

Modeling Credit Value Adjustment With Downgrade-Triggered Termination Clause Using A Ruin Theoretic Approach

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Abstract

Downgrade-triggered termination clause is a recent innovation in credit risk management to control counterparty credit risk. It allows one party of an over-the-counter derivative to close off its position at marked-to-market price when the other party's credit rating downgrades to an agreed alarming level. Although the default risk is significantly reduced, the non-defaulting party may still suffer losses in case that the other party defaults without triggering the termination clause prior to default. At the heart of the valuation of credit risk adjustment(CVA) is the computation of the probability of default. We employ techniques from ruin theory and complex analysis to provide solutions for probabilities of default, which in turn lead to very efficient and accurate algorithms for computing CVA. The underlying risk model in question is an extension of the commercially available KMV-Merton model and hence can be easily implemented. We provide a hypothetical example of CVA computation for an interest-rate swap with downgrade-triggered termination clause. The paper also contributes to ruin theory by presenting explicit solutions to finite-time ruin probabilities in a jump-diffusion model.

Key Words. credit risk management, counterparty credit risk, credit value adjustment, alternative termination event, ruin theory, complex analysis, Laplace transform inversion, finite-time ruin probability.

1 Introduction

The recent financial crisis which started in 2007 with the credit crunch in the U.S. housing market and quickly spread out to nearly every fabric of the global economy has driven financial institutions, regulators and academics around the world to investigate the root causes and to establish new policies, procedures and trading practices to address the imminent problems facing the financial markets. One area under particular scrutiny is the practice of the banking industry on monitoring and managing counterparty credit risk, which has caused a devastating rippling effect on other sectors of the global economy.

Counterparty credit risk refers to the risk of financial losses to one party of a bilateral financial arrangement when the other party fails to fulfill its contract obligation. The financial industry

has traditionally controlled counterparty credit risk by setting credit limit policies and requiring collateral on credit exposures. Thus derivative traders have to forego trading opportunities with credit exposure that exceeds a prescribed credit limit. Nevertheless, numerous examples from the 2007 crisis have shown that even the high profile institutions rated with highest credit ratings by nearly all major rating agencies could suddenly go bankrupt or otherwise suffer severe crippling losses. As the industry weathers through the crisis, many have realized that counterparty credit risk is an inherited risk in trading with each other and the cost of such risk should be reflected appropriately on their derivative books. Among many other changes in practice, many financial institutions have seen transitions from traditional credit limit policies to new accounting standards and procedures for the implementation of Credit Value Adjustment (CVA), which in essence puts a price on the counterparty credit risk. The transition is driven by many factors. (1) Changes in regulation. For instance, Financial Accounting Statement (FAS) 157 sets guidelines for how enterprise must report market or fair value and require them to account for expected losses associated with counterparty defaults. (2) Improved trading practice. Many institutions that have implemented fair value accounting procedures charge CVA from trading departments for credit exposure and hence provide incentives for traders to monitor and manage their overall credit risk. Similar in mechanism to self-insurance, the CVA also provides a buffer to absorb losses from potential counter-party default.

With the increasing complexity of product development in equity-based insurance products, more and more insurance companies entered over-the-counter trading with other financial institutions, inevitably exposing themselves to counterparty credit risk. Although the actuarial profession is well-known for expertise in quantifying, assessing and managing a wide variety of risks associated with the insurance business, there is relatively scarce research work in the actuarial literature on the issue of credit risk modeling. Hence we attempt to initiate a discussion on the modeling and valuation of counterparty credit risk using actuarial techniques.

1.1 Credit Value Adjustment

Counterparty credit risk is similar to many traditional insurable risks in that losses are contingent on random events. However, there are at least two main features of the counterparty credit risk that set it apart from other insurable risks and for which the classical severity-frequency models are not immediately applicable. (1) The risk exposure evolves over time due to the nature of financial derivatives. (2) The losses in the event of counterparty default are also uncertain. The portfolio/asset in the agreement is often marked-to-market at the time of default, which may divert far away from book values. Therefore, the time-varying risk exposure, uncertain loss at default as well as the likelihood of default are all factored into the valuation of CVA in the banking industry.

In a simplified formula, the CVA is often quoted as the expected value of possible losses throughout the term of the arrangement, which can be estimated by the product of (1) loss given default (LGD), which is a percentage of loss due to counterparty default, (2) potential future exposure (PFE), which measures the total value of exposure on each valuation date, and (3) probability of

counterparty default (PD) for each valuation period,

$$\text{CVA} = \text{LGD} \times \sum_{i=1}^n \left[\text{PFE}(t_i) \times \text{PD}(t_{i-1}, t_i) \right], \quad (1.1)$$

where $\text{PFE}(t)$ is the total value of exposure on the valuation date $t = t_0, \dots, t_n$, and $\text{PD}(s, t)$ is the probability of counterparty default between dates s and t . Interested readers may consult Canabarro and Duffie (2003), Crouhy, Galai and Mark (2001) for a variety of models used by banks and regulators to quantify and model counterparty credit risk.

Among all three components used for CVA modeling, the first two factors, namely LGD and PFE, are usually easier to measure or estimate based on market information. The LGD is often assumed in the literature to be a fixed ratio based on specific information on the nature of counterparty transactions. However, if necessary, the randomness in LGD can be accommodated in the simulation of PFE.

According to De Prisco and Rosen (2005), the most prevailing method in market practice of determining the PFE is to compute the distribution of future exposure on OTC derivatives on a set of valuation dates ($\text{PFE}(t), t = t_0, \dots, t_n$) in four steps: (1) *Scenario Generating*. As the payoffs of derivative products are often dependent on cash flows between the involved parties, it is the first and foremost task to generate all possible market scenarios of their trading positions. Each market scenario is a realization of a set of price factors that affect the values of trades in the portfolio. (2) *Instrument Valuation*. Every financial instrument involved is valued at the contractual level for every scenario generated and at every valuation date. (3) *Portfolio Aggregation*. Since financial institutions often use credit risk mitigation techniques such as requiring each other to post collateral when the uncollateralized exposure exceeds a threshold, the effect of these provisions should be considered for each scenario generated. (4) *Statistics Calculation*. The realizations of exposures on all possible scenarios are computed by aggregating all transactions with a counterparty and hence produce an empirical distribution of exposure at the counterparty level. Depending on their own practice, the institutions may choose to compute different statistics for risk monitoring and management.

The most difficult task in the valuation of CVA appears to be the determination of default probabilities. Unlike the loss distribution of other insurable risks, the probability of default for a particular counterparty may not be directly estimated from historical default rates of other firms. Even if one is willing to believe that actual default rates are equal to historical averages, there is often lack of sufficient data on default events for firms of comparable size, debt structure or exposure to similar risks, in order to make any credible estimation. Many have questioned the reliability of historical default data on companies categorized by credit ratings, which are usually defined on a qualitative scale. Readers are recommended to read Chapter 9 of Crouhy, Galai and Mark (2001) for more details.

This technical difficulty has given rise to a vast amount of research work in financial literature on the modeling of default probabilities. We can roughly group the mainstream models into three

categories: (1) Structural Models. The models often propose the asset and liability structure of the counterparty and the event of default is viewed as the first passage time of asset process downcrossing the liability level. Examples of structural models for the valuation of contingent claims and bankruptcy rates can be found in Black and Cox (1976), Leland (1994), Leland and Toft (1996), etc. There are also many empirical studies on a collection of structural models for default probabilities, such as Duffie and Singleton (1997), Huang and Huang (2003), etc. (2) Intensity-Based/Reduced-Form Models. In contrast with the structural models, the intensity-based models regard default as an exogenous random event characterized by a deterministic default intensity function or more generally a stochastic intensity process. (3) Empirical Extraction. Assuming that market prices reflect investors' perception of default rates, the probabilities can be extracted from the term structure of credit-default swap (CDS) spreads, which are directly observable in CDS markets. Interested readers can read Bielecki and Rutkowski (2002) for a comprehensive account of both structural and reduced-form models and Yi (2010) for examples of empirical extraction methods used in the banking industry.

1.2 Credit Value Adjustment Subject to ATE

In the wake of the 2007 credit crisis, new mechanisms for credit risk management have also been developed in the banking industry such as the alternative termination event (ATE) clause which allows investors to close out their positions at market value prior to maturity under certain pre-described circumstances. Among many other specific forms, one type of ATE clause that has been increasingly popular is the downgrade-triggered termination clause, under which one party of the contract may choose to terminate if the credit rating of the other party drops to or below an agreed threshold. Without the protection of such a clause, the credit quality of the counterparty may continue to worsen, eventually leading to default and causing losses to the non-defaulting party. However, the downgrade clause is not a panacea for all counterparty risks. Losses may incur when the counterparty defaults without ever triggering the clause. Hence the underwriting of such clauses complicates the valuation of the CVA and poses a new technical challenge. Information on the downgrade-triggered termination clause can be found in Carver (2011). Throughout the rest of the paper, we shall call it a downgrade trigger for short.

Despite its significance in credit risk management, there have only been very few papers on the valuation of the CVA with downgrade trigger, probably attributable to its fairly short history. It is clear that the method of extracting probabilities of default from CDS spreads is not suitable for modeling CVA with downgrade trigger, since the prices of CDS do not provide adequate information on downgrades. Yi (2010) was among the first to propose models for both unilateral and bilateral CVAs. Zhou (2011) implemented a discrete-time Markov chain model for CVAs subject to ATE with multiple credit downgrade triggers. Both have used simple models but with rather complicated computational schemes.

In this paper, we take a somewhat different approach on this issue of CVA with a downgrade trigger. Rather than starting afresh with new models, we would extend the well-known KMV-

Merton model that has been extensively studied in the literature. The structural model is chosen for this work as it possesses many analytical properties which lead to simple computational algorithms. It was brought to our attention recently that a multi-dimensional model similar to the one-dimensional model used in this paper also appeared in Lipton and Sepp (2009). Three main differences should be noted here. (1) Lipton and Sepp (2009) applies specifically to the CVA valuation for credit default swaps whereas we investigate in this paper a unilateral CVA with downgrade trigger without particular contract assumptions. (2) Their solution method is based on numerical inversion of Laplace transforms but our paper produces explicit expressions for evaluation. (3) No downgrade trigger is considered in their paper.

In what follows, we first introduce the KMV-Merton model and then extend its framework to a jump-diffusion model, which better represents credit downgrades. Then we use a ruin theoretic approach to find solutions to several default-related probabilities required for the computation of CVAs. It should be pointed out that the work also contributes to ruin literature by providing explicit formulas for finite-time ruin probabilities of a perturbed compound Poisson risk model. In the end, we present a numerical example in order to illustrate the procedure of implementing CVAs with and without a downgrade trigger.

2 KMV-Merton Model

The KMV-Merton model is one of the most successful models for default probability forecasting widely used in the financial industry. It was based on the application of option pricing theory to the valuation of corporate debt, originally proposed by Merton (1974) and later developed into commercial products by the KMV corporation, which was later acquired by Moody's. Due to the proprietary nature of this model, there are very few papers that provide technical details on how it was implemented by KMV. Nevertheless, a number of academic papers in finance literature have based their analysis on generic structures when assessing the accuracy and efficiency of the probability of default derived from the KMV-Merton model as well as other accounting-based measures, see, for example, Bharath and Shumway (2008), Hillegeist, Keating, Cram and Lundstedt (2004), etc. Since we shall not use the original KMV-Merton model for the valuation of CVA with downgrade trigger, it is not the purpose of this paper to provide any argument for or against such a model. However, as the model lays the ground for further development in this paper, we shall provide an overview of the original model as well as the pitfalls that appeared in its implementation. Among many reports published by KMV professionals, we shall use Crosbie and Bohn (2003) as our primary reference on the generic structure of the KMV-Merton model.

As with many other structural models, the KMV-Merton model assumes that the total value of a corporate consists of the market value of the firm's debt and that of the equity. Although not directly observable, the evolution of the total value of the firm, denoted by $V = \{V_t, t \geq 0\}$ is assumed to follow a geometric Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$,

$$dV_t = V_t(\mu dt + \sigma_V dW_t), \quad (2.1)$$

where $W = \{W_t, t \geq 0\}$ is a one-dimensional standard Brownian motion and μ, σ_V^2 are called drift and volatility coefficients respectively. The debt of the firm is assumed to be a zero coupon bond maturing in T periods, which is the time horizon of the default probability to be predicted later. According to Crosbie and Bohn (2003), empirical study has shown that companies in general do not default necessarily when the market value of current liability exceeds that of current assets due to the long-term nature of certain liabilities. Therefore, in the KMV model, the face amount of debt, denoted by D , is often taken to be the value of the firm's short-term debt and one half of long-term debt. Because of the protective covenant, the debt holders have the right to force a default when the total value of the firm is perceived to be lower than the value of debt. In the case of default, the debt holders are the first to claim the salvage value of the firm after liquidation. In other words, the equity holders can only claim the amount of firm value in excess of the face amount of debt. It is often argued in essence using Table 1 that the value of equity at the end of the prediction period is equivalent to the payoff of an European call option written on the debt of the firm.

Firm value	$V_T(> D)$	$V_T(\leq D)$	Stock price	$V_T(> D)$	$V_T(\leq D)$
Debt value	D	V_T	Strike price	D	D
Equity value	$V_T - D$	0	Call option	$V_T - D$	0

Table 1: Payoff Table

Therefore, the equity value of the firm can be viewed as the arbitrage-free price of a European call option with the strike equaling the book value of debt and maturity equaling the period of prediction.

$$E_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}(V_T - D)_+], \quad (2.2)$$

where $\mathbb{E}_t^{\mathbb{Q}}$ is the conditional expectation taken under the risk-neutral probability measure \mathbb{Q} with respect to \mathcal{F}_t . Using the well-known Black-Scholes option pricing formula, we obtain the equity value of the firm, denoted by $E = \{E_t, t \geq 0\}$,

$$E_t = W(V_t) := V_t N(d_1) - e^{-r(T-t)} D N(d_2), \quad (2.3)$$

where r is the risk free force of interest, N is the cumulative standard normal distribution function,

$$d_1 = \frac{\ln(V_t/D) + (r + (1/2)\sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}}, \quad d_2 = d_1 - \sigma_V \sqrt{T-t}.$$

Since the firm value V is a geometric Brownian motion, it is an easy consequence of Ito's formula that E is also a geometric Brownian motion. Although the total value of the firm is not observable, one could estimate the market value of equity by multiplying the firm's outstanding shares by its current stock price. The volatility coefficient σ_E can be obtained by the sample mean of log returns

on equity value. In theory, once σ_E is determined, the value σ_V can be inferred from the estimates of σ_E via the relationship

$$\sigma_E = \frac{V_t}{E_t} \frac{\partial E_t}{\partial V_t} \sigma_V = \frac{V_t}{E_t} N(d_2) \sigma_V, \quad (2.4)$$

which follows from the Ito's formula. However, as Crosbie and Bohn (2003, Page 16) argued, "the model linking equity and asset volatility given by [equation (2.4)] holds only instantaneously. In practice the market leverage moves around far too much for [equation (2.4)] to provide reasonable results". Instead, KMV implements an iterative procedure to solve for the asset volatility. First, the market values of the firm's equity are observed and recorded on daily basis and an initial guess of the volatility σ_E is used in (2.3) to determine the corresponding implied market values of assets. Then the implied values of assets are collected to produce new sample mean and variance of their log-values, μ and σ_E^2 respectively. The resulting volatility σ_E is then used as an input to the next round of iteration until the implied volatility converges within a certain tolerable level of estimation error.

According to Crosbie and Bohn (2003), the probability of default is considered to be the probability that the market value of the firm's assets will be less than the book value of the firm's liabilities by the time the debt matures, which is interpreted as

$$p(t, T) = \mathbb{P}_t[V_T \leq D] = \mathbb{P}_t[\ln V_T \leq \ln D], \quad (2.5)$$

where \mathbb{P}_t is the conditional probability with respect to \mathcal{F}_t . Since the stochastic differential equation (2.1) implies that

$$V_T = V_t \exp \left\{ \left(\mu - \frac{1}{2} \sigma_V^2 \right) (T - t) + \sigma_V W_{T-t} \right\},$$

then we obtain

$$p(t, T) = \mathbb{P}_t \left[\ln V_t + \left(\mu - \frac{1}{2} \sigma_V^2 \right) (T - t) + \sigma_V W_{T-t} \leq \ln D \right] = N \left(- \frac{\ln(V_t/D) + (\mu - \sigma_V^2/2)(T - t)}{\sigma_V \sqrt{T - t}} \right). \quad (2.6)$$

Then the KMV model defines the distance-to-default (DD)¹ as

$$\text{DD} = \frac{\ln(V_t/D) + (\mu - \sigma_V^2/2)(T - t)}{\sigma_V \sqrt{T - t}}. \quad (2.7)$$

Although clearly motivated by $p(t, T) = N(-\text{DD})$, it turns out that the KMV company does not use the distance-to-default (2.7) to compute the probability of default through (2.6) but rather developed an empirical mapping from the distance-to-default to historical default and bankruptcy

¹The definition of distance-to-default appears to have been used inconsistently in many reports and research papers from Moody's KMV. Take Crosbie and Bohn (2003) for example. DD is first defined as "the number of standard deviations the asset value is away from default" in formula (2) and then claimed to be formula (14) of that paper under the Black-Scholes model, which is shown as (2.7) in this paper. This inconsistency has been carried on to many books such as Crouhy, Galai and Mark (2001, Section 5.2).

frequencies of all companies with the same distance-to-default. Crosbie and Bohn (2003) claims that “the normal distribution is a very poor choice to define the probability of default”, citing one of the reasons being “the resulting empirical distribution of default rates has much wider tails than the normal distribution”. In other words, as the distance-to-default increases, the probability of default predicted by (2.6) appears to decrease much faster than that predicted by empirical data based on bankruptcy frequencies.

We want to point out two pitfalls in the KMV model that can be easily fixed. (1) The quantity $p(t, T)$ defined in (2.5) is in fact not the probability of default by the time of maturity but rather the probability of default precisely at the time of maturity. In other words, the definition (2.5) used in many KMV reports including Crosbie and Bohn (2003) does not reflect the possibilities that default occurs prior to the time of maturity.

The probability of default by the maturity of debt is the probability that the running minimum of asset value is greater than the book value of liability. Using the well-known result on the running minimum of drifted Brownian motion (c.f. Björk (2004, Proposition 18.4)), we obtain

$$\begin{aligned} \tilde{p}(t, T) &= \mathbb{P}_t \left[\inf_{t \leq s \leq T} V(s) \leq D \right] = \mathbb{P}_t \left[\inf_{t \leq s \leq T} \left\{ \left(\mu - \frac{1}{2} \sigma_V^2 \right) (s - t) + \sigma_V W_{s-t} \right\} \leq \ln \frac{D}{V_t} \right] \\ &= N \left(\frac{\ln(D/V_t) - (\mu - \sigma_V^2/2)(T - t)}{\sigma_V \sqrt{T - t}} \right) + \exp \left\{ \frac{(2\mu - \sigma_V^2) \ln(D/V_t)}{\sigma_V^2} \right\} N \left(\frac{\ln(D/V_t) + (\mu - \sigma_V^2/2)(T - t)}{\sigma_V \sqrt{T - t}} \right) \end{aligned}$$

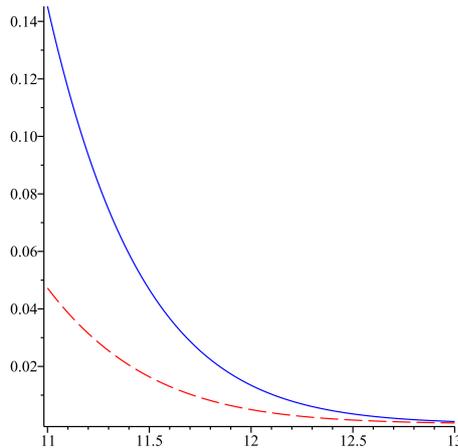


Figure 1: The probability of default $p(t, T)$ (solid) and $\tilde{p}(t, T)$ (dashed) as a function of V_t .

We show through a numerical example that the probability of default $\tilde{p}(t, T)$ is indeed significantly higher than $p(t, T)$. The parameters are taken from the example in Crosbie and Bohn (2003, page 18) where $\mu = 0.07, \sigma_V = 0.0961, D = 10, T - t = 1$. Figure 1 shows that $\tilde{p}(t, T)$ has a heavier tail than that of $p(t, T)$, as the asset value V_t increases from 11 to 13 and consequently the distance-to-default rises from 1.67 to 3.41. Using the particular example given Crosbie and Bohn (2003, page 18), when the distance-to-default defined in (2.7) is 3.0, the asset value is roughly

12.5116. The prediction from $\tilde{p}(t, T)$ is 34 basis points whereas that from $p(t, T)$ is only 13 basis points.

(2) In the same spirit, the payoff of a European call option overestimates the equity value by including the cases where the asset values drop below debt value then rise above the debt value at the end of the prediction period. Thus the valuation of the firm's equity that is consistent with the model assumptions should be given by

$$\tilde{E}_t := \mathbb{E}_t^{\mathbb{Q}} \left[e^{-r(T-t)} (V_T - D)_+ I \left(\inf_{t \leq s \leq T} V_s \geq D \right) \right], \quad (2.8)$$

which is the price of a down-and-out barrier option. It is also known from Björk (2004, Proposition 18.8) that

$$\tilde{E}_t = W(V_t) - \left(\frac{D}{V_t} \right)^{2r/\sigma^2 - 1} W\left(\frac{D^2}{V_t} \right).$$

One should be aware that \tilde{E}_t depends on the time until maturity $T - t$ as embedded in W and thus we shall write $\tilde{E}_t(s)$ as a function of the time until maturity s when necessary.

3 Modeling CVA with Downgrade Trigger

We are interested in the valuation of CVA with an additional provision termed downgrade-triggered termination clause, which has been increasingly common in credit risk management. The downgrade trigger refers to a credit rating level which is below the initial rating of the counter-party by a nationally recognized rating agency at the time of inception. In the event that the rating of the counter-party is downgraded to or below the trigger level prior to maturity, the investor closes off its position with the counter-party, thereby preventing future losses due to continued worsening of its credit quality. However, the investor is not entirely immune from potential losses. For example, if the counter-party defaults immediately after the rating drops below the trigger level, the investor still experiences a loss due to the counter-party default. Thus the key difference in modeling the CVA with and without downgrade trigger lies in the measure of the “unprotected” portion of loss due to the downgrade trigger being inactive prior to default. For brevity, we shall call the probability of default without a proceeding downgrade below downgrade trigger by the probability of jump-to-default.

Although there are a variety of models in finance literature for modeling downgrade transitions, we intend to keep the structure of the KMV-Merton model and seek for a relatively simple extension for the following reasons. (1) Jump-diffusion structural models have been extensively studied in finance literature. There are well developed statistical techniques for model calibration. (2) The KMV model utilizes both market information (stock prices) as well as accounting information (book value of liability), which means the results can be easily adapted as an interpolation tool for the pricing of CVA based on known values of CVA with comparable information.

However, it is difficult to use the KMV-Merton model directly for the CVA with downgrade trigger due to the fact that the geometric Brownian motion does not move fast enough over a short

period of time to capture sudden changes in market value, which are the primary triggering events of a downgrade clause. Hence we introduce a well-known jump-diffusion model proposed by Kou (2002), Kou and Wang (2003), which can be traced back to Merton (1976) with similar structure.

We assume that the market value of asset follows a jump-diffusion process given by

$$dV_t = V_{t-} (\nu dt + \sigma dW_t + dZ_t), \quad Z_t = \sum_{i=1}^{N_t} (U_i - 1), \quad (3.1)$$

where $\{N_t, t \geq 0\}$ is a Poisson process with intensity rate λ independent of W and $\{U_i, i \in \mathbb{N}\}$ is a sequence of independent and identically distributed non-negative random variables with $U_i - 1$ representing percentage change caused by the i th jump in the process. Furthermore, we assume that $Y_i = \ln U_i$ has a bilateral exponential distribution with the density

$$f(y) = p\eta_1 e^{-\eta_1 y} I_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} I_{\{y < 0\}}, \quad \eta_1 > 1, \eta_2 > 0, \quad (3.2)$$

where $p, q \geq 0, p + q = 1$, representing the probabilities of incurring upward and downward jumps. We shall also denote its distribution function by F . It is known from Ito's formula that

$$V_T = V_t \exp \left\{ \left(\nu - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma W_{T-t} + \sum_{k=1}^{N_{T-t}} Y_k \right\}.$$

We introduce the notation to formulate the valuation of the CVA with a downgrade trigger. Consider the market value of assets of the counter-party is driven by the process $V = \{V_t, t \geq 0\}$ defined by the SDE (3.1) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions. Denote by $A = \{A_t, t \geq 0\}$ the evaluation of net present values of future cash flows between two parties from the perspective of an investor, which is assumed to be modeled deterministically in this paper. However, the following analysis does not exclude the possibility that A is modeled by an independent stochastic process adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$. In other words, the process A represents the potential future exposure (PFE) of an OTC transaction between two counter-parties. Let B be the level of asset value corresponding to the downgrade trigger, T be the time of maturity of the contract, l be the ratio of loss given default upon the default of the counter-party. Thus the time of default of the counter-party is given by $\hat{\tau}_D = \inf\{s : V_{t+s} < D\}$ and the first time that the credit rating of a counter-party reaches the downgrade trigger is given by $\hat{\tau}_B = \inf\{s : V_{t+s} < B\}$ where $B > D$. As hinted in Crosbie and Bohn (2003), one could establish a mapping between the V_t (or equivalently DD given μ and σ_V) and the letter-based credit ratings. Consequently, one could also use the same mapping to determine the implied distance-to-default and the corresponding market value B of asset leading to a downgrade.

Hence the primary task of this paper is to provide an efficient and accurate algorithm to compute the following probabilities of default within a prediction horizon $T - t$:

1. Probability of default: $\mathbb{P}\{\hat{\tau}_D < T - t\}$
2. Probability of jump-to-default: $\mathbb{P}\{\hat{\tau}_B < T - t, \hat{\tau}_B = \hat{\tau}_D\}$.

For technical reasons, it is often more convenient to work with the logarithmic asset value, denoted by $X = \{X_s = \ln V_{t+s} - \ln B, s \geq 0\}$, which is a superposition of drifted Brownian motion and compound Poisson process

$$X_s = X_0 + \mu s + \sigma W_s + \sum_{k=1}^{N_s} Y_k, \quad (3.3)$$

with the cumulant generating function

$$\kappa(s) = \ln \mathbb{E}[\exp\{sX_1\}] = \frac{1}{2}\sigma^2 s^2 + \mu s + \lambda \left(\frac{p\eta_1}{\eta_1 - s} + \frac{q\eta_2}{\eta_2 + s} - 1 \right). \quad (3.4)$$

When necessary, we write $\kappa(s; \mu, \lambda, p, \eta_1, q, \eta_2)$ to indicate the input of parameters. Then the aforementioned random times translate to

$$\begin{aligned} \hat{\tau}_D &= \inf\{s : \ln V_{t+s} - \ln B < \ln D - \ln B\} = \inf\{s : X_s < y, X_0 = x\}; \\ \hat{\tau}_B &= \inf\{s : \ln V_{t+s} - \ln B < \ln B - \ln B\} = \inf\{s : X_s < 0, X_0 = x\}, \end{aligned}$$

where

$$x = \ln(V_t/B) > 0, \quad y = \ln(D/B) < 0.$$

For brevity, we introduce the following notation

$$\begin{aligned} \psi(t; x) &= \mathbb{P}\{\tau_0 \leq t\}, & \tau_0 &= \inf\{s : X_s < 0, X_0 = x\}; \\ \phi(t; x, y) &= \mathbb{P}\{\tau_0 \leq t, \tau_0 = \tau_y\}, & \tau_y &= \inf\{s : X_s < y, X_0 = x\}. \end{aligned}$$

These probabilities are well studied for various risk models in ruin literature. The first is often known as the finite-time ruin probability and the second is the (defective) joint probability distribution of time of ruin and deficit at ruin.

We denote the unilateral CVA with trigger level B and debt value D by $\text{CVA}(V_t; B, D)$. Therefore, using the continuous-time version of CVA formula (1.1), we obtain

$$\text{CVA}(V_t; B, D) = l \mathbb{E}[I(\hat{\tau}_B = \hat{\tau}_D \leq T - t)A_{\hat{\tau}_D}] = l \mathbb{E}^x[I(\tau_y = \tau_0 \leq T - t)A_{\tau_0}],$$

where \mathbb{E}^x corresponds to the probability measure \mathbb{P}^x under which $\mathbb{P}^x\{X(0) = x\} = 1$.

Similarly, we could also compute the unilateral CVA without downgrade trigger, denoted by $\text{CVA}(V_t; D)$, where D denotes the debt value. In this case, we define $X = \{X_s = \ln V_{t+s} - \ln D, s \geq 0\}$ and $\tau_0 := \inf\{s : X_s < 0\} = \inf\{s : V_{t+s} < D\} = \hat{\tau}_D$. Thus,

$$\text{CVA}(V_t; D) = l \mathbb{E}[I(\hat{\tau}_D \leq T - t)A_{\hat{\tau}_D}] = l \mathbb{E}^{\hat{x}}[I(\tau_0 \leq T - t)A_{\tau_0}],$$

where $\hat{x} = X_0 = \ln(V_t/D)$.

In other words, the CVA with a downgrade trigger can be determined by

$$\text{CVA}(V_t; B, D) = l \mathbb{E} \left[\int_0^{T-t} A_s d\phi(s; x, y) \right],$$

whereas the CVA without a downgrade trigger can be written as

$$\text{CVA}(V_t; D) = l \mathbb{E} \left[\int_0^{T-t} A_s d\psi(s; \hat{x}) \right].$$

In the next section, we shall first use a ruin theoretic technique to find explicit expressions for the Laplace-Stieltjes transforms for each fixed $x > 0$ and $y < 0$

$$\tilde{\phi}(\delta; x, y) = \int_0^\infty e^{-\delta s} d\phi(s; x, y) = \mathbb{E}^x[e^{-\delta\tau_0} I(X_{\tau_0} < y, \tau_0 < \infty)], \quad \delta \geq 0, \quad (3.5)$$

$$\tilde{\psi}(\delta; \hat{x}) = \int_0^\infty e^{-\delta s} d\psi(s; \hat{x}) = \mathbb{E}^{\hat{x}}[e^{-\delta\tau_0} I(\tau_0 < \infty)], \quad \delta \geq 0. \quad (3.6)$$

Note that $\tilde{\phi}$ and $\tilde{\psi}$ can be also viewed as the Laplace transform of the density of the time of default τ_0 and that of the time of jump-to-default $\tau_0 I(\tau_0 = \tau_y)$.

4 Analytic Solution to Laplace Transforms

4.1 Probabilities of Default

We provide an analytic solution to the Laplace-Stieltjes transform using techniques from ruin theory. Over the past decades, there has been development of systematic tools for analyzing ruin-related quantities. Most notable is the expected discounted penalty function introduced by Gerber and Shiu (1998), which characterizes the joint distribution of the time of default, the surplus immediately prior to ruin and the deficit at ruin. An extension of this function which proves to be convenient in our discussion is the expected present value of operating costs up to ruin, defined as

$$H(x) := \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} l(X_t) dt \right], \quad x \geq 0, \quad (4.1)$$

where the function l represents the running cost of a business modeled by the risk process X and the constant δ is interpreted as the force of interest for discounting. Introduced in Cai et al. (2009), the function is known to encompass a variety of ruin-related quantities such as the expected discounted penalty function, aggregate claims and accumulated utilities up to ruin, etc. Although not immediately evident, simple proofs can be found in Cai et al. (2009) and Feng (2011) that $\tilde{\phi}(\delta; x, y)$ is a special case of H with the cost function $l(x) = \lambda F(y - x)$ and so is $\tilde{\psi}(\delta; x)$ with $l(x) = \lambda F(-x) + \Delta(x)$ where Δ is the Dirac delta function that assigns mass one to the point zero, and F is the distribution function of the density (3.2).

Theorem 4.1. *The Laplace transforms defined in (3.5) and (3.6) with $\delta > 0$ are given by*

$$\tilde{\phi}(\delta; x, y) = \frac{(\eta_2 + \gamma_1)(\eta_2 + \gamma_2)e^{\eta_2 y}}{\eta_2(\gamma_2 - \gamma_1)} [e^{\gamma_1 x} - e^{\gamma_2 x}], \quad x \geq 0, y < 0, \quad (4.2)$$

$$\tilde{\psi}(\delta; x) = \frac{\gamma_2(\eta_2 + \gamma_1)}{\eta_2(\gamma_2 - \gamma_1)} e^{\gamma_1 x} - \frac{\gamma_1(\eta_2 + \gamma_2)}{\eta_2(\gamma_2 - \gamma_1)} e^{\gamma_2 x}, \quad x \geq 0, \quad (4.3)$$

where $\gamma_1 = \gamma_1(\delta)$ and $\gamma_2 = \gamma_2(\delta)$ are the only two negative roots of $\kappa(s) = \delta$.

Proof. For brevity, we shorten the notation $\tilde{\phi}(\delta; x, y)$ and $\tilde{\psi}(\delta; x)$ as $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ in this proof. It is easy to show using Theorem 2.1 of Feng (2011) and Proposition 3.1 of Cai et al. (2009) that $\tilde{\phi}(x)$ satisfies the integro-differential equation

$$\frac{1}{2}\sigma^2\tilde{\phi}''(x) + \mu\tilde{\phi}'(x) + \lambda \int_{-x}^{\infty} \tilde{\phi}(x+z) dF(z) - (\lambda + \delta)\tilde{\phi}(x) + l(x) = 0, \quad x > 0, \quad (4.4)$$

where

$$l(x) = \lambda \int_{-\infty}^{y-x} dF(z) = \lambda q \int_{x-y}^{\infty} \eta_2 e^{-\eta_2 t} dt = \lambda q e^{-\eta_2(x-y)}.$$

The equation is easy to solve using the properties of two operators

$$\mathcal{T}_s f(x) = e^{sx} \int_x^{\infty} e^{-sy} f(y) dy, \quad \mathcal{E}_s f(x) = e^{-sx} \int_0^x e^{sy} f(y) dy.$$

We use \mathcal{D} to denote the differentiation operator and then write the equation in terms of operators

$$\left\{ (\lambda + \delta) - \mu\mathcal{D} - \frac{1}{2}\sigma^2\mathcal{D}^2 \right\} \tilde{\phi}(x) = \lambda(p\eta_1\mathcal{T}_{\eta_1} + q\eta_2\mathcal{E}_{\eta_2})\tilde{\phi}(x) + l(x).$$

Since $(s - \mathcal{D})\mathcal{T}_s f = f$ and $(s + \mathcal{D})\mathcal{E}_s f = f$, we obtain

$$P(\mathcal{D})\tilde{\phi}(x) = (\eta_1 - \mathcal{D})(\eta_2 + \mathcal{D})l(x) = 0,$$

where

$$P(s) = (\eta_1 - s)(\eta_2 + s) \left\{ (\lambda + \delta) - \mu s - \frac{1}{2}\sigma^2 s^2 \right\} - \lambda[p\eta_1(\eta_2 + s) + q\eta_2(\eta_1 - s)].$$

It is easy to see that the equation $P(s) = 0$ is equivalent to $\kappa(s) = \delta$ and has four roots, two of which are negative, to be denoted by γ_1 and γ_2 .

Since the probability function $\tilde{\phi}$ must be bounded, the solution must be of the form

$$\tilde{\phi}(x) = Ae^{\gamma_1 x} + Be^{\gamma_2 x}, \quad x > 0.$$

Making a substitution into (4.4) and using the fact that $P(\gamma_1) = 0$ and $P(\gamma_2) = 0$ gives

$$\frac{\eta_2 A}{\eta_2 + \gamma_1} + \frac{\eta_2 B}{\eta_2 + \gamma_2} = e^{\eta_2 y}.$$

Letting $x = 0$ leads to $\tau_0 = 0$ and $X_{\tau_0} = 0 > y$ which means $\tilde{\phi}(0) = 0$, i.e. $A + B = 0$. Therefore, we can determine both A and B and hence obtain the solution (4.2).

The derivation of $\tilde{\psi}(x)$ is nearly identical to that of $\tilde{\phi}(x)$ and we must have

$$\tilde{\psi}(x) = Ae^{\gamma_1 x} + Be^{\gamma_2 x}, \quad x > 0,$$

where A and B are subject to slightly different boundary conditions. Since $l(x) = \lambda q e^{-\eta_2 x}$ for $x > 0$, we also make the substitution into the integro-differential equation for $\tilde{\psi}(x)$ similar to (4.4) and thus obtain

$$\frac{\eta_2 A}{\eta_2 + \gamma_1} + \frac{\eta_2 B}{\eta_2 + \gamma_2} = 1.$$

Note that $\tilde{\psi}(0) = 1$ due to the Dirac delta function Δ . Thus we arrive at the solution (4.3). \square

To facilitate the model calibration, we also want to have an easy way to compute the tail probability functions

$$\begin{aligned}\bar{\phi}(s; x, y) &= \mathbb{P}\{\tau_0 > s, X_{\tau_0} < y\} = \int_s^\infty d\phi(t; x, y); \\ \bar{\psi}(s; x) &= \mathbb{P}\{\tau_0 > s, \tau_0 < \infty\} = \int_s^\infty d\psi(t; x).\end{aligned}$$

It is easy to show using integration by parts that their Laplace transforms are given by

$$\int_0^\infty e^{-\delta s} \bar{\phi}(s; x, y) ds = \frac{\tilde{\phi}(0; x, y) - \tilde{\phi}(\delta; x, y)}{\delta}, \quad \int_0^\infty e^{-\delta s} \bar{\psi}(s; x) ds = \frac{\tilde{\psi}(0; x) - \tilde{\psi}(\delta; x)}{\delta}. \quad (4.5)$$

Thus it only remains to determine the values of $\tilde{\phi}(0; x, y)$ and $\tilde{\psi}(0; x)$.

Corollary 4.1. *If $\mu + \lambda(p/\eta_1 - q/\eta_2) > 0$,*

$$\begin{aligned}\tilde{\phi}(0; x, y) &= \frac{(\eta_2 + \gamma_1(0))(\eta_2 + \gamma_2(0))e^{\eta_2 y}}{\eta_2(\gamma_2(0) - \gamma_1(0))} [e^{\gamma_1(0)x} - e^{\gamma_2(0)x}], \quad x \geq 0, y < 0, \\ \tilde{\psi}(0; x) &= \frac{\gamma_2(0)(\eta_2 + \gamma_1(0))}{\eta_2(\gamma_2(0) - \gamma_1(0))} e^{\gamma_1(0)x} - \frac{\gamma_1(0)(\eta_2 + \gamma_2(0))}{\eta_2(\gamma_2(0) - \gamma_1(0))} e^{\gamma_2(0)x}, \quad x \geq 0,\end{aligned}$$

where $\gamma_1(0)$ and $\gamma_2(0)$ are the only two negative roots of $\kappa(s) = 0$. If $\mu + \lambda(p/\eta_1 - q/\eta_2) \leq 0$,

$$\begin{aligned}\tilde{\phi}(0; x, y) &= \frac{(\eta_2 + \gamma_1(0))e^{\eta_2 y}}{\gamma_1(0)} (1 - e^{\gamma_1(0)x}), \quad x \geq 0, y < 0, \\ \tilde{\psi}(0; x) &= 1, \quad x \geq 0,\end{aligned}$$

where $\gamma_1(0)$ is the only negative root of $\kappa(s) = 0$.

Proof. It is clear from the graph of κ in Figure 2(a) that if $\kappa'(0) \leq 0$, the only two negative roots $\gamma_1(\delta), \gamma_2(\delta)$ converge to $\gamma_1(0)$ and 0 respectively as $\delta \rightarrow 0$. As shown in Figure 2(b), if $\kappa'(0) > 0$, the only two negative roots $\gamma_1(\delta), \gamma_2(\delta)$ converge to $\gamma_1(0)$ and $\gamma_2(0)$ as $\delta \rightarrow 0$. Although we only draw graphs of the case where $p = 0$ and $q = 1$ in Figure 2, the same results can be shown for all general cases. The conclusion follows from that $\kappa'(0) = \mathbb{E}[X_1] = \mu + \lambda(p/\eta_1 - q/\eta_2)$. \square

We can now tell that $\phi(s; x, y)$ is a defective probability distribution function whereas $\psi(s; x)$ is a proper probability distribution function when $\mu + \lambda(p/\eta_1 - q/\eta_2) \leq 0$ and defective otherwise.

Remark 4.1. *In the case that $p = 0, \eta_2 = \eta$, we can obtain simple expressions for γ_1^* and γ_2^* .*

$$\gamma_1^* = \frac{-(\mu + \eta\sigma^2/2) - \sqrt{(\mu - \eta\sigma^2/2)^2 + 2\sigma^2\lambda}}{\sigma^2}, \quad \gamma_2^* = \frac{-(\mu + \eta\sigma^2/2) + \sqrt{(\mu - \eta\sigma^2/2)^2 + 2\sigma^2\lambda}}{\sigma^2}. \quad (4.6)$$

Note that if $\mu - \lambda/\eta > 0$, both γ_1^* and γ_2^* are negative roots of $\kappa(s) = 0$. If $\mu - \lambda/\eta \leq 0$, then γ_1^* is the only negative root of $\kappa(s) = 0$.

4.2 Valuation of Equity

In the framework of the KMV-Merton model, the market value of the firm is not observable. The parameters are usually estimated indirectly through the observable data on equity values viewed as an option on firm value. The European call option has been used due to its analytical tractability. As we have explained earlier, the valuation of equity as a down-and-out barrier option appears to be more consistent with the purpose of forecasting the probability of default over a period of time.

The pricing of barrier option under the double exponential jump-diffusion model has been developed in Kou and Wang (2003) with an example of up-and-in call option. Following the same technique as shown in Kou and Wang (2003), we can easily show that the Laplace transform of the down-and-out call option (2.8) follows immediately from the formula for the probability of default.

It is known that in general jump-diffusion models do not lead to a unique risk neutral measure. Kou (2002) employed a rational expectation equilibrium argument to derive a particular risk neutral measure. In the KMV-Merton model, the parameters are estimated from observed data on “options” and hence we do not need to specify the particular choices of measure. Under any risk neutral measure, the market value of assets is given by

$$V_T = V_t \exp \{X_{T-t}\}, \quad \text{where } X_s = r^*s + \sigma W_s + \sum_{k=1}^{N_s} Y_k,$$

where r is the risk-free force of interest earned in a bank account and

$$r^* := r - \frac{1}{2}\sigma^2 - \lambda\zeta, \quad \zeta := \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

Theorem 4.2. *If the asset value of the firm is driven by the jump-diffusion process (3.1), the Laplace transform of the asset value E_t defined by (2.8) is given by*

$$\begin{aligned} \tilde{E}(\delta) &:= \int_0^\infty e^{-\delta s} E_t(s) ds \\ &= \frac{V_t}{\delta} \left\{ 1 - \frac{\hat{\gamma}_2(\tilde{\eta}_2 + \hat{\gamma}_1)}{\tilde{\eta}_2(\hat{\gamma}_2 - \hat{\gamma}_1)} \left(\frac{V_t}{D}\right)^{\hat{\gamma}_1} + \frac{\hat{\gamma}_1(\tilde{\eta}_2 + \hat{\gamma}_2)}{\tilde{\eta}_2(\hat{\gamma}_2 - \hat{\gamma}_1)} \left(\frac{V_t}{D}\right)^{\hat{\gamma}_2} \right\} \\ &\quad - \frac{D}{\delta + r} \left\{ 1 - \frac{\bar{\gamma}_2(\eta_2 + \bar{\gamma}_1)}{\eta_2(\bar{\gamma}_2 - \bar{\gamma}_1)} \left(\frac{V_t}{D}\right)^{\bar{\gamma}_1} + \frac{\bar{\gamma}_1(\eta_2 + \bar{\gamma}_2)}{\eta_2(\bar{\gamma}_2 - \bar{\gamma}_1)} \left(\frac{V_t}{D}\right)^{\bar{\gamma}_2} \right\}, \end{aligned}$$

where $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are the two negative roots of $\kappa(s; r^*, \lambda, p, \eta_1, q, \eta_2) = \delta$, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the two negative roots of $\kappa(s; r^*, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{q}, \tilde{\eta}_2) = \delta + r$, and

$$\tilde{p} = \frac{p\eta_1}{(1 + \zeta)(\eta_1 - 1)}, \tilde{q} = \frac{q\eta_2}{(1 + \zeta)(\eta_2 + 1)}, \tilde{\lambda} = \lambda(1 + \zeta), \tilde{\eta}_1 = \eta_1 - 1, \tilde{\eta}_2 = \eta_2 + 1. \quad (4.7)$$

Proof. We apply the usual technique of a change of measure using the Esscher transform

$$\left. \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} \right|_{\mathcal{F}_T} = e^{-r(T-t)} \frac{V_T}{V_t} = \exp\{X_{T-t} - r(T-t)\}.$$

Interested readers may read Gerber and Shiu (1994) for details on the Esscher transform. It is easy to show that under $\tilde{\mathbb{Q}}_t$, the logarithmic market value process X is still a double exponential jump-diffusion process but with a different set of parameters given in (4.7). The risk free force of interest r^* and volatility coefficient σ do not change under this measure.

It follows from the definition (2.8) that

$$\begin{aligned}\tilde{E}_t(T-t) &= \mathbb{E}_t^{\tilde{\mathbb{Q}}}[e^{-r(T-t)}V_T I(V_T \geq D, \hat{\tau}_D > T-t)] - De^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[I(V_T \geq D, \hat{\tau}_D > T-t)] \\ &= V_t \mathbb{E}_t^{\tilde{\mathbb{Q}}}[I(\hat{\tau}_D > T-t)] - De^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[I(V_T \geq D, \hat{\tau}_D > T-t)].\end{aligned}$$

For notational brevity, we introduce $\hat{X}_s = \ln V_{t+s} - \ln D$ for all $s \geq 0$ and hence $\hat{X}_0 = \ln(V_t/D) > 0$. Similarly we must have

$$\tau_0 = \inf\{s \geq 0 : \hat{X}_s \leq 0\} = \hat{\tau}_D = \inf\{s \geq 0 : V_{t+s} \leq D\}.$$

Therefore,

$$\begin{aligned}\tilde{E}_t(s) &= V_t \tilde{\mathbb{Q}}_t\{\tau_0 > s\} - De^{-rs} \mathbb{Q}_t\{\tau_0 > s\} \\ &= V_t [1 - \tilde{\mathbb{Q}}_t\{\tau_0 \leq s\}] - De^{-rs} [1 - \mathbb{Q}_t\{\tau_0 \leq s\}].\end{aligned}$$

Taking the Laplace transform gives

$$\tilde{E}(\delta) = \frac{V_t}{\delta} [1 - \tilde{\psi}_1(\delta; \ln(V_t/D))] - \frac{D}{\delta + r} [1 - \tilde{\psi}_2(\delta + r; \ln(V_t/D))],$$

where $\tilde{\psi}_1(\delta, x)$ is the Laplace transform of the probability density function of the time of default under measure $\tilde{\mathbb{Q}}_t$ and $\tilde{\psi}_2(\delta, x)$ is that under measure \mathbb{Q}_t . Hence the two are of the same form but with different sets of parameters. \square

5 Laplace Transform Inversion

We give an illustration on the analytic inversion of Laplace transforms. Consider in this section the jump diffusion model (3.3) with downward jumps only. We shall write η_2 as η , $\tilde{\phi}(\delta; x, y)$ as $\tilde{\phi}(\delta)$ for short. However, the analysis in this section can be easily extended to the general model (3.3).

Let $\sigma, \mu, \lambda, \eta$ be given positive numbers. Consider the function

$$\kappa(s) = \frac{1}{2}\sigma^2 s^2 + \mu s - \frac{\lambda s}{s + \eta}.$$

We want to solve the equation $\kappa(s) = \delta$ for s . Note that this equation is equivalent to

$$g(s; \delta) := (s + \eta)(\kappa(s) - \delta) = 0,$$

where $g(s; \delta)$ is a polynomial in s and δ . It is of degree 3 in s . Therefore, the inverse function of κ is an algebraic function defined on a Riemann surface of 3 sheets; see Knopp (1975), Part II. Typical graphs of κ are shown in Figure 2.

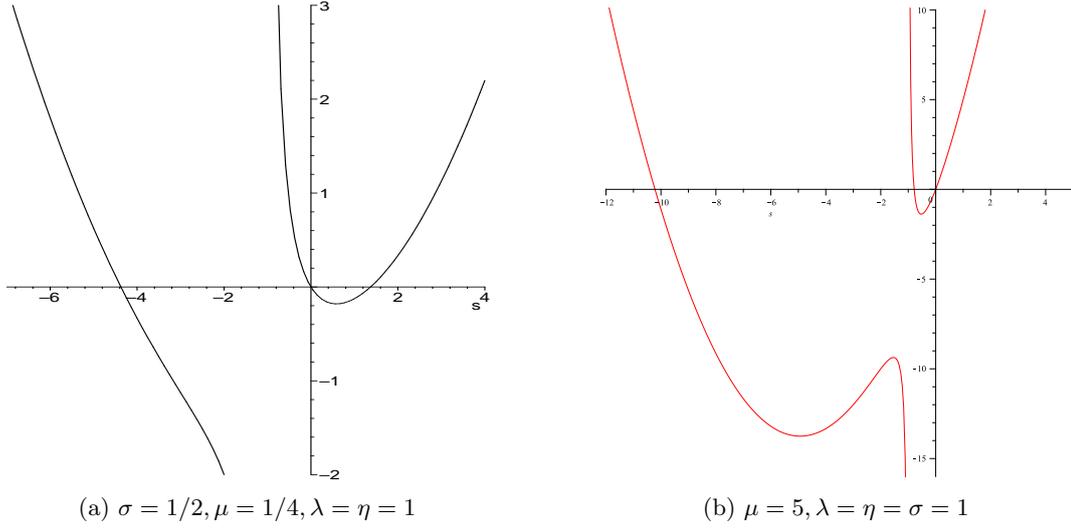


Figure 2: Graphs of $\kappa(s)$ where $p = 0$ and $q = 1$

If $\delta > 0$ then the equation $\kappa(s) = \delta$ admits three real solutions $s = \gamma_i(\delta)$, $i = 1, 2, 3$, that we order according to

$$\gamma_1(\delta) < -\eta < \gamma_2(\delta) < 0 < \gamma_3(\delta).$$

Let a be the minimum value of $\kappa(s)$ for $s > -\eta$. If $a < \delta \leq 0$ then there are still three real solutions, and $\delta = a$ is a branch point between $\gamma_2(\delta)$ and $\gamma_3(\delta)$.

If $\delta < a$ there are two possibilities to be considered.

1. There is only one real root of $\kappa(s) = \delta$ for $\delta \in (-\infty, a)$ as in Figure 2(a);
2. Let c and b be the local minimum and maximum values of $\kappa(s)$ for $s < -\eta$. Then there are three roots of $\kappa(s) = \delta$ for $\delta \in [b, c]$ as in Figure 2(b). Note that the borderline cases $\kappa(s) = b$ and $\kappa(s) = c$ both of which have double roots are included.

The inverse function of κ has branch points in the complex δ -plane which can be found as the zeros of the discriminant of $g(s; \delta) = 0$ with respect to s . This discriminant is a cubic polynomial in δ . Therefore, there are at most three branch points such as $\delta = a$ in Figure 2(a) and $\delta = a, b, c$ in Figure 2(b). The values of a, b, c are determined in Remark 6.1. Consider a path in the complex δ -plane starting at a point $\delta_0 > 0$ but avoiding the branch points. We also assume that the path lies in the cut plane $\mathbb{C}^* = \mathbb{C} \setminus (-\infty, 0]$. Along such a path each function $\gamma_i(\delta)$ can be continued analytically. Consider the Taylor expansion

$$\gamma_1(\delta) = \gamma_1(\delta_0) + \gamma_1'(\delta_0)(\delta - \delta_0) + O((\delta - \delta_0)^2)$$

as $\delta \rightarrow \delta_0$. In particular, as $\epsilon \rightarrow 0$,

$$\gamma_1(\delta_0 + \epsilon i) = \gamma_1(\delta_0) + \gamma_1'(\delta_0)\epsilon i + O(\epsilon^2). \quad (5.1)$$

Since $\gamma_1(\delta)$ is a decreasing function of $\delta > 0^2$ (as is evident from the fact that $s = \gamma_1(\delta) < -\eta$ is the only solution of $\kappa(s) = \delta > 0$), we have $\gamma_1'(\delta_0) < 0$. Therefore, (5.1) shows that, for ϵ sufficiently close to 0,

$$\epsilon \Im \gamma_1(\delta_0 + \epsilon i) < 0.$$

It is obvious that solutions s of $\kappa(s) = \delta$ are never real if $\Im \delta \neq 0$. Therefore, by the intermediate value theorem, the sign of the imaginary part of the analytic continuation of $\gamma_1(\delta)$ is opposite to that of $\Im \delta$ whenever $\Im \delta \neq 0$. The same argument holds with $\gamma_2(\delta)$ in place of $\gamma_1(\delta)$. Analogously, $\gamma_3(\delta)$ is an increasing function of $\delta > 0$ and so its analytic continuation has an imaginary part of the same sign as $\Im \delta \neq 0$. Therefore, $s = \gamma_3(\delta)$ is the only solution of $g(s; \delta) = 0$ whose imaginary part has the same sign as that of $\Im \delta \neq 0$. This shows that $\gamma_3(\delta)$ cannot have branch points in \mathbb{C}^* , and so, by the monodromy theorem, $\gamma_3(\delta)$ is an analytic function on \mathbb{C}^* . Moreover, while $\gamma_1(\delta), \gamma_2(\delta)$ may have branch points in \mathbb{C}^* , the unordered pair $\{\gamma_1(\delta), \gamma_2(\delta)\}$ is well-defined for $\delta \in \mathbb{C}^*$. This pair consists of the two solutions of $g(s; \delta) = 0$ whose imaginary parts have signs opposite to that of $\Im \delta \neq 0$.

For $\delta > 0$ we consider the function

$$\tilde{\phi}(\delta) = \frac{(\gamma_1(\delta) + \eta)(\gamma_2(\delta) + \eta)e^{y\eta}}{\eta(\gamma_2(\delta) - \gamma_1(\delta))} \left(e^{x\gamma_1(\delta)} - e^{x\gamma_2(\delta)} \right), \quad (5.2)$$

where $x > 0$ and y are given constant. This function has the form

$$\tilde{\phi}(\delta) = K(\gamma_1(\delta), \gamma_2(\delta)),$$

where K is an analytic function on \mathbb{C}^2 which is symmetric, that is, $F(u, v) = F(v, u)$. Therefore, by analytic continuation, $\tilde{\phi}(\delta)$ becomes an analytic function in \mathbb{C}^* (the branch points of the inverse function of κ lying in \mathbb{C}^* are removable singularities of $\tilde{\phi}$.) We can actually compute $\tilde{\phi}(\delta)$ as follows. For given $\delta \in \mathbb{C}$ with $\Im \delta \neq 0$, we compute the zeros of $g(s; \delta) = 0$. These zeros can be found numerically or using Cardano's formula for the solution of the cubic equation. Two of them have imaginary part opposite to that of $\Im \delta$. We use these solutions as γ_1 and γ_2 in (5.2). We note that the calculation of $\tilde{\phi}(\delta)$ for complex values of δ is required for several of the numerical inversion methods of the Laplace transform of $\tilde{\phi}(\delta)$; see Abate and Whitt (1995).

Since we know that $\tilde{\phi}(\delta)$ is analytic on \mathbb{C}^* we can find the inverse Laplace transform $f(t)$ of $\tilde{\phi}(\delta)$ in a more convenient way as follows. We start with the well known formula

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\delta t} \tilde{\phi}(\delta) d\delta. \quad (5.3)$$

²To prove this directly one needs to show that $\kappa(s)$ is decreasing for $s < -\eta$ as long as $\kappa(s) > 0$. We assume that $s < -\eta < 0$ and $\kappa(s) > 0$. Then $\frac{1}{2}\sigma^2 s + \mu - \frac{\lambda}{s+\eta} < 0$, so $\mu < \frac{\lambda}{s+\eta} - \frac{1}{2}\sigma^2 s$. Using this inequality we obtain

$$\kappa'(s) = \sigma^2 s + \mu - \frac{\lambda\eta}{(s+\eta)^2} < \frac{1}{2}\sigma^2 s + \frac{\lambda s}{(s+\eta)^2} < 0.$$

In the integral (5.3) we deform the path of integration towards the negative δ -axis as described in Doetsch (1974, Section 25, page 161). We obtain that

$$f(t) = \frac{1}{2\pi i} \int_C e^{\delta t} \tilde{\phi}(\delta) d\delta, \quad (5.4)$$

where C is the path coming from $-\infty$ following the lower boundary of the cut $(-\infty, 0]$ until 0 and then returning to $-\infty$ along the upper boundary of the cut. We can show that $\delta \tilde{\phi}(\delta)$ is a bounded function on \mathbb{C}^* (the proof is omitted). Therefore, Doetsch (1974, Theorem 25.1, page 162) justifies the step from (5.3) to (5.4). If $\tilde{\phi}(\delta \pm i0)$ denote the values of $\tilde{\phi}$ on the cut for $\delta < 0$ then $\tilde{\phi}(\delta + i0)$ is conjugate to $\tilde{\phi}(\delta - i0)$. Therefore, we can simplify (5.4) to

$$f(t) = \frac{1}{\pi} \int_{-\infty}^0 e^{\delta t} \Im \tilde{\phi}(\delta - i0) d\delta. \quad (5.5)$$

The function $\Im \tilde{\phi}(\delta - i0)$ can be computed for given $\delta < 0$ as follows. If the equation $g(s; \delta) = 0$ has only real zeros (as happens for $a \leq \delta < 0$) then $\Im \tilde{\phi}(\delta - i0) = 0$ and there is nothing to compute. If $g(s, \delta) = 0$ has one real zero (which is negative) and two complex conjugate zeros then we evaluate $\tilde{\phi}(\delta - i0)$ by using (5.2) with γ_1 and γ_2 replaced by the negative zero and the zero with positive imaginary part. A typical graph of $\Im \tilde{\phi}(\delta - i0)$ is shown in Figure 3.

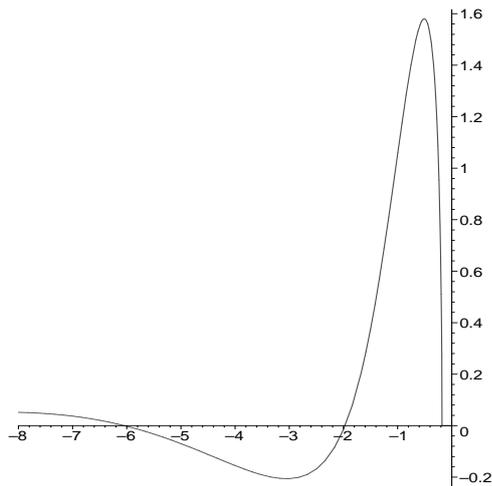


Figure 3: Graph of $\Im \tilde{\phi}(\delta - i0)$ for $\sigma = \frac{1}{2}, \mu = \frac{1}{4}, \lambda = \eta = 1$

Numerical experiments show that the computation of the function $f(t)$ based on formula (5.5) is very effective especially if t is large. A similar analysis can be carried out for $\tilde{\psi}(\delta; x)$ as well as the Laplace transforms of $\bar{\psi}$ and $\bar{\phi}$ defined in (4.5).

6 Finite-time Ruin Probability

In this section, we divert to show a by-product of our analysis on the inverse Laplace transform. In recent ruin literature, explicit solutions have been found to finite-time ruin probabilities in a

variety of risk models, such as Asmussen (1984), Dickson and Willmot (2005), etc. for compound Poisson risk models; Dickson and Li (2010), Borovkov and Dickson (2008), Landriault et al. (2011), etc. for the Sparre Andersen risk models. The list is far from being comprehensive. However, it appears that finite-time ruin probabilities were not previously known for jump-diffusion models such as (3.3). We shall now provide a new addition to the list of explicit solutions in the framework of (3.3) with downward exponential jumps only.

Theorem 6.1. *The finite-time ruin probability is given by*

$$\begin{aligned} \psi(t; x) &= \tilde{\psi}(0; x) \\ &+ \frac{1}{\pi} \int_S \frac{s_1 \{ (s_1 + \eta) z e^{wx} \cos(zx) - (s_1 + \eta) z e^{s_1 x} - [(\eta + w)(w - s_1) + z^2] e^{wx} \sin(zx) \} e^{\delta t}}{\delta \eta [(w - s_1)^2 + z^2]} d\delta, \end{aligned} \quad (6.1)$$

where $S = \{\delta < 0 : D(\delta) > 0\}$ and all other expressions are real and given by

$$\begin{aligned} s_1 &= u + v - \frac{A}{3}, w = -\frac{1}{2}(u + v) - \frac{A}{3}, z = \frac{\sqrt{3}}{2}(u - v), A = \frac{2\mu + \eta\sigma^2}{\sigma^2}, B = \frac{2(\eta\mu - \lambda - \delta)}{\sigma^2}, C = \frac{-2\eta\delta}{\sigma^2}, \\ D(\delta) &= \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, u = \sqrt[3]{-\frac{q}{2} + \sqrt{D(\delta)}}, v = \sqrt[3]{-\frac{q}{2} - \sqrt{D(\delta)}}, p = B - \frac{1}{3}A^2, q = \frac{2}{27}A^3 - \frac{1}{3}AB + C. \end{aligned}$$

Proof. We can compute $\psi(t; x)$ by $\psi(t; x) = \tilde{\psi}(0; x) - \bar{\psi}(t; x)$. The first term $\tilde{\psi}(0; x)$ is determined in Corollary 4.1. In view of (4.5), the second term $\bar{\psi}(t; x)$ can be found by inverting the Laplace transform

$$\bar{\psi}(t; x) = \frac{1}{\pi} \int_{-\infty}^0 \Im \left(\frac{\tilde{\psi}(0; x) - \tilde{\psi}(\delta; x)}{\delta} \right) e^{\delta t} d\delta = -\frac{1}{\pi} \int_{-\infty}^0 \Im \left(\tilde{\psi}(\delta; x) \right) \frac{1}{\delta} e^{\delta t} d\delta.$$

Thus the proof is completed when the imaginary part of $\tilde{\psi}(\delta; x)$ is identified. To this end, we introduce Cardano's formula for cubic roots. Since $\gamma_1(\delta)$ and $\gamma_2(\delta)$ are both solutions to the cubic equation $\alpha s^3 + \beta s^2 + \theta s + \zeta = 0$, where $\alpha = \sigma^2/2, \beta = \mu + \eta\sigma^2/2, \theta = \eta\mu - \lambda - \delta, \zeta = -\eta\delta$, we can work with the equivalent cubic equation $s^3 + As^2 + Bs + C = 0$ and then set $s = y - A/3$ to obtain $y^3 + py + q = 0$ with the discriminant $D(\delta)$. If $D(\delta) < 0$ there are three real roots and if $D(\delta) > 0$ there is one real root and a pair of complex conjugate roots. If $D(\delta) < 0$, then $\tilde{\psi}(\delta - i0)$ is real so $\Im \tilde{\psi}(\delta - i0) = 0$ and we do not have to compute anything. If $D(\delta) > 0$ then we introduce real numbers u and v . Thus the three roots are

$$y_1 = u + v, y_2 = -\frac{1}{2}(u + v) + i\frac{\sqrt{3}}{2}(u - v), y_3 = -\frac{1}{2}(u + v) - i\frac{\sqrt{3}}{2}(u - v).$$

The solutions of the original cubic equation are $s_i = y_i - a/3$. It is clear that s_1 is real and s_2, s_3 are conjugate with $\Im s_2 > 0$. Therefore, we substitute the two expressions s_1 for $\gamma_1(\delta)$ and s_2 for $\gamma_2(\delta)$ into (4.3) for $\tilde{\psi}(\delta - i0)$ with negative δ . We can now calculate the imaginary part of this expression since we have formulas for real part and imaginary part of s_2 while all other numbers are real. Knowing that $s_2 = w + iz$, we obtain the formula (6.1) after algebraic simplification. \square

Remark 6.1. The constant $\tilde{\psi}(0; x)$ is given in Corollary 4.1 with the constants in (4.6). The domain of integration S can be determined easily by either of the following two cases. Note that for some constants a^*, b^*, c^*, d^* , we can write

$$D(\delta) = a^* \delta^3 + b^* \delta^2 + c^* \delta + d^*.$$

These coefficients can be obtained easily using computer algebra systems such as Maple. Similar to previous proof, we rewrite $D(\delta) = 0$ as $\delta^3 + A^* \delta^2 + B^* \delta + C^* = 0$ with $A^* = b^*/a^*, B^* = c^*/a^*, C^* = d^*/a^*$. By setting $\delta = t - A^*/3$, we obtain the depressed cubic equation

$$t^3 + p^* t + q^* = 0, \quad \text{where } p^* = B^* - \frac{1}{3}(A^*)^2, q^* = \frac{2}{27}(A^*)^3 - \frac{1}{3}A^*B^* + C^*,$$

which has three zeros

$$t_k = 2\sqrt{-\frac{p^*}{3}} \cos\left(\frac{1}{3} \arccos\left(\frac{3q^*}{2p^*} \sqrt{\frac{-3}{p^*}}\right) - k \frac{2\pi}{3}\right), \quad k = 0, 1, 2.$$

Let $D^* = (q^*/2)^2 + (p^*/3)^3$. If $D^* > 0$, then $\kappa(s) = \delta$ has only one real root for $\delta \in (-\infty, 0]$ as shown in Figure 2(a) and the unique real root $a = \sqrt[3]{-q^*/2 + \sqrt{D^*}} + \sqrt[3]{-q^*/2 - \sqrt{D^*}} - A^*/3$ and hence $S = (-\infty, a)$. If $D^* \leq 0$, then $\kappa(s) = \delta$ has three real roots for $\delta \in (-\infty, 0]$ as appeared in Figure 2(b), $a = t_0 - A^*/3, b = t_1 - A^*/3, c = t_2 - A^*/3$ and $S = (-\infty, c) \cup (b, a)$.

The formula (6.1) can be viewed as a generalization of the well-known formulas for finite-time ruin probability in the classical compound Poisson model with exponential claims. See for example, Ch. V. Prop. 1.3 of Asmussen and Albrecher (2010), Theorem 5.6.4 of Rolski et al. (1998) for the explicit formulas. Since the classical compound Poisson model can be viewed as the limiting case of the model (3.3) as σ goes to zero, we give a numerical example in Table 2 to illustrate the convergence of $\psi(t; x)$ to its counterpart finite-time ruin probability in the classical model, denoted by $\psi_c(t; x)$. The values of $\psi_c(t; x)$ are computed using both (5.6.11) of Rolski et al. (1998) and Ch.V (1.6) of Asmussen and Albrecher (2010).

σ	$\psi(t; x)$	$\psi_c(t; x)$
0.1	0.2470612116	0.2459378310
0.01	0.2459490849	0.2459378310
0.001	0.2459379434	0.2459378310
0.0001	0.2459378320	0.2459378310

Table 2: Comparison of finite-time ruin probabilities ($t = x = \mu = \lambda = \eta = 1$)

We could also reproduce the classical result using the method described in Section 5. In the classical compound Poisson model, the function κ reduces to

$$\kappa(s) = \mu s - \frac{\lambda s}{s + \eta},$$

where $\lambda, \mu, \eta > 0$. Taking the limit of (4.3) as σ goes to zero, the Laplace transform of finite-time ruin probability is given by

$$\tilde{\psi}_c(\delta; x) = \frac{\eta + \gamma_2}{\eta} e^{\gamma_2 x},$$

where γ_2 is the non-positive root of $k(s) = 0$. Note that $k(s) = 0$ is equivalent to

$$g(s) = \mu s^2 + \mu \eta s - \lambda s - \delta s - \delta \eta = 0. \quad (6.2)$$

Thus,

$$\gamma_2 = \frac{(\lambda + \delta - \mu \eta) - \sqrt{(\lambda + \delta - \mu \eta)^2 + 4\mu \delta \eta}}{2\mu}.$$

Since the discriminant of (6.2) is given by

$$\Delta(\delta) = \delta^2 + 2(\lambda + \mu \eta)\delta + (\lambda - \mu \eta)^2,$$

then γ_2 must have a non-zero imaginary part when

$$\delta \in S := (-(\sqrt{\lambda} + \sqrt{\mu \eta})^2, -(\sqrt{\lambda} - \sqrt{\mu \eta})^2),$$

and be real when $\delta \in \mathbb{R} \setminus S$. Therefore, using the same method as in the proof of Theorem 6.1, we obtain

$$\psi_c(t; x) = \tilde{\psi}_c(0; x) + \frac{1}{\pi} \int_S \left\{ \frac{\eta + w(\delta)}{\delta \eta} e^{w(\delta)x} \sin(z(\delta)x) + \frac{z(\delta)}{\delta \eta} e^{w(\delta)x} \cos(z(\delta)x) \right\} e^{\delta t} d\delta, \quad (6.3)$$

where

$$w(\delta) := \frac{\lambda + \delta - \mu \eta}{2\mu}, \quad z(\delta) := \frac{\sqrt{|\Delta(\delta)|}}{2\mu}.$$

Following Corollary 4.1, we note that if $\mu > \lambda/\eta$, then

$$\tilde{\psi}_c(0; x) = \frac{\lambda}{\mu \eta} e^{(\lambda/\mu - \eta)x}, \quad x \geq 0,$$

otherwise,

$$\tilde{\psi}_c(0; x) = 1, \quad x \geq 0.$$

For consistency, we can verify the formula (6.3) is equivalent to the formulas in Rolski et al. (1998). However, we find that Theorem 5.6.4 of Rolski et al (1998) contains a few typos which we corrected in Appendix. In the integral of (6.3), we substitute

$$\delta = -\lambda - \mu \eta + 2\sqrt{\lambda \mu \eta} \cos y, \quad 0 \leq y \leq \pi.$$

Then

$$|\Delta(\delta)| = 4\lambda \mu \eta (1 - \cos^2 y) = 4\lambda \mu \eta \sin^2 y.$$

We can then use the addition formula for sine under the integral, and some simplifications lead to the formula in Theorem 5.6.4 of Rolski et al. (1998) upon correction.

Similarly, we can also obtain explicit formulas for the finite-time probability of ruin with deficit greater than a certain level. This formula was not previously derived even for the classical compound Poisson risk model. The proof is very similar to that of Theorem 6.1 and hence omitted.

Theorem 6.2. *The finite-time probability of jump-to-ruin is given by*

$$\phi(t; x, y) = \tilde{\phi}(0; x, y) + \frac{1}{\pi} \int_S \frac{(s_1 + \eta)e^{\eta y} \{[(w + \eta)(w - s_1) + z^2]e^{wx} \sin(xz) + z(s_1 + \eta)(e^{s_1 x} - e^{wx} \cos(zx))\}}{\delta \eta [(w - s_1)^2 + z^2]} e^{\delta t} d\delta,$$

7 Numerical Implementation of CVAs

In this section we illustrate the whole procedure for implementation from model calibration to the computation of CVAs using a specific numerical example.

Effective date	5-Mar-2007
Termination date	5-Mar-2010
Notional principal	USD 100 million
Payment dates	Each 5-Mar and 5-Sep, commencing 5-Sep-2007, up to and including 5-Mar-2010.
<i>Fixed amounts</i>	
Fixed-rate payer	Microsoft
Fixed rate	5% per annum
<i>Floating amounts</i>	
Floating-rate payer	Intel
Floating rate	USD 6-month LIBOR

Table 3: Exact from hypothetical swap confirmation

Suppose we want to put the price of counterparty default risk on a hypothetical interest rate swap between Microsoft and Intel by computing the CVA subject to early determination due to credit rating downgrades. This hypothetical contract is described in Hull (2008, Chapter 7, page 148) which we summarize in Table 3 for the sake of completeness.

Step 1: Estimating PFE on each valuation date

The agreement specifies that payments are to be exchanged every 6 months and all rates are quoted semiannual nominal. Since we shall assess the impact of counterparty risk of Intel from the standpoint of Microsoft, we first reproduce the cash flows to Microsoft in Table 4.

We use the method of valuation in terms of bond prices discussed in Hull (2008, Section 7.7) to price the interest rate swaps. For simplicity, we only assess the impact of default on semiannual basis, although the valuation can be done with higher frequency or even continuously. In order to calculate the present values, we also make assumptions on yield rates for periods up until maturity, which could be extracted from Eurodollar futures. We consider the market values of the interest rate swap as the total exposure of Microsoft to the credit risk from Intel during each period. Here we list the assumptions on the LIBOR zero curve as well as the corresponding value of interest rate swaps in Table 5. Again all rates in this table are quoted semiannual nominal.

Date	6m LIBOR (%)	Received	Paid	Net
5-Mar-2007	4.20			
5-Sep-2007	4.80	+2.10	-2.50	-0.40
5-Mar-2008	5.30	+2.40	-2.50	-0.10
5-Sep-2008	5.50	+2.65	-2.50	+0.15
5-Mar-2009	5.60	+2.75	-2.50	+0.25
5-Sep-2009	5.90	+2.80	-2.50	+0.45

Table 4: Cash flows to Microsoft

k	Date	6m LIBOR	12m LIBOR	18m LIBOR	24m LIBOR	30m LIBOR	PFE $_k$
0	5-Mar-2007	4.20	-	-	-	-	0
1	5-Sep-2007	4.80	4.90	5.20	5.30	5.50	0.48525954
2	5-Mar-2008	5.30	5.50	5.80	6.00	-	1.83211570
3	5-Sep-2008	5.50	5.70	5.90	-	-	1.264783917
4	5-Mar-2009	5.60	5.90	-	-	-	0.858142332
5	5-Sep-2009	5.90	-	-	-	-	0.437105391

Table 5: The value of Interest Rate Swap (IRS) to Microsoft

Step 2: Calibrating model parameters for PD

In principle, the calibration of the model (3.3) can be done similarly to that of the KMV-Merton model. Taking advantage of the well-developed estimation methods described in Section 2, we can assume that the drift and volatility coefficients are taken from the KMV-Merton model without jump components. Then the parameters from the jump component, namely $\lambda, p, \eta_1, \eta_2$ can be chosen to adjust the default probabilities. As stated in Crosbie and Bohn (2003), the predictions on probabilities of default from the geometric Brownian motion are often too low to be credible for commercial use. With the freedom of extra parameters from the jump component, the model (3.3) is equipped with additional leverage to match the probabilities of default to an empirical level within a certain time horizon. This technique known as quantile matching is often used in the insurance industry for calibrating parameters of asset pricing models. Readers may read Hardy (2003, Chapter 4) for detailed discussion.

In this numerical example, we assume that the best estimates $\mu = 0.07$ and $\sigma = 0.0961$ are already known from data analysis of Intel's stock prices using KMV-Merton iterative methods and the estimated asset value of Intel is $V_t = \$12.5116$ billions and the book value of its liability is $D = \$9.0948$ billion, which corresponds to the distance-to-default of 4.0 according to (2.7). (Some of these parameters were quoted from Crosbie and Bohn (2003, page 18) used for an unknown entity.) Since Crosbie and Bohn (2003) argued that empirical study shows the distance-of-default of four corresponds to a default rate of around 100 basis points, we want to ensure that the remaining parameters are chosen so that the one-year probability of default matches the empirical estimate.

In this case we use the model (3.3) with downward jumps only, which represent hazardous

events that significantly reduce the credit quality of Intel. We assume that $\lambda = 1/10$ which can be roughly interpreted as saying hazardous events come once every ten years on average. We use an iterative algorithm based on a bisection method to seek for the parameter value η such that $\psi(t; \hat{x}) = 0.01$ where $t = 1, \hat{x} = \ln(V_t/D) = \ln(12.5116/9.0948)$. Our algorithm produces $\eta = 8.0518$ with accuracy up to four decimal places.

We want to point out that in this example the parameter λ is rather arbitrary. When implementing this model, one could numerically determine both λ, η in the model with downward jumps only or either $\lambda, p, \eta_1, \eta_2$ using more complex iterative schemes so that more points of probability of default can be matched with empirical estimates.

Step 3: Computing PD

Having obtained all the parameters by quantile matching, we are now ready to compute the probability of default and the probability of jump-to-default using the Laplace inversion developed earlier.

Suppose when entering the interest rate swap with Intel, Microsoft attached an alternative terminating event clause which states the swap should be closed out at market value in the event that Intel's credit rating downgrades to BBB on the Standard & Poor's scale. Assume that we have the KMV's empirical mapping from credit rating to the DD available and can hence determine that the downgrade trigger corresponds to a distance-to-default of around 3, which translates to the market value $B = 10$ using the formula (2.7). Then we determine in Table 6 both probabilities of default and jump-to-default on half-yearly basis up to the maturity of the interest rate swap.

Valuation date	$t_1 = 1/2$	$t_2 = 1$	$t_3 = 3/2$	$t_4 = 2$	$t_5 = 5/2$	$t_6 = 3$
$\psi(t; \hat{x})$	47.73	100	154.33	208.46	259.82	306.96
$\phi(t; x, y)$	36.39	68.55	96.15	119.49	139.22	155.93

Table 6: The probabilities of default on Intel (in bps)

Step 4: Valuation of CVAs

In the last step, we combine the information collected on PD and PFE to compute the CVAs. In this example, we assume the loss to Microsoft regarding this contract at the default of Intel is the market value of the swap and hence the LGD is taken to be one. Therefore, the CVA with the downgrade trigger is given by

$$\text{CVA}(12.5116; 10, 9.0948) = \sum_{k=1}^5 \text{PFE}_k \times [\phi(t_k; x, y) - \phi(t_{k-1}; x, y)] = 0.013882054(\text{millions}).$$

Similarly, we can compute the CVA without the downgrade trigger

$$\text{CVA}(12.5116; 9.0948) = \sum_{k=1}^5 \text{PFE}_k \times [\psi(t_k; \hat{x}) - \psi(t_{k-1}; \hat{x})] = 0.025654281(\text{millions}).$$

The Maple code and spreadsheets used in this numerical example can be provided upon request.

Appendix: Erratum to Theorem 5.6.4 of Rolski et al. (1998)

Readers should be reminded that the notation in Rolski et al. (1998) is different from ours. For better readability, we provide the erratum with the original notation from their book. The integral at the bottom of page 203 should read: If $0 < b \leq 1$ then

$$\frac{1}{\pi} \int_0^\pi \frac{\sin y \sin(my)}{1 + b^2 - 2b \cos y} dy = \begin{cases} 2^{-1}b^{m-1} & \text{if } m \geq 1, \\ 0 & \text{if } m = 0, \\ -2^{-1}b^{-m-1} & \text{if } m \leq -1. \end{cases}$$

If $b > 1$ then

$$\frac{1}{\pi} \int_0^\pi \frac{\sin y \sin(my)}{1 + b^2 - 2b \cos y} dy = \begin{cases} 2^{-1}b^{-m-1} & \text{if } m \geq 1, \\ 0 & \text{if } m = 0, \\ -2^{-1}b^{m-1} & \text{if } m \leq -1. \end{cases}$$

Theorem 5.6.4 should read as follows. If $c = \delta\beta/\lambda \geq 1$, then

$$\psi(u; x) = c^{-1}e^{-(c-1)c^{-1}\delta u} - e^{-\delta u - (1+c)\lambda x} \frac{1}{\pi} \int_0^\pi g(\delta u, \lambda x, y) dy,$$

where $c = \delta\beta/\lambda$ and

$$g(w, \theta, y) = 2 \frac{e^{(2\sqrt{c}\theta + w/\sqrt{c}) \cos y}}{1 + c - 2\sqrt{c} \cos y} \left(\sin y \sin\left(y + \frac{w}{\sqrt{c}} \sin y\right) \right).$$

If $0 < c < 1$, then

$$\psi(u; x) = 1 - e^{-\delta u - (1+c)\lambda x} \frac{1}{\pi} \int_0^\pi g(\delta u, \lambda x, y) dy.$$

It can be shown analytically that the corrected formulas agree with the formula (6.3).

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