A Proof of the Celebrated Goldbach's Theorem

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Theorem (Goldbach [3], see also [1][2][4])

\[ \sum_{m,n \geq 2} \frac{1}{m^n - 1} = 1, \]

where \( \sum' \) means that every term only occurs once (for example \( 1/15 = 1/(16 - 1) \) is only added once even though \( 16 = 4^2 = 2^4 \)).

Proof: Let \( R \) denote the set of all integers larger than 1 that are not perfect powers: \( R = \{2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, \ldots \} \). Since every perfect power can be written uniquely as \( r^s (r \in R, s \geq 2) \) and every integer \( \geq 2 \) can be written uniquely as \( r^s (r \in R, s \geq 1) \), we have

\[ \sum_{m,n \geq 2} \frac{1}{m^n - 1} = \sum_{r \in R} \sum_{s=2}^{\infty} \frac{1}{r^s - 1} = \sum_{r \in R} \sum_{i=1}^{\infty} \sum_{s=2}^{\infty} \frac{1}{r^s i} = \sum_{r \in R} \sum_{i=1}^{\infty} \left( \frac{1}{r^i} \right)^s = \sum_{r \in R} \sum_{i=1}^{\infty} \left( \frac{1}{r^i} \right)^2 = \sum_{r \in R} \sum_{i=1}^{\infty} \frac{1}{r^i (r^i - 1)} = \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left( \frac{1}{m-1} - \frac{1}{m} \right) = 1. \quad \square \]

References

1. E. Catalan, Note sur la sommation de quelques séries, Journal de Mathématiques Pures et Appliquées 7 (1842), 1-12.


3. C. Goldbach, Letter to L. Euler, 1737.


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