A matrix operator approach to the analysis of ruin-related quantities in the phase-type renewal risk model

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Corrected Version

Abstract

It is well-known in ruin theory that the expected present value of penalty at ruin satisfies a defective renewal equation in the Erlang-n renewal risk model. This paper presents a new matrix operator approach to derive a parallel defective renewal equation for the expected present value of total operating costs in a phase-type renewal risk model and hence provides explicit matrix analytic solutions to a variety of ruin-related quantities.

Keywords: ruin theory; Gerber-Shiu function; total operating costs; defective renewal equation; phase-type distribution; compound geometric distribution.

1 Introduction

In classical renewal risk models, it is assumed that the incoming cash flows of an insurance company are solely generated by continuous premium income collected at the constant rate c per time unit and the outgoing cash flows are determined by a sequence of insurance claims. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying usual conditions, we consider that the arrival times of individual claims $\{T_1, T_2, T_3, \cdots\}$ follow an adapted renewal process. In other words, the inter-claim times $\{T_1, T_2 - T_1, T_3 - T_2, \cdots\}$ are mutually independent and identically distributed. The sequence of claims $\{Y_1, Y_2, \cdots\}$ is independent of the sequence of arrival times and all claims are mutually independent with a common density function $q(y)$ with Laplace transform $\tilde{q}(s) = \int_0^\infty e^{-sy} q(y) dy$. Hence, on $(\Omega, \mathcal{F})$ together with a family of probability measures $\{\mathbb{P}^x, x \in \mathbb{R}\}$, we consider the insurer's surplus process $X = \{X_t, t \geq 0\}$ as the balance of the two opposing cash flows , i.e.

$$X_t = x + ct - \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0,$$

where the number of claims up to time t is given by $N(t) = \max\{n : T_n < t\}$.

Furthermore, if we impose the condition that the inter-claim times follow the generalized Erlang-n distribution with the Laplace transform

$$\tilde{k}(s) = \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + s}, \tag{1.1}$$

many interesting results were known in the literature about this Erlang-n renewal risk model.

As a crucial objective of ruin theory is to quantify and measure the risk of insolvency associated with the insurance business, we are particularly interested in quantities pertaining to the event of ruin. One of such quantities was introduced by Gerber and Shiu (1998) in the context of classical compound Poisson risk model. It is the *expected present value of penalty at ruin*, often referred to as the *Gerber-Shiu function*, defined by

$$m(x) = \mathbb{E}^x[e^{-\delta\tau_0} w(X_{\tau_0-}, |X_{\tau_0}|) I(\tau_0 < \infty)], \qquad x \geq 0,$$

where the time of ruin τ_0 is determined by $\tau_0 = \inf\{t : X_t < 0\}$, the constant $\delta \geq 0$ is the force of interest for discounting purpose and the penalty function w is a functional of both the surplus immediately prior to ruin X_{τ_0-} and the deficit at ruin $|X_{\tau_0}|$. The function I is an indicator such that $I(A) = 1$ if A is true and $I(A) = 0$ otherwise.

An operator known as the Dickson-Hipp operator played an important role in the Erlang-n renewal risk model. For any $s > 0$ and function f for which such an integral exists, the Dickson-Hipp operator is defined by

$$\mathcal{T}_s f(x) = e^{sx} \int_x^\infty e^{-sy} f(y)\, \mathrm{d}y.$$

It was shown in Li and Garrido (2004), Gerber and Shiu (2005) that the Gerber-Shiu function satisfies the defective renewal equation, known as Li's renewal equation in the literature,

$$m(x) = \frac{\lambda_1 \cdots \lambda_n}{c^n} \int_0^x m(x-y) \prod_{i=1}^{n} \mathcal{T}_{\rho_i} q(y)\, \mathrm{d}y + \frac{\lambda_1 \cdots \lambda_n}{c^n} \prod_{i=1}^{n} \mathcal{T}_{\rho_i} \alpha(x), \tag{1.2}$$

where

$$\alpha(x) = \int_0^\infty w(x, y-x) q(y)\, \mathrm{d}y$$

and the constants $\rho_1, \cdots, \rho_n$ are the n roots with non-negative real parts of the equation

$$\tilde{k}(\delta - cs)\tilde{q}(s) = 1.$$

A recent paper by Cai *et al.* (2009) studied another quantity with analogy to the expected present value of penalty at ruin. Cai *et al.* argued that it is sensible for an insurer to be able to quantify and control business costs in long term. The suggested measure of business costs was the *expected present value of total operating costs up to default*, defined by

$$H(x) = \mathbb{E}^x \left[\int_0^{\tau_d} e^{-\delta t} l(X_t)\, \mathrm{d}t \right], \qquad x \geq 0,$$

where the time of default τ_d is determined by $\tau_d = \inf\{t : X_t < d\}$ and the cost function l is a functional of the surplus level X_t. It was shown in the classical compound Poisson model that the function H encompasses the entire family of Gerber-Shiu function and furthermore includes a variety of other ruin-related quantities such as dividends paid up to ruin, etc.

Having seen such a generalization in the classical compound Poisson model, one may wonder whether a similar defective renewal equation to (1.2) also exists for the function H in the Erlang-n renewal risk model. This paper indeed provides an affirmative answer. However, due to its mathematical tractability, this problem is best pursued in a more general renewal risk model in which inter-claim times are phase-type distributed.

The phase-type renewal risk model is a topic of active research in the current literature, which includes, among others, Asmussen (2000), Jacobsen (2003), Schmidli (2005), Ren (2007), Li (2008). Albrecher and Boxma (2005) proposed a more general renewal risk model from which many solutions in this paper can be retrieved as well. For a more detailed account of phase-type renewal models, readers are referred to Asmussen (2000).

The phase-type renewal risk model assumes that the inter-claim time is determined by the time till absorption of a homogenous continuous-time Markov chain with an absorbing state. Throughout the paper, we shall use $\mathbf{0}, \mathbf{1}, \mathbf{I}$ for a column vector of all 0's, 1's and an identity matrix, all of required dimension, respectively. Suppose the Markov chain denoted by $J = \{J_t, t \geq 0\}$ has the transient state space $E = \{1, 2, \cdots, m\}$ with a non-singular sub-intensity matrix $\mathbf{\Lambda}$. Hence, in contrast with (1.1), the Laplace transform of the inter-claim time distribution in the phase-type renewal risk model is given by

$$\tilde{k}(s) = -\mathbf{a}^\top (\mathbf{\Lambda} - s\mathbf{I})^{-1} \boldsymbol{\eta},$$

where $\boldsymbol{\eta} = -\mathbf{\Lambda}\mathbf{1}$ is the exit rate vector and $\mathbf{a}$ is the entrance law vector which specifies the probability mass distribution of the state in which J starts off at time 0.

Note that the sojourn time in each transient state of J is exponentially distributed. Hence introducing an auxiliary state variable will enable us to work in the desirable Markovian structure. For this precise reason, Feng (2009) introduced the expected present value of total operating costs with the state variable, defined by

$$H(x,i)=\mathbb{E}^{(x,i)}\left[\int_0^{\tau_d} e^{-\delta t} l(X_t,J_t)\,\mathrm{d}t\right], \tag{1.3}$$

where $l(x,i)$ represents the operating costs incurred when the surplus X is at the level x and the Markov chain J is in the state i. We often denote $\mathbf{H}(x)=\Big(H(x,1),\cdots,H(x,n)\Big)^{\top}$ and $\mathbf{l}(x)=\Big(l(x,1),\cdots,l(x,n)\Big)^{\top}$. Hence, if J starts at random according to the entrance law $\mathbf{a}$, then the expected present value of total operating costs in the usual sense is given by

$$H(x)=\mathbb{E}^{x}\left[\int_0^{\tau_d} e^{-\delta t} l(X_t,J_t)\,\mathrm{d}t\right]=\mathbf{a}^{\top}\mathbf{H}(x). \tag{1.4}$$

All specific examples of the function $\mathbf{H}$ and $\mathbf{l}$ considered in this paper are bounded.

In Section 2, we shall introduce a matrix version of Dickson-Hipp operator $\mathcal{T}$. The main result is presented in Section 3 that the function H defined in (1.4) satisfies the defective renewal equation in the phase-type renewal risk model

$$H(x)=\frac{1}{c}\int_0^{x} H(x-y)\mathbf{a}^{\top}\mathcal{T}_{\mathbf{R}}q(y)\boldsymbol{\eta}\,\mathrm{d}y+\frac{1}{c}\mathbf{a}^{\top}\mathcal{T}_{\mathbf{R}}\mathbf{l}(x), \tag{1.5}$$

with the matrix $\mathbf{R}$ to be discussed in Section 4. Section 5 is devoted to the derivation of matrix analytic solutions to many ruin-related quantities through the defective renewal equation (1.5).

While the main result in Feng (2009) shows that the function H defined in (1.3) satisfies a unified integro-differential equation in a general piecewise-deterministic Markov risk model, this paper works out solutions to the function H from the perspective of defective renewal equations in the smaller class of phase-type renewal model. Despite their similarities, Feng (2009) provides solutions through solving systems of differential equations but the solutions in this paper are direct results of the defective renewal equation. As a result of the two distinctive approaches, most solutions in Feng (2009) rely solely on roots of negative real parts of a matrix Lundberg equation whereas all matrix analytic solutions in this paper depends only on its roots of non-negative real parts.

2 Matrix Dickson-Hipp operator

The Dickson-Hipp operator was introduced by Dickson and Hipp (2001) in the context of Erlang-2 renewal risk model and later extensively exploited in more general renewal risk models by Li and Garrido (2004a, 2004b), Gerber and Shiu (2005), etc. Albrecher *et al.*

(2009) further developed a new operator algebraic approach involving both the Dickson-Hipp operator and other Green operators in a general Sparre Andersen model.

As it shall become clear soon, we need a matrix version of the Dickson-Hipp operator which facilitates an efficient derivation of a general defective renewal equation.

Definition 2.1. For any function f, the matrix Dickson-Hipp operator is defined by

$$\mathcal{T}_{\mathbf{S}} f(x) = e^{\mathbf{S}x} \int_x^\infty e^{-\mathbf{S}u} f(u)\,\mathrm{d}u,$$

with the matrix $\mathbf{S}$ for which the integral exists.

A sufficient condition for the existence of the matrix Dickson-Hipp operator is that f is bounded and all eigenvalues of $\mathbf{S}$ have non-negative real parts.

Lemma 2.1. For all matrices $\mathbf{S}_1$ and $\mathbf{S}_2$, we have

$$\mathcal{T}_{\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)\mathcal{T}_{\mathbf{S}_2}\mathbf{f} = \mathcal{T}_{\mathbf{S}_1}\mathbf{f} - \mathcal{T}_{\mathbf{S}_2}\mathbf{f}.$$

Proof: Substituting $w = u + v$, we have

$$\begin{aligned}
\mathcal{T}_{\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)\mathcal{T}_{\mathbf{S}_2}\mathbf{f}(x) &= \int_0^\infty \int_0^\infty e^{-u\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)e^{-v\mathbf{S}_2}\mathbf{f}(x+u+v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^\infty \left(\int_0^w e^{-u\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)e^{u\mathbf{S}_2}\,\mathrm{d}u \right) e^{-w\mathbf{S}_2}\mathbf{f}(x+w)\,\mathrm{d}w.
\end{aligned}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}u} e^{-u\mathbf{S}_1} e^{u\mathbf{S}_2} = e^{-u\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)e^{u\mathbf{S}_2},$$

which implies

$$\int_0^w e^{-u\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)e^{u\mathbf{S}_2}\,\mathrm{d}u = e^{-w\mathbf{S}_1}e^{w\mathbf{S}_2} - \mathbf{I}.$$

Therefore, we get

$$\mathcal{T}_{\mathbf{S}_1}(\mathbf{S}_2 - \mathbf{S}_1)\mathcal{T}_{\mathbf{S}_2}\mathbf{f}(x) = \int_0^\infty e^{-w\mathbf{S}_1}\mathbf{f}(x+w)\,\mathrm{d}w - \int_0^\infty e^{-w\mathbf{S}_2}\mathbf{f}(x+w)\,\mathrm{d}w$$

completing the proof. □

For reasons to be seen later, we need to be cautious about two representations of the convolution operator for which we use distinct notations $*$ and $\star$. For integrable functions f, q and Q defined on $[0, \infty)$, we have

$$\begin{aligned}
f * Q(x) &= \int_0^x f(x-y)\,\mathrm{d}Q(y), \qquad x \geq 0, \\
f \star q(x) &= \int_0^x f(x-y)q(y)\,\mathrm{d}y, \qquad x \geq 0.
\end{aligned}$$

Lemma 2.2. Suppose q is a scalar function and let $\tilde{q}(\mathbf{S}) = \int_0^\infty e^{-\mathbf{S}y} q(y)\,\mathrm{d}y$, then

$$\mathcal{T}_{\mathbf{S}}\{f \star q\}(x) = \tilde{q}(\mathbf{S})\mathcal{T}_{\mathbf{S}} f(x) + \mathcal{T}_{\mathbf{S}} q \star f(x).$$

Proof: For any scalar $z > 0$ for which $z\mathbf{I} - \mathbf{S}$ is invertible, we must have

$$\begin{aligned}
\mathcal{T}_z\{\mathcal{T}_{\mathbf{S}}\{f \star q\}\}(0) &= (z\mathbf{I} - \mathbf{S})^{-1}[\mathcal{T}_{\mathbf{S}}\{f \star q\}(0) - \mathcal{T}_z\{f \star q\}(0)] \\
&= (z\mathbf{I} - \mathbf{S})^{-1}[\tilde{q}(\mathbf{S})\tilde{f}(\mathbf{S}) - \tilde{q}(z)\tilde{f}(z)] \\
&= (z\mathbf{I} - \mathbf{S})^{-1}[\tilde{q}(\mathbf{S})\tilde{f}(\mathbf{S}) - \tilde{q}(\mathbf{S})\tilde{f}(z) + \tilde{q}(\mathbf{S})\tilde{f}(z) - \tilde{q}(z)\tilde{f}(z)] \\
&= \tilde{q}(\mathbf{S})\mathcal{T}_z\mathcal{T}_{\mathbf{S}} f(0) + \mathcal{T}_z\mathcal{T}_{\mathbf{S}} q(0)\tilde{f}(z).
\end{aligned}$$

Taking inverse Laplace transform on both sides, we obtain the desired equality. □

Lemma 2.3. Let $\tilde{q}(s) = \int_0^\infty e^{-sy}\,\mathrm{d}Q(y)$ and $\overline{B}(x) = \mathcal{T}_{\mathbf{S}}\overline{Q}(x)$, then

$$\mathcal{T}_{\mathbf{S}}\{f * Q\}(x) = \tilde{q}(\mathbf{S})\mathcal{T}_{\mathbf{S}} f(x) + f * B(x).$$

Proof: Recall that $f \star q = f * Q$ if $\overline{Q}(x) = \mathcal{T}_0 q(x)$. The desired equality is obtained from Lemma 2.2 with the fact that $\overline{B}(x) = \mathcal{T}_0\mathcal{T}_{\mathbf{S}} q(x) = \mathcal{T}_{\mathbf{S}}\mathcal{T}_0 q(x) = \mathcal{T}_{\mathbf{S}}\overline{Q}(x)$. □

3 Defective renewal equation

Theorem 3.1. In the phase-type renewal risk model, the vector function $\mathbf{H}$ defined in (1.3) satisfies a matrix defective renewal equation

$$\mathbf{H}(x) = \frac{1}{c}\int_0^x \left[\mathcal{T}_{\mathbf{R}} q(y)\boldsymbol{\eta}\mathbf{a}^\top\right]\mathbf{H}(x-y)\,\mathrm{d}y + \frac{1}{c}\mathcal{T}_{\mathbf{R}}\mathbf{l}(x), \tag{3.1}$$

where the $m \times m$ symmetric matrix $\mathbf{R}$ is uniquely determined by

$$\left[\int_0^\infty e^{-\mathbf{R}y}\,\mathrm{d}Q(y)\right]\boldsymbol{\eta}\mathbf{a}^\top = \delta\mathbf{I} - \boldsymbol{\Lambda} - c\mathbf{R}. \tag{3.2}$$

Proof: Let $\mathbf{P}_{ij}(t) = \mathbb{P}[J_t = j | J_0 = i]$ be the probability that J starts off in state i and happens to be in state j at time t. It is easy to show that $\mathbf{P}(t) = (\mathbf{P}_{ij}(t)) = \exp(\boldsymbol{\Lambda}t)$. To fix the initial state of J in state i, we let $\mathbf{a}^\top = (\mathbf{0}, 1, \mathbf{0})$ be the unit vector with only i-th element being 1 and hence the probability density functions of the time till absorption of J with $J_0 = i$ is given by $k_i(t) = \sum_{j\in E}\mathbf{P}_{ij}(t)\boldsymbol{\eta}_j$ for all $t \geq 0$. For brevity, we put the density of inter-claim time for all possible initial states in a matrix form $\mathbf{k}(t) = \mathbf{P}(t)\boldsymbol{\eta}$.

We use the law of total probability to consider two scenarios. (1) If there is a claim, we condition on the time t and size y of the first claim to find the present value of operating costs to be paid in the future as the surplus restarts at $x+ct-y$ and the Markov chain regenerates itself according to $\mathbf{a}$. (2) If there is no claim, we still have to count the operating costs at all surplus levels. Then we arrive at the sum of two integrals

$$\begin{aligned}\mathbf{H}(x) &= \int_0^\infty e^{-\delta t}\left\{\mathbf{k}(t)\int_0^{x+ct} \mathbf{a}^\top \mathbf{H}(x+ct-y)\,\mathrm{d}Q(y) + \mathbf{P}(t)\mathbf{l}(x+ct)\right\}\mathrm{d}t \\ &= \int_0^\infty e^{(\boldsymbol{\Lambda}-\delta)t}\left\{\int_0^{x+ct} \boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}(x+ct-y)\,\mathrm{d}Q(y) + \mathbf{l}(x+ct)\right\}\mathrm{d}t. \qquad (3.3)\end{aligned}$$

Let $\mathbf{S}=(1/c)(\delta\mathbf{I}-\boldsymbol{\Lambda})$. Since $\boldsymbol{\Lambda}$ is non-singular subintensity matrix, all of its eigenvalues are negative and hence all eigenvalues of $\delta\mathbf{I}-\boldsymbol{\Lambda}$ are positive. Therefore, the matrix $\mathbf{S}$ is positive definite.

Making a change of variables yields

$$\mathbf{H}(x) = \frac{1}{c}\mathcal{T}_{\mathbf{S}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H} * Q + \mathbf{l}\}(x). \qquad (3.4)$$

Suppose there exists a matrix $\mathbf{R}$ for which $\mathcal{T}_{\mathbf{R}}\mathbf{H}$ exists. Applying Lemma 2.1 and 2.3, we obtain

$$\begin{aligned}&\mathcal{T}_{\mathbf{S}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H} * Q + \mathbf{l}\}(x) = \mathcal{T}_{\mathbf{R}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H} * Q + \mathbf{l}\}(x) - \mathcal{T}_{\mathbf{R}}(\mathbf{S}-\mathbf{R})\mathcal{T}_{\mathbf{S}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H} * Q + \mathbf{l}\}(x) \\ &= \tilde{q}(\mathbf{R})\mathcal{T}_{\mathbf{R}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}\}(x) + \mathcal{T}_{\mathbf{R}}q \star \boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}(x) + \mathcal{T}_{\mathbf{R}}\mathbf{l}(x) - \mathcal{T}_{\mathbf{R}}(\mathbf{S}-\mathbf{R})\mathcal{T}_{\mathbf{S}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H} * Q + \mathbf{l}\}(x) \\ &= \tilde{q}(\mathbf{R})\mathcal{T}_{\mathbf{R}}\{\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}\}(x) + \mathcal{T}_{\mathbf{R}}q \star \boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}(x) + \mathcal{T}_{\mathbf{R}}\mathbf{l}(x) - c\mathcal{T}_{\mathbf{R}}(\mathbf{S}-\mathbf{R})\mathbf{H}(x)\end{aligned}$$

with the last equality from (3.4).

Note that if we further require that

$$\tilde{q}(\mathbf{R})\boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}(x) = c(\mathbf{S}-\mathbf{R})\mathbf{H}(x),$$

which implies (3.2), we then arrive at the defective renewal equation

$$\mathbf{H}(x) = \frac{1}{c}\mathcal{T}_{\mathbf{R}}q \star \boldsymbol{\eta}\mathbf{a}^\top \mathbf{H}(x) + \frac{1}{c}\mathcal{T}_{\mathbf{R}}\mathbf{l}(x),$$

which is exactly (3.1). □

Furthermore, under the usual definition (1.4), we arrive at the defective renewal equation (1.5) which is a generalization of Li's renewal equation.

Theorem 3.2. In the phase-type renewal risk model, the function H defined in (1.4) satisfies the defective renewal equation

$$H(x) = p\int_0^x H(x-y)\,\mathrm{d}Q_\delta(y) + \frac{1}{c}\mathbf{a}^\top \mathcal{T}_{\mathbf{R}}\mathbf{l}(x), \qquad (3.5)$$

where $p = (1/c)\mathbf{a}^\top \mathcal{T}_\mathbf{R}\overline{Q}(0)\boldsymbol{\eta}$ and $\overline{Q}_\delta(y) = 1/(cp)\mathbf{a}^\top \mathcal{T}_\mathbf{R}\overline{Q}(y)\boldsymbol{\eta}$. Its explicit solution is given by the Riemann-Stieltjes integral

$$H(x) = \frac{1}{c(1-p)} \int_0^x \mathbf{a}^\top \mathcal{T}_\mathbf{R}\mathbf{l}(x-y)\,\mathrm{d}G_\delta(y),$$

where the compound geometric distribution G_δ is defined by

$$G_\delta(x) = \sum_{k=0}^{\infty} (1-p)\, p^k Q_\delta^{*k}(x).$$

Proof: Pre-multiplying both sides of (3.1) by $\mathbf{a}^\top$ and letting $H(x) = \mathbf{a}^\top \mathbf{H}(x)$ gives (3.5). The solution follows immediately from the defective renewal equation (3.5) according to Theorem 2.1 of Lin and Willmot (1999). □

4 Representation of the matrix R

The matrix $\mathbf{R}$ was first derived in Ren (2007) and Li (2008) through a different approach in the analysis of the Gerber-Shiu function. We shall now adopt their arguments to find an explicit representation of the matrix $\mathbf{R}$.

Suppose the matrix $\mathbf{R}$ is diagonalizable, then we shall seek its eigenvectors $\boldsymbol{\Phi}_1, \cdots, \boldsymbol{\Phi}_m$ such that $\boldsymbol{\Phi}^{-1}\mathbf{R}\boldsymbol{\Phi} = \boldsymbol{\Delta} = \mathrm{diag}(\rho_1, \cdots, \rho_m)$, where $\rho_1, \cdots, \rho_m$ are the eigenvalues of $\mathbf{R}$ and $\boldsymbol{\Phi} = (\boldsymbol{\Phi}_1, \cdots, \boldsymbol{\Phi}_m)$. Or equivalently, the matrix $\mathbf{R}$ can be represented as

$$\mathbf{R} = \boldsymbol{\Phi}\boldsymbol{\Delta}\boldsymbol{\Phi}^{-1}. \tag{4.1}$$

By the definition of matrix exponential, we can show that

$$\boldsymbol{\Phi}^{-1} e^{\mathbf{R}y} \boldsymbol{\Phi} = e^{\boldsymbol{\Delta} y} = \mathrm{diag}(e^{\rho_1 y}, \cdots, e^{\rho_m y}).$$

Hence $e^{\rho_1 y}, \cdots, e^{\rho_m y}$ are eigenvalues of the matrix exponential $e^{\mathbf{R}x}$ corresponding to the eigenvectors $\boldsymbol{\Phi}_1, \cdots, \boldsymbol{\Phi}_m$ respectively.

Transposing all terms in (3.2) and pre-multiply both sides of the the equation by $\boldsymbol{\Phi}_i$ gives

$$\boldsymbol{\Phi}_i\{\mathbf{a}\boldsymbol{\eta}^\top \tilde{q}(\rho_i) + \boldsymbol{\Lambda} + (c\rho_i - \delta)\mathbf{I}\} = \mathbf{0}.$$

Since $\boldsymbol{\Phi}_i$'s are non-zero vectors, we must have

$$\det\{\mathbf{a}\boldsymbol{\eta}^\top \tilde{q}(\rho_i) + \boldsymbol{\Lambda} + (c\rho_i - \delta)\mathbf{I}\} = 0. \tag{4.2}$$

The vector $\boldsymbol{\Phi}_i$ can be obtained from the eigenvectors of the matrix $\{\mathbf{a}\boldsymbol{\eta}^\top \tilde{q}(\rho_i) + \boldsymbol{\Lambda} + (c\rho_i - \delta)\mathbf{I}\}$ corresponding to eigenvalue 0.

Assuming that the matrix $\mathbf{\Lambda} - (\delta - c\rho_i)\mathbf{I}$ is invertible, we obtain

$$\begin{aligned}
& \det\{\mathbf{a}\boldsymbol{\eta}^\top \tilde{q}(\rho_i) + \mathbf{\Lambda} + (c\rho_i - \delta)\mathbf{I}\} \\
= \; & \det\{\mathbf{\Lambda} - (\delta - c\rho_i)\mathbf{I}\} \det\{\mathbf{I} + [\mathbf{\Lambda} + (c\rho_i - \delta)\mathbf{I}]^{-1}\mathbf{a}\boldsymbol{\eta}^\top \tilde{q}(\rho_i)\} \\
= \; & \det\{\mathbf{\Lambda} - (\delta - c\rho_i)\mathbf{I}\} \det\{1 + \mathbf{a}^\top [\mathbf{\Lambda} + (c\rho_i - \delta)\mathbf{I}]^{-1}\boldsymbol{\eta}\tilde{q}(\rho_i)\} \\
= \; & \det\{\mathbf{\Lambda} - (\delta - c\rho_i)\mathbf{I}\}\{1 - \tilde{k}(\delta - c\rho_i)\tilde{q}(\rho_i)\},
\end{aligned}$$

where $\tilde{k}$ is the Laplace transform of the phase-type distribution $(\mathbf{a}, \mathbf{\Lambda})$. The last equality is obtained from the determinant identity $\det\{\mathbf{I} + \mathbf{XY}\} = \det\{\mathbf{I} + \mathbf{YX}\}$. Hence the condition (4.2) leads to the generalized Lundberg fundamental equation $\tilde{k}(\delta - c\rho_i)\tilde{q}(\rho_i) = 1$.

In order for the matrix Dickson-Hipp operator $\mathcal{T}_{\mathbf{R}}$ to exist for all bounded functions involved in the previous derivations, we require all ρ_i's to have non-negative real parts. It is shown in Proposition 2.1 of Albrecher and Boxma (2005) that the generalized Lundberg equation has m roots with non-negative real parts, the number of which match exactly the required dimension of the matrix $\mathbf{R}$. Hence we obtain both $\mathbf{\Phi}$ and ρ_i's which enable us to represent $\mathbf{R}$ in the explicit form (4.1).

5 Applications

5.1 Gerber-Shiu functions

It is shown in Lemma 4.1 of Feng (2009) that in the phase-type renewal risk model the Gerber-Shiu function, defined with the auxiliary state variable by

$$m(x, i) = \mathbb{E}^{(x,i)}[e^{-\delta\tau_0} w(X_{\tau_0}, |X_{\tau_0 -}|) I(\tau_0 < \infty)], \qquad x \geq 0,$$

is a special case of the function H defined in (1.3) with $d = 0$ and

$$\mathbf{l}(x) = \boldsymbol{\eta} \int_x^\infty w(x, y - x)\, \mathrm{d}Q(y). \tag{5.1}$$

Therefore, the Gerber-Shiu function defined in the usual sense is given by

$$m(x) = \mathbb{E}^x[e^{-\delta\tau_0} w(X_{\tau_0}, |X_{\tau_0 -}|) I(\tau_0 < \infty)] = \sum_{j \in E} a_j m(x, j). \tag{5.2}$$

Corollary 5.1. In the phase-type renewal risk model, the function m defined in (5.2) satisfies the defective renewal equation

$$m(x) = p \int_0^x m(x - y)\, \mathrm{d}Q_\delta(y) + \frac{1}{c}\mathbf{a}^\top \mathcal{T}_{\mathbf{R}} \alpha(x) \boldsymbol{\eta}. \tag{5.3}$$

Hence, it admits the explicit solution

$$m(x) = \frac{1}{c(1-p)} \int_0^x \mathbf{a}^\top \mathcal{T}_{\mathbf{R}} \alpha(x-y)\boldsymbol{\eta}\, \mathrm{d}G_\delta(y). \tag{5.4}$$

Proof: We can obtain the results by inserting (5.1) into (3.5) in Theorem 3.2. □

Example 5.1. *Discounted distribution of first drop in surplus*

Since the first drop in surplus below initial level can occur in a continuum of time which determines different discounted values, we consider the drop size distribution discounted to time zero from the time at which the drop occurs. Owing to space homogeneity, the size of first drop below initial surplus must have the same distribution as the size of deficit when the surplus starts at level zero. Therefore, we introduce the discounted density f as

$$f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \mathbb{E}^0[e^{-\delta\tau_0} I(|X_{\tau_0}| \le z)].$$

We now use a common technique in probability, see for example Section 3.6 of Karatzas and Shreve (1998), to treat $I(z \ge 0) = \int_0^z \delta(y)\,\mathrm{d}y$ where δ is a Dirac delta function. Hence,

$$f(z) = \mathbb{E}^0[e^{-\delta\tau_0}\delta(|X(\tau_0)| - z)],$$

which is indeed a special case of the Gerber-Shiu function with the penalty function $w(x,y) = \delta(y-z)$ with the initial surplus $X_0 = 0$. Inserting the expression for w into (5.1) gives

$$\mathbf{l}(x) = \boldsymbol{\eta} \int_x^\infty \delta[y - (x+z)]\,\mathrm{d}Q(y) = \boldsymbol{\eta} q(x+z).$$

Letting $x = 0$ and inserting the expression for $\mathbf{l}$ in (3.5), we obtain

$$f(z) = \frac{1}{c}\mathbf{a}^\top \mathcal{T}_{\mathbf{R}} q(z)\boldsymbol{\eta}.$$

Since $p = \int_0^\infty f(z)\,\mathrm{d}z$, the constant can indeed be interpreted as the discounted probability with which the first drop in surplus occurs. The function Q_δ is in fact the discounted distribution of the size of first drop in surplus given that the drop does occur, i.e. $Q_\delta(y) = (1/p)\int_0^y f(z)\,\mathrm{d}z$. Therefore, the first term in renewal equations (3.5) and (5.3) can now be interpreted as a direct application of the law of total probability conditioning on the size of the first drop in surplus below its initial level.

Corollary 5.2. In the phase-type renewal risk model in which the claim sizes are phase-type distributed with characteristics $(\mathbf{b}, \mathbf{Q})$, the distribution Q_δ is also phase-type with characteristics $(\mathbf{d}, \mathbf{Q})$, where

$$p = \frac{1}{c}\left(\int_0^\infty \mathbf{a}^\top e^{-\mathbf{R}u}\boldsymbol{\eta}\mathbf{b}^\top e^{\mathbf{Q}u}\,\mathrm{d}u\right)\mathbf{1}, \tag{5.5}$$

and

$$\mathbf{d}^\top = \frac{1}{cp}\int_0^\infty \mathbf{a}^\top e^{-\mathbf{R}u}\boldsymbol{\eta}\mathbf{b}^\top e^{\mathbf{Q}u}\,\mathrm{d}u.$$

Proof: Let $\mathbf{q} = -\mathbf{Q}\mathbf{1}$. Since the claim size distribution $Q(y) = 1 - \mathbf{b}^\top e^{\mathbf{Q}y}\mathbf{1}$, the Laplace transform of $Q_\delta(y)$ is given by

$$\begin{aligned}\int_0^\infty e^{-sy}\,\mathrm{d}Q_\delta(y) &= \frac{1}{cp}\int_0^\infty e^{-sy}\mathbf{a}^\top \mathcal{T}_{\mathbf{R}}q(y)\boldsymbol{\eta}\,\mathrm{d}y \\ &= \frac{1}{cp}\int_0^\infty e^{-sy}\mathbf{a}^\top\left(\int_0^\infty e^{-\mathbf{R}u}\mathbf{b}^\top e^{\mathbf{Q}(u+y)}\mathbf{q}\,\mathrm{d}u\right)\boldsymbol{\eta}\,\mathrm{d}y \\ &= \frac{1}{cp}\int_0^\infty \mathbf{a}^\top e^{-\mathbf{R}u}\boldsymbol{\eta}\mathbf{b}^\top e^{\mathbf{Q}u}\int_0^\infty e^{-sy}e^{\mathbf{Q}y}\,\mathrm{d}y\mathbf{q}\,\mathrm{d}u \\ &= \frac{1}{cp}\left(\int_0^\infty \mathbf{a}^\top e^{-\mathbf{R}u}\boldsymbol{\eta}\mathbf{b}^\top e^{\mathbf{Q}u}\,\mathrm{d}u\right)(s\mathbf{I}-\mathbf{Q})^{-1}\mathbf{q}.\end{aligned}$$

We obtain the expression (5.5) immediately from the fact that the Laplace transform at zero equals one. We also recognize by the one-to-one correspondence of distributions and their Laplace transforms that $Q_\delta(y)$ is also phase-type distributed with characteristics $(\mathbf{d}, \mathbf{Q})$. □

Example 5.2. *Laplace transform of the time of ruin*

We may further establish a connection between the compound geometric distribution G_δ and the Laplace transform of the time of ruin defined by $\psi_\delta(x) = \mathbb{E}^x[e^{-\delta\tau_0}I(\tau_0 < \infty)], x \geq 0$.

Since the function ψ_δ is a special case of the function m by choosing $w(x, y) = 1$ for all x, y and hence $\alpha(x) = \overline{Q}(x)$. Inserting the expression for α into (5.4) gives the solution

$$\begin{aligned}\psi_\delta(x) &= \frac{p}{1-p}\int_0^x \overline{Q}_\delta(x-y)\,\mathrm{d}G_\delta(y) \\ &= \frac{p}{1-p}\left[G_\delta(x) - \sum_{k=0}^\infty (1-p)\,p^k Q_\delta^{\star(k+1)}(x)\right] \\ &= \frac{p}{1-p}\left\{G_\delta(x) - \frac{1}{p}\left[G_\delta(x) - (1-p)\right]\right\} = 1 - G_\delta(x).\end{aligned}$$

Corollary 5.3. In the phase-type renewal risk model in which the claim sizes are phase-type distributed with characteristics $(\mathbf{b}, \mathbf{Q})$, the compound geometric distribution G_δ is also phase-type distributed with $(p\mathbf{d}, \mathbf{Q} + p\mathbf{q}\mathbf{d}^\top)$.

Proof: Since G_δ is a compound geometric distribution with characteristics (p, Q_δ), it follows immediately from Corollary 5.2 and Lemma 8.3.2 of Rolski *et al.* (1999) that $G_\delta(y)$ is also phase-type distributed with characteristics $(p\mathbf{d}, \mathbf{Q} + p\mathbf{q}\mathbf{d}^\top)$. □

Therefore, in this particular case, the Laplace transform of the time of ruin is given by

$$\psi_\delta(x) = p\mathbf{d}^\top e^{(\mathbf{Q}+p\mathbf{q}^\top\mathbf{d})x}\mathbf{1}, \qquad x \geq 0,$$

which generalizes formula (8.3.4) of Rolski *et al.* (1999).

Example 5.3. *Expected present value of penalty at ruin depending on deficit only*

The expected present value of penalty at ruin depending on deficit only is given by

$$P(x) = \mathbb{E}^x[e^{-\delta\tau_0} g(|X_{\tau_0}|)], \qquad x \geq 0.$$

Hence it is also a special case of the function H with $\mathbf{l}(x) = \boldsymbol{\eta} \int_x^\infty g(y-x)\,\mathrm{d}Q(y)$.

Corollary 5.4. In the phase-type renewal risk model in which the claim sizes are phase-type distributed with characteristics $(\mathbf{b}, \mathbf{Q})$, the solution to P is given by

$$P(x) = \frac{1}{c}\left(\mathbf{t}^\top, \mathbf{0}^\top\right) \exp\left\{\begin{pmatrix} \mathbf{Q} & p\mathbf{q}\mathbf{d}^\top \\ \mathbf{0} & \mathbf{Q}+p\mathbf{q}\mathbf{d}^\top \end{pmatrix} x\right\}\begin{pmatrix} \mathbf{q} \\ \mathbf{q} \end{pmatrix},$$

where

$$\mathbf{t}^\top = \int_0^\infty \mathbf{a}^\top e^{-\mathbf{R}u} \boldsymbol{\eta} \mathbf{b}^\top \left(\int_0^\infty g(z) e^{\mathbf{Q}z}\,\mathrm{d}z\right) e^{\mathbf{Q}u}\,\mathrm{d}u.$$

Proof: Since

$$\mathbf{l}(x) = \boldsymbol{\eta} \int_x^\infty g(y-x) \mathbf{b}^\top e^{\mathbf{Q}y} \mathbf{q}\,\mathrm{d}y = \boldsymbol{\eta} \mathbf{b}^\top \left(\int_0^\infty g(z) e^{\mathbf{Q}z}\,\mathrm{d}z\right) e^{\mathbf{Q}x} \mathbf{q},$$

then the Laplace transform of $\mathbf{a}^\top \mathcal{T}_\mathbf{R} \mathbf{l}$ is given by

$$\begin{aligned}
\int_0^\infty e^{-sy} \mathbf{a}^\top \mathcal{T}_\mathbf{R} \mathbf{l}(y)\,\mathrm{d}y &= \int_0^\infty e^{-sy} \mathbf{a}^\top \int_0^\infty e^{-\mathbf{R}u} \boldsymbol{\eta} \mathbf{b}^\top \left(\int_0^\infty g(z) e^{\mathbf{Q}z}\,\mathrm{d}z\right) e^{\mathbf{Q}(u+y)} \mathbf{q}\,\mathrm{d}u\,\mathrm{d}y \\
&= \int_0^\infty \mathbf{a}^\top e^{-\mathbf{R}u} \boldsymbol{\eta} \mathbf{b}^\top \left(\int_0^\infty g(z) e^{\mathbf{Q}z}\,\mathrm{d}z\right) e^{\mathbf{Q}u} \int_0^\infty e^{-sy} e^{\mathbf{Q}y} \mathbf{q}\,\mathrm{d}y\,\mathrm{d}u \\
&= \mathbf{t}^\top (s\mathbf{I} - \mathbf{Q})^{-1} \mathbf{q}.
\end{aligned}$$

According to Theorem 3.2, the solution to P is a convolution of $\mathbf{a}^\top \mathcal{T}_\mathbf{R} \mathbf{l}$ and $G_\delta/(1-p)$. Hence P is given by $\|\mathbf{t}\|/(1-p)$ times the convolution of the proper distributions $\mathrm{PH}(\mathbf{t}/\|\mathbf{t}\|, \mathbf{Q})$ and $\mathrm{PH}(p\mathbf{d}, \mathbf{Q}+p\mathbf{q}\mathbf{d}^\top)$. The desired solution is obtained from Theorem 8.2.6 of Rolski *et al.* (1999) together with the fact that $-(\mathbf{Q}+p\mathbf{q}\mathbf{d}^\top)\mathbf{1} = (1-p)\mathbf{q}$. □

5.2 Expected present value of total claim costs up to ruin

Suppose that we measure the actual cost of each claim by a function $\varpi[(x,i),(y,j)]$, which depends on both the surplus level and state variable immediately prior to the claim payment, (x,i), and the surplus level and state variable after the claim payment, (y,j). Then the expected present value of claim costs up to ruin with an auxiliary state variable is given by

$$C(x,i) = \mathbb{E}^{(x,i)}\left[\sum_{k=1}^N e^{-\delta T_k} \varpi[(X_{T_k-}, J_{T_k-}),(X_{T_k}, J_{T_k})]\right], \qquad x \geq 0,$$

where $N = \max\{k : T_k \le \tau_0\}$.

It is shown in Feng (2009) that the function C can be retrieved from the function H by letting $d = 0$ and choosing

$$l(x, i) = \sum_{j \in E} \eta_i a_j \int_0^\infty \varpi[(x, i), (x - y, j)] \,\mathrm{d}Q(y),$$

It is also known that the function $m(x, i)$ is a special case of the function $C(x, i)$. We shall now investigate an example of interest to the analysis of aggregate claim models.

Example 5.4. *Expected present value of aggregate claims*

The expected present value of the aggregate claims up to ruin given by

$$K(x) = \mathbb{E}^x \left\{ \sum_{k=1}^{N} e^{-\delta T_k} Y_k \right\} = \mathbb{E}^x \left\{ \sum_{k=1}^{N} e^{-\delta T_k} (X_{T_k -} - X_{T_k}) \right\}, \qquad x \ge 0, \tag{5.6}$$

is a special case of C with $\varpi[(x, i), (y, j)] = x - y$ for all $i, j \in E$ and hence is a special case of H by taking the cost function

$$\mathbf{l} = \boldsymbol{\eta} \mathbf{a}^\top \int_0^\infty y \mathbf{1} \,\mathrm{d}Q(y) = \mu \boldsymbol{\eta}, \tag{5.7}$$

where μ is the mean of claim sizes.

Corollary 5.5. In the phase-type renewal risk model, the solution to K defined in (5.6) is given by

$$K(x) = \frac{\mu}{c(1-p)} \mathbf{a}^\top \mathbf{R}^{-1} \boldsymbol{\eta} G_\delta(x), \qquad x \ge 0.$$

Proof: Applying Theorem 3.2 with $\mathbf{l}$ given in (5.7) yields the result immediately. □

Note that the result generalizes formula (6.2) of Cai *et al.* (2009).

5.3 Insurer's accumulated utilities up to default

The notion of utility has long been introduced to determine insurance premium in the literature of risk theory, see for example, Kaas *et al.* (1994), Gerber and Pafumi (1996), etc. In recent years, there has been growing interest in applying the concept of utility as a measure or criterion in the decision making of ruin-related stochastic control problems. For detailed accounts, readers may refer to Browne (1995) and Young (2004), etc.

Assume that an insurer's perceived value of its surplus is not necessarily measured by absolute counts of money but rather better described by a utility function. Then we are interested in an indicator of the overall perceived value of the insurance business - the accumulated utilities of surplus at all times up to the time of ruin. Such a quantity can be given by the expected present value of utilities up to the default of business

$$U(x) = \mathbb{E}^x \left[\int_0^{\tau_d} u(X_t)\,\mathrm{d}t \right], \qquad x \geq 0,$$

where u is the utility function of the insurer in question.

Example 5.5. *Accumulated exponential utilities up to ruin*

As an illustration, we shall use the exponential utility function $u(x) = -e^{-ax}/a$ with $a > 0$, which is often a favorite in economics and actuarial science due to its mathematical tractability. Even though this example specifically deals with the exponential utility, the method is general to apply for other choices of utility functions as long as the existence of the matrix Dickson-Hipp operator is properly addressed.

It suffices for us in this example to search a solution to the function given by

$$W(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-aX_t}\,\mathrm{d}t \right] = \sum_{i \in E} a_i W(x, i).$$

The function W is obviously a special case of the function H with the cost function $\mathbf{l}(x) = e^{-ax}\mathbf{1}$. Therefore, $\mathbf{a}^\top \mathcal{T}_{\mathbf{R}} \mathbf{l}(x) = \mathbf{a}^\top (\mathbf{R} + a\mathbf{I})^{-1} e^{-ax}\mathbf{1}$. The general solution to W is given by

$$W(x) = \frac{1}{c(1-p)} \mathbf{a}^\top (\mathbf{R} + a\mathbf{I})^{-1} \mathbf{1} \int_0^x e^{-a(x-y)}\,\mathrm{d}G_\delta(y).$$

Corollary 5.6. In the phase-type renewal risk model in which claim sizes are phase-type distributed with characteristics $(\mathbf{b}, \mathbf{Q})$, the solution to W is given by

$$W(x) = \frac{1}{ac} \mathbf{a}^\top (\mathbf{R} + a\mathbf{I})^{-1} (1, \mathbf{0}^\top) \exp\left\{ \begin{pmatrix} -a & ap\mathbf{d}^\top \\ \mathbf{0} & \mathbf{Q} + p\mathbf{q}\mathbf{d}^\top \end{pmatrix} x \right\} \begin{pmatrix} a \\ \mathbf{q} \end{pmatrix}.$$

Proof: We may treat the exponential utility function e^{-ax} as a multiple of an exponential density function and hence the solution to W becomes a multiple of the convolution of an exponential distribution and Q_δ. □

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