_Tp_x \Phi(\ln(G/F_0); (\mu - m)T, \sigma^2 T).$$

Throughout the paper we denote by $\Phi(x; a, b)$ the distribution function of a normal random variable with mean a and variance b. Most computing software packages offer built-in functions for normal distribution of this form.

Risk measures are generally much easier to compute when no cash flows from margin offsets are included (i.e. $M_s = 0$ for $0 \le s \le T$), as the insurer's liability results only from the benefit at maturity. Under this assumption, risk measures serve as direct tools to assess the gross liabilities of

guaranteed benefits. Although most literature on the subject has focused on arbitrage-free prices of guaranteed benefits, it is debatable whether no-arbitrage theory is really applicable as guaranteed benefits are not tradable derivatives and there is no sizable market for hedging mortality risk. We want to point out that risk measures of future gross liabilities can be used as alternative pricing principles for rider charges. A comprehensive account of premium principles in traditional life insurance can be found in Rolski et al. [23] and Goovaerts et al. [13].

Proposition 3.1. *The quantile risk measure V_α with $\alpha > \xi_e$ for the gross liability of GMMB is given by*

$$V_\alpha = e^{-rT}G - F_0 \exp\{(\mu - r - m)T + \sigma\sqrt{T}z_\beta\}, \tag{3.2}$$

where z_β is the $100\beta\%$ percentile of a standard normal distribution and $\beta = (1-\alpha)/\,{}_Tp_x$.

Proposition 3.2. *The conditional tail expectation CTE_α with $\alpha > \xi_e$ for the gross liability of GMMB is given by*

$$\mathrm{CTE}_\alpha = e^{-rT}G - {}_Tp_x \frac{F_0}{1-\alpha} e^{(\mu-r-m)T+\sigma^2T/2}\Phi(z_\beta; \sigma\sqrt{T}, 1). \tag{3.3}$$

Note that (3.2) and (3.3) are equivalent to (9.7) and (9.17) in Hardy [16] for which no contract exits due to mortality are considered, i.e. ${}_Tp_x = 1$.

3.1.2 Net Liability

While most research papers in the current literature focus on the liability side for pricing and valuation, we think it is also important to assess the revenue risk on the asset side since most fees and charges are based on equity-linking account values. In the extreme cases where equity prices are low for a prolonged period, not only is the guarantee liability high at maturity, the revenues generated from rider charges are also persistently low, which exacerbates the severity of losses to the insurer.

The distribution of liability net of margin offsets provides a tool to analyze the impact of financial risk on both the liability side and the asset side. Although the payoffs of variable annuity guarantees bear no resemblance to an Asian option, it turns out that the net liability can be viewed as a similar path-dependent "derivative". As we can see from the formulation (2.4) and (2.5), the distribution of net liability is completely determined by future lifetime random variable, the geometric Brownian motion representing fund values at the time of payment, and its integral representing the accumulation of rider charges.

The work by Yor [21] on the pricing of Asian option has led to the study of the joint distribution of the geometric Brownian motion and its integral, which forms the basis of our analysis on variable annuity guaranteed benefits. A parallel study on the integral of geometric Brownian motion is done by the European school of actuarial scientists led by Goovaerts and coauthors with applications in modeling annuities with random investment returns. Two distinct approaches are used to derive a

double Laplace transform of the integral of geometric Brownian motion in De Schepper, Goovaerts, Delbaen [9] and De Schepper et al. [10]. Interestingly, Yor [22] provides an analysis connecting the aforementioned and other related results developed in both actuarial and finance literature.

Interested readers are referred to Carr and Schröder [8] for comprehensive accounts of pricing methods for Asian options. Many other applications of these techniques can be seen in Geman and Yor [12]. With analogy to Asian options, we provide two methods of computing risk measures of guaranteed benefits.

As one shall see in the proofs, a key quantity to the calculation of quantile risk measure is the following probability distribution.

$$P(t,w) := \mathbb{P}\left[e^{-rt}\frac{F_t}{F_0} + \int_0^t e^{-rs}\frac{M_s}{F_0}\,\mathrm{d}s < w\right]. \tag{3.4}$$

Note that for $y \geq 0$,

$$\begin{aligned}\mathbb{P}[L_0 > y|\tau_x > T] &= \mathbb{P}\left[e^{-rT}(G - F_T) - \int_0^T e^{-rs}M_s\,\mathrm{d}s > y\right] \\ &= \mathbb{P}\left[e^{-rT}\frac{F_T}{F_0} + \int_0^T e^{-rs}\frac{M_s}{F_0}\,\mathrm{d}s < \frac{e^{-rT}G - y}{F_0}\right] = P\left(T, \frac{e^{-rT} - y}{F_0}\right).\end{aligned}$$

Thus, we can determine the probability of non-positive liabilities by

$$\xi_e = 1 - {}_Tp_x\mathbb{P}[L_0 > 0|\tau_x > T] = 1 - {}_Tp_xP\left(T, \frac{e^{-rT}}{F_0}\right).$$

Proposition 3.3. *The quantile risk measure V_α with $\alpha > \xi_e$ for the net liability of GMMB is determined implicitly by $1 - \alpha = {}_Tp_xP(T, B(T))$ with $B(T) = (e^{-rT}G - V_\alpha)/F_0$ and*

$$\begin{aligned}P(t,w) &= \sqrt{\frac{2}{\pi^3\sigma^2 t}}\exp\left(\frac{2\pi^2}{\sigma^2 t} - \frac{\nu^2\sigma^2 t}{8}\right) \\ &\times \int_0^\infty \exp\left(-\frac{2w^2}{\sigma^2 t}\right)\sinh y \sin\left(\frac{4\pi y}{\sigma^2 t}\right)\int_0^{\sqrt{w}} \frac{2\rho^\nu}{1+\rho^2+2\rho\cosh y}\exp\left(-\frac{A(1+\rho^2+2\rho\cosh y)}{2(w-\rho^2)}\right)\mathrm{d}\rho\,\mathrm{d}y,\end{aligned} \tag{3.5}$$

where $\nu = 2(\mu - m - r)/\sigma^2$, $A = 4m_e/\sigma^2$.

Remark 3.1. Note that $P(T, B(T))$ increases with V_α, as $P(t,w)$ increases with w due to (3.4) and $B(T)$ decreases with V_α. One can use the Newton-Raphson method to determine V_α, for which we need

$$\begin{aligned}\frac{\partial P(T,B(T))}{\partial V_\alpha} &= -\sqrt{\frac{2}{\pi^3\sigma^2 T}}\exp\left(\frac{2\pi^2}{\sigma^2 T} - \frac{\nu^2\sigma^2 T}{8}\right) \\ &\times \int_0^\infty \exp\left(-\frac{2y^2}{\sigma^2 T}\right)\sinh y \sin\left(\frac{4\pi y}{\sigma^2 T}\right)\int_0^{\sqrt{B(T)}} \frac{2\rho^\nu}{(B(T)-\rho^2)^2 F_0}\exp\left(-\frac{A(1+\rho^2+2\rho\cosh y)}{2(B(T)-\rho^2)}\right)\mathrm{d}\rho\,\mathrm{d}y.\end{aligned}$$

The computation of conditional tail expectation hinges on the value of the following expectation.

$$Z(t,w) := \mathbb{P}\left[\left\{e^{-rt}\frac{F_t}{F_0} + \int_0^t e^{-rs}\frac{M_s}{F_0}\,\mathrm{d}s\right\} I_{\{e^{-rt}\frac{F_t}{F_0}+\int_0^t e^{-rs}\frac{M_s}{F_0}\,\mathrm{d}s<w\}}\right].$$

Proposition 3.4. *The conditional tail expectation* CTE_α *with* $\alpha > \xi_e$ *for the net liability of GMMB is given by*

$$\mathrm{CTE}_\alpha = e^{-rT}G - {}_Tp_x\frac{F_0}{1-\alpha}Z(T,B(T)), \tag{3.6}$$

where Z *is given by*

$$\begin{aligned} Z(t,w) &= \sqrt{\frac{2}{\pi^3\sigma^2 t}}\exp\left(\frac{2\pi^2}{\sigma^2 t} - \frac{\nu^2\sigma^2 t}{8}\right)\int_0^\infty \exp\left(-\frac{2y^2}{\sigma^2 t}\right)\sinh y \sin\left(\frac{4\pi y}{\sigma^2 t}\right) \\ &\times \int_0^{\sqrt{w}}\left[\frac{2\rho^{\nu+2}}{1+\rho^2+2\rho\cosh y}\exp\left(-\frac{A(1+\rho^2+2\rho\cosh y)}{2(w-\rho^2)}\right) + A\rho^\nu E_1\left(\frac{A(1+\rho^2+2\rho\cosh y)}{2(w-\rho^2)}\right)\right]\mathrm{d}\rho\,\mathrm{d}y \end{aligned} \tag{3.7}$$

and $E_1(z)$ *is the exponential integral defined by* $E_1(z) = \int_z^\infty e^{-t}/t\,\mathrm{d}t$.

A few words of caution seem to be necessary regarding the computation of integrals such as (3.5) and (3.7). As pointed out in Carr and Schröder [8] to similar integrals in the context of Asian options, we observe that the integrals are multiplied by a constant

$$\sqrt{\frac{2}{\pi^3\sigma^2 t}}\exp\left\{\frac{2\pi^2}{\sigma^2 t}\right\}, \tag{3.8}$$

which could be very large when σ and t are small and consequently causes overflow problems for computing in extreme cases. In those cases, the integrals have to be computed with very high accuracy which could be difficult with limited computing resources. However, it does not seem to cause much concern for the GMMB since variable annuity contracts are generally long-term, i.e. $T \gg 0$. Under the circumstances when very small t and σ are required, we do have an alternative approach, which is to numerically invert the Laplace transforms of P and Z. This approach is analogous to the Laplace transform approach for pricing Asian option as well.

Proposition 3.5. *The quantile risk measure* V_α *for the net liability of GMMB is implicitly determined by* $1-\alpha = {}_Tp_xP(T,B(T))$ *with* $\alpha > \xi_e, B(T) = (e^{-rT}G - V_\alpha)/F_0$ *and* $P(t,w)$ *can be obtained by numerically inverting its Laplace transform* $\tilde{P}(s,w) := \int_0^\infty e^{-st}P(t,w)\,\mathrm{d}t$ *given by*

$$\tilde{P}(s,w) = \frac{4}{\sigma^2}\int_0^{\sqrt{w}}\int_0^{(w-\rho^2)/A}\frac{\rho^{\nu-1}}{u}\exp\left\{-\frac{1}{2u}(1+\rho^2)\right\}I_{2\eta}\left(\frac{\rho}{u}\right)\mathrm{d}u\,\mathrm{d}\rho, \tag{3.9}$$

where $2\eta = \sqrt{8s/\sigma^2+\nu^2}, A = 4m_e/\sigma^2$, *and* $I_{2\eta}(\cdot)$ *is the modified Bessel function of the first kind.*

Similarly, we can provide an alternative approach to the computation of conditional tail expectation risk measure.

Proposition 3.6. *The conditional tail expectation* CTE_α *with* $\alpha > \xi_e$ *for the net liability of GMMB is determined by* (3.6) *in which the function* $Z(t,w)$ *can be obtained by inverting the Laplace transform* $\tilde{Z}(s,w) := \int_0^\infty e^{-st} Z(t,w)\,\mathrm{d}t$ *with*

$$\tilde{Z}(s,w) = \frac{4}{\sigma^2}\int_0^{\sqrt{w}}\int_0^{(w-\rho^2)/A}\left(\frac{\rho^{\nu+1}}{u} + A\rho^{\nu-1}\right)\exp\left\{-\frac{1}{2u}(1+u^2z^2)\right\} I_{2\eta}\left(\frac{\rho}{u}\right)\,\mathrm{d}u\,\mathrm{d}\rho \quad (3.10)$$

where $2\eta = \sqrt{8s/\sigma^2+\nu^2}$, $A = 4m_e/\sigma^2$, *and* $I_{2\eta}(\cdot)$ *is the modified Bessel function of the first kind.*

Remark 3.2. For variable annuity products, the guaranteed benefit G is always quoted as a percentage of purchase payment F_0. It is clear from the constant $B(T)$ in Propositions 3.3 and 3.5 that for each fixed α, the quantile risk measure V_α must be a linear function of F_0 when all else are held constant. Similarly, the conditional tail expectation CTE_α is also a linear function of F_0 according to Propositions 3.4 and 3.6. Therefore, this remark may serve as a theoretical justification of the usual practice of calculating risk measures in proportion to purchase payments.

3.2 GMDB

With the techniques shown in the previous subsection, we can also obtain the risk measures for both the gross and net liabilities of the GMDB.

3.2.1 Gross Liability

Note that the probability that there is no guarantee payment over the term of the GMDB, denoted by ξ_d, is given by

$$\xi_d = 1 - \int_0^T {}_tp_x\mu_{x+t}\mathbb{P}(e^{\delta t}G > F_t)\,\mathrm{d}t = 1 - \int_0^T {}_tp_x\mu_{x+t}\Phi(\ln(G/F_0);(\mu-m-\delta)t,\sigma^2 t)\,\mathrm{d}t.$$

We are interested in risk measures under which the gross liability is positive.

Proposition 3.7. *The quantile risk measure* V_α *with* $\alpha > \xi_d$ *for the gross liability of GMDB is determined implicitly by*

$$1-\alpha = \int_0^T {}_tp_x\mu_{x+t}\Phi\left(\ln\frac{e^{(\delta-r)t}G - V_\alpha}{F_0};(\mu-r-m)t,\sigma^2 t\right)\,\mathrm{d}t.$$

Proposition 3.8. *The conditional tail expectation* CTE_α *with* $\alpha > \xi_d$ *for the gross liability of GMDB is given by*

$$\begin{aligned}\mathrm{CTE}_\alpha = {} & \frac{G}{1-\alpha}\int_0^T e^{(\delta-r)t}\,{}_tp_x\mu_{x+t}\Phi\left(\ln\frac{e^{(\delta-r)t}G - V_\alpha}{F_0};(\mu-r-m)t,\sigma^2 t\right)\,\mathrm{d}t \\ & -\frac{F_0}{1-\alpha}\int_0^T {}_tp_x\mu_{x+t}\,e^{(\mu-r-m)t+\sigma^2 t/2}\Phi\left(\ln\frac{e^{(\delta-r)t}G - V_\alpha}{F_0};(\mu-r-m+\sigma^2)t,\sigma^2 t\right)\,\mathrm{d}t\end{aligned}$$

3.2.2 Net Liability

Similar to the actuarial practice of payment periods in life insurance, we can also develop risk measures for the net liability based on death benefit payable at the end of one nth of a year, denoted by $L_0^{(n)}$ in (2.6). The net liability under consideration in this subsection refers to $L_0^{(n)}$. The formulas can be easily extended to the net liability based on death benefit payable continuously, denoted by L_0 in (2.5). However, in general, the implementation of risk measures for $L_0^{(n)}$ requires less computational efforts and can serve as good estimates for risk measures of L_0.

The expressions for $P(t,w)$ in the case of the GMDB are the same as those for $P(t,w)$ in the case of the GMMB except that m_e is always replaced with m_d. Note that the probability of non-positive liabilities is determined by

$$\begin{aligned}
\xi_d &= \mathbb{P}[L_0^{(n)} \le 0] = 1 - \sum_{k=1}^{\lceil nT \rceil} \mathbb{P}\left[\kappa_x^{(n)} = \frac{k}{n}\right] \mathbb{P}\left[L_0^{(n)} > 0 \middle| \kappa_x^{(n)} = \frac{k}{n}\right] \\
&= 1 - \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} \mathbb{P}\left[e^{-rk/n}(e^{\delta k/n}G - F_{k/n}) - \int_0^{k/n} e^{-rs} M_s \, \mathrm{d}s > 0\right] \\
&= 1 - \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} \mathbb{P}\left[e^{-rk/n}\frac{F_{k/n}}{F_0} + \int_0^{k/n} e^{-rs}\frac{M_s}{F_0} \, \mathrm{d}s < \frac{e^{(\delta-r)k/n}G}{F_0}\right] \\
&= 1 - \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} P\left(\frac{k}{n}, \frac{e^{(\delta-r)k/n}G}{F_0}\right).
\end{aligned}$$

Proposition 3.9. *The quantile risk measure V_α with $\alpha > \xi_d$ for the net liability of GMDB is determined implicitly by*

$$1 - \alpha = \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} P\left(k/n, B(k/n)\right),$$

where $P(t,w)$ is given by (3.5) *with $A = 4m_d/\sigma^2$ and $B(k/n) = (e^{(\delta-r)k/n}G - V_\alpha)/F_0$.*

Proposition 3.10. *The conditional tail expectation* CTE_α *with $\alpha > \xi_d$ for the net liability of GMDB is given by*

$$\mathrm{CTE}_\alpha = \frac{1}{1-\alpha} \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} \left[e^{(\delta-r)k/n}GP(k/n, B(k/n)) - F_0 Z(k/n, B(k/n))\right],$$

where $P(t,w)$ and $Z(t,w)$ are given by (3.5) *and* (3.7) *respectively, both with $A = 4m_d/\sigma^2$ and $B(k/n) = (e^{(\delta-r)k/n}G - V_\alpha)/F_0$.*

Since the computation of risk measures for GMDB requires the evaluation of $P(k/n, V_\alpha)$ and $Z(k/n, V_\alpha)$ at multiple points, there may potentially be overflow issues with the expression (3.8) in the cases where $t = k/n$ and σ are both very small. When that happens, we recommend the inverse Laplace transform approach which we shall reiterate in the following remarks for clarification.

Remark 3.3. *The quantile risk measure V_α with $\alpha > \xi_d$ for the net liability of GMDB can be alternatively determined by*

$$1-\alpha = \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} P\left(k/n, B(k/n)\right)$$

where $P(t,w)$ is obtained by numerically inverting $\tilde{P}(s,w)$ in (3.9) *with $A = 4m_d/\sigma^2, B(k/n) = (e^{(\delta-r)k/n}G - V_\alpha)/F_0$.*

Remark 3.4. *The conditional tail expectation* CTE_α *with $\alpha > \xi_d$ for the net liability of GMDB can be alternatively determined by*

$$\mathrm{CTE}_\alpha = \frac{1}{1-\alpha} \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} \left[e^{(\delta-r)k/n} G P(k/n, B(k/n)) - F_0 Z(k/n, B(k/n)) \right],$$

where $P(t,w)$ and $Z(t,w)$ are obtained by numerically inverting $\tilde{P}(s,w)$ and $\tilde{Z}(s,w)$ in (3.9) *and* (3.10) *respectively, both with $A = 4m_d/\sigma^2, B(k/n) = (e^{(\delta-r)k/n}G - V_\alpha)/F_0$.*

4 Numerical Example

In this section, we provide examples in which analytical calculations of quantile risk measure and conditional tail expectation for both GMDB and GMMB are implemented. Consider a variable annuity product with both GMDB and GMMB riders designed for policyholders of age 65 at policy issue. The term of the variable annuity is 10 years, i.e. $T = 10$.

1. The GMMB rider provides the policyholder with the greater of a guaranteed benefit, typically a percentage of initial purchase payment, and the accumulated value in the subaccount at maturity.

2. The GMDB rider offers a death benefit equal to the greater of a guaranteed benefit earning interest at a roll-up rate and the accumulated value in the subaccount. Both the death benefit and account value are accumulated until the end of the year in which the policyholder deceases. The death benefit is payable at the year end.

To model the future lifetime of policyholders, we use the life tables published in an actuarial study by the U.S. Social Security Administration [7] in 2005. The study compiles a comprehensive set of period and cohort life tables by sex and calendar year based on historic and projected mortality data in the areas covered by the US social security program. For illustrative purpose, we use an excerpt of the period life table for male and calendar year 2010 with the additional calculation of survival probabilities as shown in Table 1. See the study [7, page 68] for the complete life table.

x	q_x	k	${}_kp_{65}$
65	0.01753	0	1.00000
66	0.01932	1	0.98246
67	0.02122	2	0.96348
68	0.02323	3	0.94304
69	0.02538	4	0.92113
70	0.02785	5	0.89775
71	0.03059	6	0.87275
72	0.03343	7	0.84606
73	0.03633	8	0.81778
74	0.03942	9	0.78807
75	0.04299	10	0.75700

Table 1: Predicted mortality rates of a male at the age of 65

4.1 GMMB

The recommended risk measure for the determination of additional asset requirement for equity-linking products by the American Academy of Actuaries [1] is $\text{CTE}_{90\%}$. According to SFTF [19], the recommended risk measures for the calculation of reserve and solvency capital by the Canadian Institute of Actuaries under various circumstances are between $\text{CTE}_{80\%}$ and $\text{CTE}_{95\%}$. Hence, in this example we provide results on both quantile and conditional tail expectation risk measures at the 80, 90 and 95 percentile levels.

We use the lognormal model (2.1) for asset prices of the investment fund in which the policyholder invests. The valuation basis for this numerical example is given as follows.

- Mean and standard deviation of log-returns per annum are $\mu = 0.09$ and $\sigma = 0.3$ respectively;
- Risk-free discount rate per annum $r = 0.04$;
- M&E fees and rider charges per annum $m = 0.01$;
- GMMB rider charge is 35 basis points per annum of the separate account, i.e. $m_e = 0.0035$.

The range of annual rider charges for various types of guaranteed benefits in the North American market can be seen in Sun [20]. Since modeling assumptions of parameters are proprietary information and not available to us, the set of parameters we chose is arbitrary but within a reasonable range. Practitioners are encouraged to test the methods and algorithms proposed in this paper with their company-specific model assumptions.

The calculation of risk measures for an individual contract is carried out for various combinations of guaranteed benefit and initial purchase payment in Table 2. The guaranteed death benefit and risk measures are all represented as percentages of initial account value. A simple justification for

G/F_0	75%	100%	120%
$V_{80\%}/F_0$	0%*	0%*	0%*
$\mathrm{CTE}_{80\%}/F_0$	6.9110%*	16.429%*	29.673%*
$V_{90\%}/F_0$	0%*	12.550%	25.956%
$\mathrm{CTE}_{90\%}/F_0$	13.822%*	30.296%	43.702%
$V_{95\%}/F_0$	12.177%	28.935%	42.341%
$\mathrm{CTE}_{95\%}/F_0$	23.283%	40.041%	53.448%

Table 2: Computation of risk measures for the GMMB rider

this practice can be seen in Remark 3.2. In this example, we used the direct calculation method introduced in Proposition 3.3 to determine quantile risk measures. In particular, the Newton-Raphson method is the root search procedure used in our program for the exact results of quantiles. All figures are given in Table 2 with five decimal places with no rounding. We used ten digits in actual computations and quantile risk measures with all ten digits are used in the calculation of conditional tail expectation based on Proposition 3.4.

The cases marked with asterisks correspond to negative risk measures of net liability L_0, the calculation of which requires an extension of analysis from previous sections. However, since risk capitals are always held in non-negative amount in practice, exact amount of negative risk measure is of no relevance for real-life applications. In these cases, we instead compute the quantile risk measure and conditional tail expectation of $L_0^* := \max\{L_0, 0\}$. In the case where L_0^* has a probability mass at zero and $\alpha < \xi_e = \mathbb{P}[L_0 \leq 0]$, we shall define the conditional tail expectation of L_0^* as follows.

$$\mathrm{CTE}_\alpha(L_0^*) := \frac{\mathbb{E}[L_0^* I(L_0^* > 0)]}{1-\alpha} = \frac{(1-\xi_e)\mathbb{E}[L_0 | L_0 > 0]}{1-\alpha} = \frac{(1-\xi_e)\mathrm{CTE}_{\xi_e}(L_0)}{1-\alpha}.$$

For example, in the case where $G/F_0 = 75\%$, we observe that $P(10,0) = 0.1143463552$, which implies $\xi_e = 1 - {}_{10}p_{65} P(10,0) = 91.34398\%$. Thus, the L_0^* has a probability mass at 0 and the 80% conditional tail expectation of L_0^* is computed by

$$\mathrm{CTE}_{80\%}(L_0^*) = \frac{(1-0.9134398)\mathrm{CTE}_{91.34398\%}(L_0)}{1-0.80} = 6.9110\%.$$

The computing time may vary greatly according to computing software, computer configuration as well as the choices of initial value for root search procedures. All computations in this example are done with the Maple 14 software package on a personal computer with Intel Corel 2 Duo CPU (3.0GHz) and 3.25 GB of RAM. It takes between one and three minutes to produce each result in Table 2 with this computer set-up.

4.1.1 Accuracy and Efficiency Test

We provide an example to compare the accuracy and efficiency of three different methods, which are (1) direct calculation, proposed in Propositions 3.3 and 3.4; (2) numerical inversion of Laplace

transform, proposed in Propositions 3.5 and 3.6; (3) Monte Carlo simulations, which is the equivalent of current industrial practice, tailored to individual contract valuation. All methods are tested under the same valuation basis as given in previous examples with $G/F_0 = 100\%$.

For the second method, the Laplace transform $\tilde{P}(s, y)$ is calculated according to Proposition 3.5 and the expression (B.1). We then invert the Laplace transform numerically using the Gaver-Stehfest method. In simple words, the Gaver-Stehfest method approximates the value of $P(t, y)$ by

$$P^*(t, y) := \sum_{k=1}^{n} w(k, n)\tilde{P}_k(t, y),$$

where

$$w(k, n) := (-1)^{n-k}\frac{k^n}{k!(n-k)!}, \qquad \tilde{P}_k(t, y) := \frac{\ln 2}{t}\frac{(2k)!}{k!(k-1)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\tilde{P}\Big(\frac{(k+j)\ln 2}{t}, y\Big).$$

According to Abate and Whitt [2], the number of digits precision required for programming with the Gaver-Stehfest method increases with the number n. For small n we need roughly $2n$ digits precision for calculation. In our computing routines using Maple, we often invoke the NAG numerical integration procedures, d01amc and d01akc, which only allow precision up to 15 digits. Therefore, we only use $n = 7$ in the following example. A shortfall of this method is that there is no known error analysis according to Abate and Whitt [2]. Nevertheless, we can also perform the first method of direct calculation for verification. Because the two methods are sufficiently different, we can be certain of its accuracy if the results from both methods agree up to a sufficient number of digits. The computation of CTE is carried out similarly by inverting the Laplace transform $\tilde{Z}$ given by Proposition 3.6 with the Gaver-Stehfest method.

For the third method, we estimate the risk measures through repeated sampling in two steps. First, $10,000$ observations of the future lifetime random variable are generated for each basic experiment. For each of those values that surpass the term of the GMMB rider, we simulate the stock prices for 250 trading dates each year for the whole term(10 years) and then determine the net liabilities according to (2.4). Since the negative values of net liabilities do not affect the risk measures, we do not generate stock prices for those future lifetimes shorter than maturity in order to improve simulation efficiency. Then we use the $9,000$-th order statistics and the sample mean of the $9,000$-th through $10,000$-th order statistics as the estimators for $V_{90\%}$ and $\text{CTE}_{90\%}$ respectively and both estimators are applied to the same data set of net liabilities for each basic experiment. Second, the basic experiments are repeated 50 times. The third column of Table 3 reports the means of 50 observations of both risk measures. Since the true values of risk measures are confirmed by the first two columns, we can evaluate the mean squared error(MSE) of the estimators. The MSEs of the quantile and CTE estimators are 0.00008268515736 and 0.00004663449632 respectively. Since both estimators are produced simultaneously, we only state the running time once under the quantile estimator.

Table 3 reports on final results from all three methods as well as computational times in minutes which indicate their relative time consumption. The Newton-Raphson method is used in search for

Methods	(1)	(2)	(3)
$V_{90\%}/F_0$	12.550%	12.550%	12.763%
Initial value	10%	$(12\%, 14\%)$	-
Time (mins)	3.6757	3.3419	2373.8
$\text{CTE}_{90\%}/F_0$	30.296%	30.296%	30.463%
Time (mins)	1.8708	0.3114	-

Table 3: A comparison of three computational methods

quantile in direct calculation and bisection method in inversion of Laplace transforms. It is clear from the comparison that the analytical methods are by far more efficient and accurate than Monte Carlo simulations.

4.1.2 Sensitivity Test

In what follows, we perform sensitivity tests on each of the parameters involved in the computation of risk measures while all else being the same as listed in the valuation basis. The guaranteed benefit level is set to be 100% of the initial purchase payment, i.e. $G = F_0$. Only the Newton-Raphson method is employed in the following computations. The middle column corresponds to the base case used for the accuracy and efficiency test.

μ	0.08	0.09	0.10
$V_{90\%}/F_0$	17.592%	12.550%	6.9829%
$\text{CTE}_{90\%}/F_0$	33.667%	30.296%	26.574%
σ	**0.24**	0.3	0.4
$V_{90\%}/F_0$	**0.40771**%	12.550%	28.189%
$\text{CTE}_{90\%}/F_0$	**19.165**%	30.296%	43.370%
m	0.035	0.01	0.015
$V_{90\%}/F_0$	8.9946%	12.550%	15.133%
$\text{CTE}_{90\%}/F_0$	27.919%	30.296%	32.023%
m_e	0.0015	0.0035	0.005
$V_{90\%}/F_0$	14.123%	12.550%	11.372%
$\text{CTE}_{90\%}/F_0$	31.599%	30.296%	29.323%
r	0.02	0.04	0.06
$V_{90\%}/F_0$	15.676%	12.550%	10.008%
$\text{CTE}_{90\%}/F_0$	37.307%	30.296%	24.570%

Table 4: Sensitivity analysis of risk measures for the GMMB rider

The results on sensitivity analysis in Table 4 can be interpreted as follows. The insurer's

liability rises ($V, \text{CTE} \uparrow$) due to higher costs of guaranteed benefits when the expected value of log-returns in the separate account decreases ($\mu \downarrow$), or the equity prices become more volatile ($\sigma \uparrow$). Higher roll-up rates ($\delta \uparrow$) give rise to more generous benefits and hence induce higher levels of risk ($V, \text{CTE} \uparrow$). Although it seems counter-intuitive that higher M&E charges ($m \uparrow$) lead to higher risk measure ($V, \text{CTE} \uparrow$), one should be reminded that only the margin offset allocated to maturity benefit m_e is included in the calculation of margin offset income M_s. While the margin offset m_e is held constant, higher M&E charges do not count towards the incoming cash flow for the GMMB rider but rather reduce the fund value in the separate account and hence lower the incoming cash flow. Consequently higher guaranteed minimum payments are more likely to be paid out, which raise both risk measures. The higher margin offset ($m_e \uparrow$) increases the incoming cash flow and hence leads to less liability and smaller risk measures ($V, \text{CTE} \downarrow$). We also change the values of risk-free discount rate δ to illustrate the impact of discounting factor. As one would expect, the present values of liability become smaller ($V, \text{CTE} \downarrow$) as the discount rate increases ($r \uparrow$). The risk measures appear to be very sensitive to the volatility coefficient σ. In the case where $\sigma = 0.2$, we set the digits of precision in Maple to be 20 and the computation takes about five minutes for each of the two risk measures. For the rest of cases we use the 10 digits of precision by default and the running times vary between one and three minutes.

The 90% CTEs in this example appear to be higher than what are often observed for C3 risk-based capitals in practice. However, this does not necessarily suggest that the risk capitals are generally underestimated in the variable annuity industry. It should be noted that risk measures are typically calculated at the aggregate level in practice as opposed to the individual level considered in this paper. The diversification of risks among contracts may reduce risk measures significantly. One should also note from the numerical test that the risk measures are very sensitive to the volatility coefficient σ. A choice of smaller σ would lead to much smaller CTE amounts.

4.2 GMDB

We now present the evaluation of risk measures for the GMDB rider. One should note that the computational efforts required by the GMDB rider are significantly increased as $P(k/n, V_\alpha)$ in (**??**) is computed multiple times and for very small k/n. Note that the constant (3.8) (with $t = 1$ and $\sigma = 0.3$) is on the order of 10^{96} and hence the double integrals in Propositions 3.9 and 3.10 need to be computed accurately up to at least the order of 10^{-101} in order to maintain precision up to five decimal places in the final results. Therefore, we shall use numerical inversion of Laplace transforms outlined in Remarks 3.3 and 3.4, which demands much less computational efforts in this case.

The valuation basis for the base numerical example is given as follows.

- Mean and standard deviation of log-returns per annum are $\mu = 0.09$ and $\sigma = 0.3$ respectively;
- Risk-free discount rate per annum $r = 0.07$;

- Roll-up rate per annum $\delta = 0.06$;
- M&E fees and rider charges per annum $m = 0.01$;
- GMDB rider charge is 35 basis points per annum of the separate account, i.e. $m_d = 0.0035$;
- Survival model follows Table 1.

G/F_0	75%	100%	120%
$V_{80\%}/F_0$	0%*	0%*	0%*
$\mathrm{CTE}_{80\%}/F_0$	7.0185%*	16.871%*	27.981%*
$V_{90\%}/F_0$	0%*	2.1353%	21.144%
$\mathrm{CTE}_{90\%}/F_0$	14.037%*	33.706%	52.568%
$V_{95\%}/F_0$	8.1979%	31.825%	50.732%
$\mathrm{CTE}_{95\%}/F_0$	26.965%	50.390%	69.140%

Table 5: Computation of risk measures for the GMDB rider

Table 5 reports on the risk measures for various initial guarantee levels. We used the bisection method to search for the quantile with accuracy up to five decimal places. In a manner similar to the previous subsection, the cases with asterisks correspond to conditional tail expectations of $L_0^{(n)*} := \max\{L_0^{(n)}, 0\}$. For example, the probability of non-positive net liability for the case of $G/F_0 = 120\%$ is given by $\xi_d = 1 - 0.1309995120 = 0.8690004880$. Hence,

$$\mathrm{CTE}_\alpha(L_0^{(n)*}) = \frac{(1-0.869)\mathrm{CTE}_{86.9\%}(L_0^{(n)})}{1-0.80} = 27.981\%.$$

The large impact of initial guarantee on risk measures is due to the fact that the roll-up rate is set at a relatively high level and hence the effect of higher initial guarantee are more pronounced as the guarantee amounts at later policy anniversaries accumulate rapidly. The big difference between quantile and CTE risk measures seems to suggest that the distribution of net liabilities may have a "heavy" tail.

Since the 90% CTE amounts are used for determining capital requirements in practice, we perform sensitivity tests on various parameters for the 90% level as shown in Table 6. We change one parameter at a time while holding all other parameters constant as in the base case. The initial guarantee is set at $G/F_0 = 1$. The middle column corresponds to the base case.

As one would expect, the pattern of risk measures with changes in parameters for the GMDB rider is very similar to that for the GMMB rider. Here we point out the impact of the roll-up rates. The risk measures increase ($V, \mathrm{CTE} \uparrow$) with roll-up rates, since higher roll-up ($\delta \uparrow$) leads to higher guaranteed benefit on all policy anniversaries and hence higher present value of liabilities.

The Maple codes for all numerical examples are available upon request.

μ	0.08	0.09	0.10
$V_{90\%}/F_0$	6.645%	2.1353%	0%*
$\mathrm{CTE}_{90\%}/F_0$	36.939%	33.706%	30.398%*
δ	0.05	0.06	0.07
$V_{90\%}/F_0$	0%*	2.1353%	7.0565%
$\mathrm{CTE}_{90\%}/F_0$	28.505%*	33.706%	39.471%
σ	0.16	0.3	0.4
$V_{90\%}/F_0$	0%*	2.1353%	6.3296%
$\mathrm{CTE}_{90\%}/F_0$	16.800%*	33.706%	42.797%
m	0.035	0.01	0.015
$V_{90\%}/F_0$	0%*	2.1353%	4.4149%
$\mathrm{CTE}_{90\%}/F_0$	31.543%*	33.706%	35.338%
m_d	0.0015	0.0035	0.005
$V_{90\%}/F_0$	3.1521%	2.1353%	1.3735%
$\mathrm{CTE}_{90\%}/F_0$	34.632%	33.706%	33.016%
r	0.06	0.07	0.08
$V_{90\%}/F_0$	2.3098%	2.1353%	1.9674%
$\mathrm{CTE}_{90\%}/F_0$	35.993%	33.706%	31.586%

Table 6: Sensitivity analysis of risk measures for the GMDB rider

5 Conclusion and Future Work

With the fierce competition in the financial market, life insurers have offered increasingly complex option-like investment-combined products. Traditional deterministic modeling is insufficient to capture the characteristics of equity-linked guarantees. Over the past few years, stochastic scenario modeling such as Monte Carlo simulation has been gradually accepted as the major tool to analyze the risk profiles of these products. However, according to recent industry survey, the implementation of stochastic modeling appears to impose prohibitive costs on many small and medium sized insurers.

The purpose of this work is to initiate attempts to enhance the accuracy and efficiency of stochastic modeling by using alternative approaches. We have been able to produce two analytical methods for the computations of quantile and conditional tail expectation risk measures for two types of variable annuity guaranteed benefits, namely, guaranteed minimum maturity benefit (GMMB) and guaranteed minimum death benefit (GMDB) at the individual contract level. The analytical methods have been shown through numerical experiments to be by far more efficient and accurate than the Monte Carlo simulations, which seems to suggest that further development of analytical methods may provide more economically viable and efficient solutions for industrial practice on the computation of risk capitals.

As the research on this matter is in its infant stage, we have only investigated product designs of basic forms. The work can be continued in multiple directions. One natural extension is to calculate or approximate risk measures at the aggregate level where multiple contracts of various sizes are issued. It is yet to be seen how analytical techniques may be used to model other popular riders such as guarantee minimum accumulation benefits (GMAB) and guarantee minimum withdrawal benefits (GMWB).

There has also been rapid development in actuarial literature on the so-called comonotonic approximations of risk measures for annuities with random investment. For example, Vanduffel et al. [25] provided analytic lower and upper bounds for the Asian option, which seems to be rather efficient as double or triple integrals no longer appear. Due to the close connection between the Asian option and the net liability of guaranteed benefit, we envision further development of the approximation techniques may lead to even more efficient computation of risk measures.

We assumed in this work that the dynamics of equity prices is driven by geometric Brownian motion. Another possible line of extension would be the study of risk measures of variable annuities based on other underlying equity models, such as exponential Lévy processes.

A Appendix: Proofs

Proof of Proposition 3.1:

If $\alpha > \xi_e$, then

$$\begin{aligned} 1-\alpha &= {}_Tp_x\,\mathbb{P}[e^{-rT}(G - F_0\frac{S_T}{S_0}e^{-mT}) > V_\alpha] \\ &= {}_Tp_x\,\mathbb{P}\left[\frac{S_T}{S_0}e^{-(r+m)T} < \frac{e^{-rT}G - V_\alpha}{F_0}\right]. \end{aligned} \tag{A.1}$$

Since $\ln e^{-(r+m)T}S_T/S_0 \sim \Phi(\cdot;(\mu-r-m)T,\sigma^2T)$, then

$$\ln\frac{e^{-rT}G - V_\alpha}{F_0} = (\mu-r-m)T + \sigma\sqrt{T}z_\beta,$$

which yields the desired expression after rearrangement. □

Proof of Proposition 3.2:

Note that for any real number x,

$$\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{y}e^{-(y-\mu)^2/(2\sigma^2)}\,\mathrm{d}y = e^{\mu+\sigma^2/2}\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-(y-\mu-\sigma^2)^2/(2\sigma^2)}\,\mathrm{d}y. \tag{A.2}$$

By definition, if $\alpha > \xi_e$,

$$\begin{aligned} \mathrm{CTE}_\alpha &= \frac{{}_Tp_x}{1-\alpha}\mathbb{E}[e^{-rT}(G-F_T)I(e^{-rT}(G-F_T) > V_\alpha)] \\ &= \frac{{}_Tp_x}{1-\alpha}\mathbb{E}\left[(e^{-rT}G - F_0e^{Y})I(e^{Y} < c)\right], \end{aligned}$$

where $Y = \ln(S_T/S_0) - (r+m)T$ with the distribution function $\Phi(\cdot;(\mu - r - m)T, \sigma^2 T)$ and $c = (e^{-rT}G - V_\alpha)/F_0$. Therefore,

$$\mathrm{CTE}_\alpha \;=\; \frac{1}{1-\alpha}\,{}_Tp_x \int_{-\infty}^{\ln c} (e^{-rT}G - F_0 e^y)\,\mathrm{d}\Phi(y;(\mu - r - m)T, \sigma^2 T),$$

which yields the following expression in view of (A.1) and (A.2).

$$\mathrm{CTE}_\alpha = e^{-rT}G - {}_Tp_x \frac{F_0}{1-\alpha} e^{(\mu-r-m)T+\sigma^2 T/2}\Phi\Big(\ln\frac{e^{-rT}G - V_\alpha}{F_0}; (\mu - r - m + \sigma^2)T, \sigma^2 T\Big), \quad \text{(A.3)}$$

We obtain (3.3) by inserting (3.2) for V_α in (A.3) and re-scaling the parameters of the normal distribution. □

Proof of Proposition 3.3:

Define

$$B_t^{(\nu)} = \nu t + B_t; \qquad A_t^{(\nu)} = \int_0^t \exp\{2(\nu s + B_s)\}\,\mathrm{d}s. \quad \text{(A.4)}$$

It is easy to see that

$$e^{-rt}F_t = F_0 \exp\{(\mu - m - r)t + \sigma B_t\} = F_0 \exp\left\{2\left[\frac{2(\mu - m - r)}{\sigma^2}\frac{\sigma^2 t}{4} + B_{\sigma^2 t/4}\right]\right\} = F_0 \exp\{2B^{(\nu)}_{\sigma^2 t/4}\}$$

and

$$\int_0^t e^{-rs} m_e F_s\,\mathrm{d}s = m_e F_0 \int_0^t e^{2B^{(\nu)}_{\sigma^2 s/4}}\,\mathrm{d}s = \frac{4m_e}{\sigma^2} F_0 A^{(\nu)}_{\sigma^2 t/4}.$$

In view of (3.1) and (2.4), we obtain

$$\begin{aligned} 1-\alpha &= {}_Tp_x\,\mathbb{P}\Big[(e^{-rT}G - e^{-rT}F_T) - \int_0^T e^{-rs} m_e F_s\,\mathrm{d}s - V_\alpha > 0\Big] \\ &= {}_Tp_x\,\mathbb{P}\Big[e^{-rT}G - F_0\exp\{2B^{(\nu)}_{\sigma^2 T/4}\} - \frac{4m_e}{\sigma^2}F_0 A^{(\nu)}_{\sigma^2 T/4} - V_\alpha > 0\Big]. \end{aligned}$$

Recall from (6.a') and Proposition 2 of Yor [21] that

$$\mathbb{P}[A_t^{(\nu)} \in \mathrm{d}u | B_t^{(\nu)} = x] = a_t(x,u)\,\mathrm{d}u,$$

where

$$\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right) a_t(x,y) = \frac{1}{u}\exp\left(-\frac{1}{2u}(1+e^{2x})\right)\theta_{e^x/u}(t),$$

and

$$\theta_r(u) = \frac{r}{\sqrt{2\pi^3 u}}\exp\{\frac{\pi^2}{2u}\}\int_0^\infty \exp\{-\frac{y^2}{2u}\}\exp\{-r\cosh y\}(\sinh y)\sin(\frac{\pi y}{u})\,\mathrm{d}y.$$

Let $s = \sigma^2 T/4$. We note that

$$
\begin{aligned}
& P(T, B(T)) = \mathbb{P}\left[e^{-rT}(G - F_T) - \int_0^T e^{-rs} M_s \,\mathrm{d}s > V_\alpha\right] \\
= \; & \mathbb{P}\Big[(e^{-4rs/\sigma^2} G - F_0 \exp\{2B_s^{(\nu)}\}) - \frac{4m_e}{\sigma^2} F_0 A_s^{(\nu)} - V_\alpha > 0\Big] \\
= \; & \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi s}} \exp\{-\frac{(x-\nu s)^2}{2s}\} a_s(x,u) I_{\{e^{-4rs/\sigma^2} G - F_0 e^{2x} - \frac{4m_e}{\sigma^2} F_0 u - V_\alpha > 0\}} \,\mathrm{d}x \,\mathrm{d}u \\
= \; & \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi s}} \exp\{-\frac{x^2}{2s}\} a_s(x,u) e^{\nu x - \nu^2 s/2} I_{\{e^{-4rs/\sigma^2} G - F_0 e^{2x} - \frac{4m_e}{\sigma^2} F_0 u - V_\alpha > 0\}} \,\mathrm{d}x \,\mathrm{d}u \\
= \; & \int_0^\infty \int_{-\infty}^\infty \frac{1}{u} \exp\{-\frac{1}{2u}(1+e^{2x})\} \theta_{e^x/u}(s) e^{\nu x - \nu^2 s/2} I_{\{e^{-4rs/\sigma^2} G - F_0 e^{2x} - \frac{4m_e}{\sigma^2} F_0 u - V_\alpha > 0\}} \,\mathrm{d}x \,\mathrm{d}u \\
= \; & \int_0^\infty \int_0^\infty \frac{\rho^{\nu-1} e^{-\nu^2 s/2}}{u} \exp\{-\frac{1}{2u}(1+\rho^2)\} \theta_{\rho/u}(s) I_{\{e^{-4rs/\sigma^2} G - F_0 \rho^2 - \frac{4m_e}{\sigma^2} F_0 u - V_\alpha > 0\}} \,\mathrm{d}\rho \,\mathrm{d}u.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& P(T, B(T)) = \int_0^{\sqrt{B(T)}} \int_0^{(B(T)-\rho^2)/A} \frac{\rho^{\nu-1} e^{-\nu^2\sigma^2 T/8}}{u} \exp\{-\frac{1}{2u}(1+\rho^2)\} \theta_{\rho/u}(\frac{\sigma^2 T}{4}) \,\mathrm{d}u \,\mathrm{d}\rho \qquad \text{(A.5)} \\
= \; & \sqrt{\frac{2}{\pi^3 \sigma^2 T}} \exp\left(-\frac{\nu^2\sigma^2 T}{8} + \frac{2\pi^2}{\sigma^2 T}\right) \int_0^\infty \exp\left(-\frac{2y^2}{\sigma^2 T}\right) \sinh y \sin\left(\frac{4\pi y}{\sigma^2 T}\right) \int_0^{\sqrt{B(T)}} I(\rho, y) \,\mathrm{d}\rho \,\mathrm{d}y,
\end{aligned}
$$

where

$$
\begin{aligned}
I(\rho, y) &= \int_0^{(B(T)-\rho^2)/A} \frac{\rho^\nu}{u^2} \exp\left(-\frac{1}{2u}(1+\rho^2+2\rho\cosh y)\right) \mathrm{d}u \qquad \text{(A.6)} \\
&= \frac{2\rho^\nu}{1+\rho^2+2\rho\cosh y} \exp\left(-\frac{A(1+\rho^2+2\rho\cosh y)}{2(B(T)-\rho^2)}\right).
\end{aligned}
$$

□

Proof of Proposition 3.4: By definition,

$$
\begin{aligned}
\mathrm{CTE}_\alpha &= \frac{{}_Tp_x}{1-\alpha} \mathbb{E}\left[\left\{e^{-rT} G - e^{-rT} F_T - \int_0^T e^{-rs} M_s \,\mathrm{d}s\right\} I_{\{e^{-rT} G - e^{-rT} F_T - \int_0^T e^{-rs} M_s \,\mathrm{d}s > V_\alpha\}}\right] \\
&= e^{-rT} G - \frac{{}_Tp_x}{1-\alpha} \mathbb{E}\left[\left\{e^{-rT} F_T + \int_0^T e^{-rs} m_e F_s \,\mathrm{d}s\right\} I_{\{e^{-rT} G - e^{-rT} F_T - \int_0^T e^{-rs} m_e F_s \,\mathrm{d}s > V_\alpha\}}\right] \\
&= e^{-rT} G - {}_Tp_x \frac{F_0}{1-\alpha} Z(T, B(T)).
\end{aligned}
$$

Note that under the GMMB,

$$
\begin{aligned}
& Z(T, B(T)) = \mathbb{E}\left[\left\{e^{-rT} \frac{F_T}{F_0} + \int_0^T e^{-rs} m_e \frac{F_s}{F_0} \,\mathrm{d}s\right\} I_{\{e^{-rT} G - e^{-rT} F_T - \int_0^T e^{-rs} M_s \,\mathrm{d}s > V_\alpha\}}\right] \\
= \; & \mathbb{E}\left[\left\{\exp\{2B_{\sigma^2 T/4}^{(\nu)}\} + \frac{4m_e}{\sigma^2} A_{\sigma^2 T/4}^{(\nu)}\right\} I_{\{e^{-rT} G - F_0 \exp\{2B_{\sigma^2 T/4}^{(\nu)}\} - \frac{4}{\sigma^2} m_e F_0 A_{\sigma^2 T/4}^{(\nu)} > V_\alpha\}}\right] \\
= \; & \int_0^{\sqrt{B}} \int_0^{(B-\rho^2)/A} \left(\rho^2 + \frac{4m_e}{\sigma^2} u\right) \frac{\rho^{\nu-1} e^{-\nu^2\sigma^2 T/8}}{u} \exp\{-\frac{1}{2u}(1+\rho^2)\} \theta_{\rho/u}(\frac{\sigma^2 T}{4}) \,\mathrm{d}\rho \,\mathrm{d}u
\end{aligned}
$$

which splits into two integrals denoted by I_1 and I_2 respectively. Note

$$I_1 = \sqrt{\frac{2}{\pi^3\sigma^2 T}} \exp\left(-\frac{\nu^2\sigma^2 T}{8} + \frac{2\pi^2}{\sigma^2 T}\right) \int_0^\infty \exp\left(-\frac{2y^2}{\sigma^2 T}\right) \sinh y \sin\left(\frac{4\pi y}{\sigma^2 T}\right) \int_0^{\sqrt{B(T)}} \rho^2 I(\rho, y)\, \mathrm{d}\rho\, \mathrm{d}y,$$

where $I(\rho, y)$ is defined in (A.6). Similarly, we have

$$I_2 = \sqrt{\frac{2}{\pi^3\sigma^2 T}} \exp\left(-\frac{\nu^2\sigma^2 T}{8} + \frac{2\pi^2}{\sigma^2 T}\right) \frac{4m_e}{\sigma^2} \int_0^\infty \exp\left(-\frac{2y^2}{\sigma^2 T}\right) \sinh y \sin\left(\frac{4\pi y}{\sigma^2 T}\right) \int_0^{\sqrt{B(T)}} J(\rho, y)\, \mathrm{d}\rho\, \mathrm{d}y,$$

where

$$\begin{aligned} J(\rho, y) &= \int_0^{(B(T)-\rho^2)/A} \frac{\rho^\nu}{u} \exp\left(-\frac{1}{2u}(1+\rho^2+2\rho\cosh y)\right) \mathrm{d}u \\ &= \rho^\nu \int_{A(1+\rho^2+2\rho\cosh y)/2/(B(T)-\rho^2)}^\infty \frac{1}{t} e^{-t}\, \mathrm{d}t = \rho^\nu E_1\left(\frac{A(1+\rho^2+2\rho\cosh y)}{2(B(T)-\rho^2)}\right). \end{aligned}$$

□

Proof of Proposition 3.5:

For notational brevity, we let $B(T) = (e^{-rT}G - V_\alpha)/F_0$ in this proof. In a manner similar to the derivation of (A.5), we can show that under GMMB,

$$\begin{aligned} P(t, B(T)) &= \mathbb{P}\left[e^{-rT}G - e^{-rt}F_t - m_e \int_0^t e^{-rs} F_s\, \mathrm{d}s > V_\alpha\right] \\ &= \mathbb{P}\left[\exp\{2B_t^{(\nu)}\} + \frac{4m_e}{\sigma^2} A_t^{(\nu)} < B(T)\right] \\ &= \int_0^{\sqrt{B(T)}} \int_0^{(B(T)-\rho^2)/A} \frac{\rho^{\nu-1} e^{-\nu^2\sigma^2 t/8}}{u} \exp\left(-\frac{1}{2u}(1+\rho^2)\right) \theta_{\rho/u}\left(\frac{\sigma^2 t}{4}\right) \mathrm{d}u\, \mathrm{d}\rho. \end{aligned}$$

It follows from (2.e') and (6.b') of Yor [21] that

$$\int_0^\infty \exp\left(-\frac{\nu^2 s}{2}\right) \theta_r(s)\, \mathrm{d}s = I_{|\nu|}(r).$$

Taking the Laplace transform yields

$$\begin{aligned} &\int_0^\infty e^{-st} P(t, B(T))\, \mathrm{d}t \\ = &\int_0^{\sqrt{B(T)}} \int_0^{(B(T)-\rho^2)/A} \frac{\rho^{\nu-1}}{u} \exp\left(-\frac{1}{2u}(1+\rho^2)\right) \int_0^\infty e^{-4sw/\sigma^2 - \nu^2 w/2} \theta_{\rho/u}(w) \frac{4}{\sigma^2}\, \mathrm{d}w\, \mathrm{d}u\, \mathrm{d}\rho \\ = &\frac{4}{\sigma^2} \int_0^{\sqrt{B(T)}} \int_0^{(B(T)-\rho^2)/A} \frac{\rho^{\nu-1}}{u} \exp\left(-\frac{1}{2u}(1+\rho^2)\right) I_{2\eta}\left(\frac{\rho}{u}\right) \mathrm{d}u\, \mathrm{d}\rho, \end{aligned}$$

where $2\eta = \sqrt{8s/\sigma^2 + \nu^2}$. □

Proof of Proposition 3.6:

Note that in the case of GMMB we have

$$
\begin{aligned}
& Z(t, B(T)) = \mathbb{E}\left[\left\{e^{-rt}\frac{F_t}{F_0} + m_e\int_0^t e^{-rs}\frac{F_s}{F_0}\,\mathrm{d}s\right\} I_{\{e^{-rT}G - e^{-rt}F_t - m_e\int_0^t e^{-rs}F_s\,\mathrm{d}s > V_\alpha\}}\right] \\
= & \int_0^{\sqrt{B(T)}}\int_0^{(B(T)-\rho^2)/A}\left(\rho^2 + \frac{4m_e}{\sigma^2}u\right)\frac{\rho^{\nu-1}e^{-\nu^2\sigma^2 t/8}}{u}\exp\left(-\frac{1}{2u}(1+\rho^2)\right)\theta_{\rho/u}\left(\frac{\sigma^2 t}{4}\right)\,\mathrm{d}u\,\mathrm{d}\rho.
\end{aligned}
$$

Taking the Laplace transform yields

$$
\begin{aligned}
& \int_0^\infty e^{-st}Z(t, B(T))\,\mathrm{d}t \\
= & \int_0^{\sqrt{B(T)}}\int_0^{(B(T)-\rho^2)/A}\left(\frac{\rho^{\nu+1}}{u} + \frac{4m_e}{\sigma^2}\rho^{\nu-1}\right)\exp\left(-\frac{1}{2u}(1+\rho^2)\right) \\
& \times\int_0^\infty e^{-4sw/\sigma^2 - \nu^2 w/2}\theta_{\rho/u}(w)\,\frac{4}{\sigma^2}\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}\rho \\
= & \frac{4}{\sigma^2}\int_0^{\sqrt{B(T)}}\int_0^{(B(T)-\rho^2)/A}\left(\frac{\rho^{\nu+1}}{u} + \frac{4m_e}{\sigma^2}\rho^{\nu-1}\right)\exp\left(-\frac{1}{2u}(1+\rho^2)\right)I_{2\eta}\left(\frac{\rho}{u}\right)\,\mathrm{d}u\,\mathrm{d}\rho.
\end{aligned}
$$

□

Proof of Proposition 3.7:

Let $f_{\tau_x}(t) = {}_tp_x\mu_{x+t}$. It is easy to see that

$$
\begin{aligned}
1-\alpha &= \int_0^T \mathbb{P}[e^{-rt}(e^{\delta t}G - F_t) > V_\alpha]f_{\tau_x}(t)\,\mathrm{d}t \\
&= \int_0^T \mathbb{P}\left[\frac{S_t}{S_0}e^{-(r+m)t} < \frac{e^{-(r-\delta)t}G - V_\alpha}{F_0}\right]f_{\tau_x}(t)\,\mathrm{d}t,
\end{aligned}
$$

which yields the result by noting that $\ln e^{-(r+m)t}S_t/S_0 \sim \Phi(\cdot; (\mu - r - m)t, \sigma^2 t)$ for all $0 \le t \le T$.□

Proof of Proposition 3.8: We follow the proof of Proposition 3.2 to show that

$$
\begin{aligned}
\mathrm{CTE}_\alpha &= \frac{1}{1-\alpha}\int_0^T \mathbb{E}[e^{-rt}(e^{\delta t}G - F_t)I(e^{-rt}(e^{\delta t}G - F_t) > V_\alpha)]f_{\tau_x}(t)\,\mathrm{d}t \\
&= \frac{1}{1-\alpha}\int_0^T \mathbb{E}[(e^{-(r-\delta)t}G - F_0e^{Y(t)})I\{e^{Y(t)} < c(t)\}]f_{\tau_x}(t)\,\mathrm{d}t,
\end{aligned}
$$

where $Y(t) = \ln(S_t/S_0) - (r+m)t$ with the distribution function $\Phi(\cdot; (\mu - r - m)t, \sigma^2 t)$ and $c(t) = (e^{-(r-\delta)t}G - V_\alpha)/F_0$. The desired expression follows by using the identity (A.2). □

Proof of Proposition 3.9:

The proof is similar to that of Proposition 3.3. Note that if $\alpha > \xi_d$, then

$$\begin{aligned}
1-\alpha &= \sum_{k=1}^{\lceil nT \rceil} \mathbb{E}\left\{ I\Big[(e^{-(r-\delta)k/n}G - e^{-rk/n}F_{k/n}) - \int_0^{k/n} e^{-rs} m_d F_s \,\mathrm{d}s - V_\alpha > 0\Big]\right\} \mathbb{P}(\kappa_x^{(n)} = k/n) \\
&= \sum_{k=1}^{\lceil nT \rceil} \mathbb{E}\left\{ I\Big[(e^{-(r-\delta)k/n}G - e^{-rk/n}F_{k/n}) - \int_0^{k/n} e^{-rs} m_d F_s \,\mathrm{d}s - V_\alpha > 0\Big]\right\} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} \\
&= \sum_{k=1}^{\lceil nT \rceil} P(k/n, B(k/n)) \, {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n},
\end{aligned}$$

where P is given by (A.5) with $A = 4m_d/\sigma^2$ and $B(k/n) = (e^{-(r-\delta)k/n}G - V_\alpha)/F_0$. □

Proof of Proposition 3.10:

For notational brevity, we let

$$D_k = \left\{ e^{-rk/n}(e^{\delta k/n}G - F_{k/n}) - m_d \int_0^{k/n} e^{-rs} F_s \,\mathrm{d}s > V_\alpha \right\}.$$

By definition, the CTE for the net liability of the GMDB,

$$\begin{aligned}
\mathrm{CTE}_\alpha &= \frac{1}{1-\alpha} \sum_{k=1}^{\lceil nT \rceil} \mathbb{E}\left[\left\{ e^{-rk/n}(e^{\delta k/n}G - F_{k/n}) - m_d \int_0^{k/n} e^{-rs} F_s \,\mathrm{d}s \right\} I_{D_k} \right] \mathbb{P}(\kappa_x^{(n)} = k/n) \\
&= \frac{1}{1-\alpha} \sum_{k=1}^{\lceil nT \rceil} \mathbb{E}\left[\left\{ e^{-(r-\delta)k/n}G - \left(e^{-rk/n}F_{k/n} + m_d \int_0^{k/n} e^{-rs} F_s \,\mathrm{d}s \right) \right\} I_{D_k} \right] {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n} \\
&= \frac{1}{1-\alpha} \\
&\quad \times \sum_{k=1}^{\lceil nT \rceil} \left\{ e^{-(r-\delta)k/n} G \, \mathbb{P}(D_k) - F_0 \mathbb{E}\left[\left(e^{-rk/n} \frac{F_{k/n}}{F_0} + m_d \int_0^{k/n} e^{-rs} \frac{F_s}{F_0} \,\mathrm{d}s \right) I_{D_k} \right] \right\} {}_{(k-1)/n}p_x \, {}_{1/n}q_{x+(k-1)/n},
\end{aligned}$$

The desired expression follows immediately by definitions of $P(T, V_\alpha)$ and $Z(T, V_\alpha)$. □

B Appendix: Alternative Expressions for Integral (3.9)

Depending the methods of numerical integration, (3.9) may not be the optimal expression to be implemented. Here we investigate alternative but equivalent expressions of (3.9). Note that (3.9) can be rewritten as

$$\tilde{P}(s, w) = \frac{4}{\sigma^2} \int \int_{C(w)} \frac{\rho^{\nu-1}}{u} \exp\left\{ -\frac{1}{2u}(1+\rho^2) \right\} I_{2\eta}\left(\frac{\rho}{u}\right) \mathrm{d}u \, \mathrm{d}\rho,$$

where

$$C(w) = \Big\{ (u, \rho) : u, \rho > 0, \rho^2 + Au < w \Big\}.$$

Let $z = \rho/u$. Then

$$\tilde{P}(s,w) = \frac{4}{\sigma^2} \int \int_{C^*(w)} (uz)^{\nu-1} \exp\left\{-\frac{1}{2u}(1+u^2z^2)\right\} I_{2\eta}(z)\, \mathrm{d}u\, \mathrm{d}z,$$

where

$$C^*(w) = \Big\{(z,\rho) : z, \rho > 0, u^2z^2 + Au < w\Big\}.$$

Hence, we obtain the alternative expression

$$\tilde{P}(s,w) = \frac{4}{\sigma^2} \int_0^\infty \int_0^{h(z)} (uz)^{\nu-1} \exp\left\{-\frac{1}{2u}(1+u^2z^2)\right\} I_{2\eta}(z)\, \mathrm{d}u\, \mathrm{d}z,$$

where

$$h(z) = \frac{\sqrt{A^2+4z^2w} - A}{2z^2} = \frac{2w}{\sqrt{A^2+4z^2w}+A}.$$

In order to avoid the overflow problem, one might use

$$\tilde{P}(s,w) = \frac{4}{\sigma^2} \int_0^\infty z^{\nu-1} F(z,w) I_{2\eta}(z) \exp(-z)\, \mathrm{d}z, \tag{B.1}$$

where

$$F(z,w) = \int_0^{h(z)} u^{\nu-1} \exp\left\{-\frac{1}{2u}(uz-1)^2\right\} \mathrm{d}u.$$

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