Euler’s Solution of the Basel Problem – The Longer Story

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Abstract:

Most accounts of Euler’s brilliant summing of the reciprocals of the square numbers describe only his final solution to the problem. In fact, the usual solution is only the third of three solutions given in that 1736 paper. We describe some of Euler’s earlier results that led up to the 1736 paper, give all three of the solutions given there, as well as other interesting results in the same paper, and describe some related results that Euler gave later in his career. (Received January 30, 2003)

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Article:

An Historical Schema

1. Thesis – problems
2. Synthesis – theories and techniques that solve those problems
3. Antithesis – problems that resist solution by those techniques

Circles and lines
Euclidean geometry
Appollonian curves

Appollonian curves
Analytic geometry
Subtangents, quadratures

Diophantine problems
Infinite descent
Fermat’s last theorem

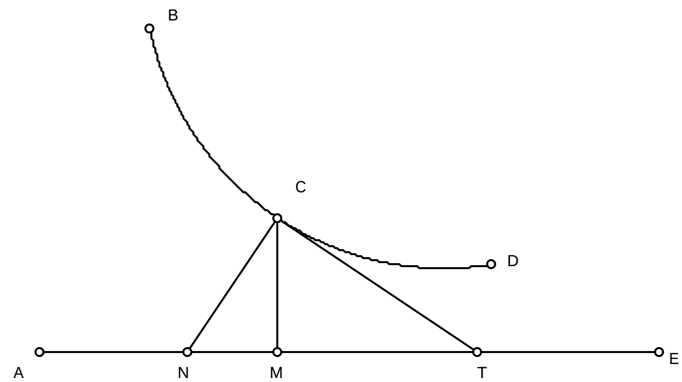
Fermat’s Last Theorem
L-functions, modular forms and elliptic curves
???

Tangents, areas and series
Calculus
Basel problem

1. Basel Problem
2. Euler’s methods
3. ???

Mathematicians in the 17^{th} Century had posed their own distinctive kinds of problems. Descartes’ new geometry and Viète’s new algebra gave them the tools of analytic geometry. Mathematicians, or “Geometers”, as they called themselves, used these new tools to solve the old problems in conics from Apollonius and to inspire new problems of their own. Most of the new problems of the era submitted easily to the new methods. A few of the more difficult ones were based on the ideas of arc length, tangent lines and areas. Most of these precursors to calculus either were solved using clever and difficult applications of the early and undeveloped ideas of calculus, or they waited for the discovery of calculus itself for their solutions.

Let us pause to look at how some of these early problems were posed. In the figure below, BCD is a curve and AE is an axis. The segment CT is tangent to BCD at the point C and the segment CN is normal to the curve at C.



In the vocabulary of the 17th Century, these segments have names, and often no distinction was made between the name of the segment and its length. We still call CM an ordinate. CT was called the tangent. Note that the tangent is only the segment from the curve to the axis. MT was called the subtangent. CN was the normal and NM was the subnormal. There were a variety of problems posed about these various segments.

The "Problem of the Subtangent" asked what curve had the property that, for any point C on the curve, the subtangent, MT was a given constant length. In his great 1686 paper, the "Nova Methodus", Leibniz showed that the curve with a constant subtangent was what he called a "logarithmic" curve, but we would call an exponential curve. The related "Problem of the Tractrix" was posed by Claude Perrault, brother of Charles Perrault, author of such children's classics as "Cinderella" and "Puss in Boots." A tractrix is a curve with a constant tangent, and Leibniz was also the first to give an explicit solution to that problem.

There were a number of other problems about curves. One was the brachistochrone problem, to find the curve between two given points down which a particle will slide in the shortest time. Another was the isoperimetric problem, to find the curve of a given length that encloses the largest possible area. Both problems yielded to the tools of calculus.

Some problems involved series. Huygens and Leibniz independently solved the problem of summing the reciprocals of the triangular numbers. Triangular numbers are numbers of the form $\frac{n(n+1)}{2}$, and are so named because they give the number of objects arranged in a triangle with 1 in the first row, 2 in the second row, 3 in the third row, and so on down to n in the last row. The first several triangular numbers, starting with $n = 1$, are 1, 3, 6, 10, 15 and 21. We give Leibniz's solution in modern notation rather than his 17th Century notation.

Let $S=\sum_{n=1}^{\infty}\frac{2}{n(n+1)}=\frac{2}{1(1+1)}+\frac{2}{2(2+1)}+\frac{2}{3(3+1)}+\ldots$. By partial fractions, $\frac{2}{n(n+1)}=\frac{2}{n}-\frac{2}{n+1}$, so the series telescopes. That is, $S=\left(\frac{2}{1}-\frac{2}{1+1}\right)+\left(\frac{2}{2}-\frac{2}{2+1}\right)+\left(\frac{2}{3}+\frac{2}{3+1}\right)+\ldots$. The second term in each group cancels the first term in the next group. Leibniz does not know to worry about issues of convergence, so he blithely regroups, leaving the sum $S=\frac{2}{1}=2$.

A few problems became important when they resisted solution with the tools of calculus. We should note that other problems, for example Fermat's problems in number theory, were mostly ignored for several decades or forgotten altogether. Not all unsolved problems become important unsolved problems.

One important unsolved problem of the time was the so-called Basel problem, posed by Pietro Mengoli (1625-1686) in 1644. Mengoli asked for the sum of the reciprocals of the perfect squares, that is $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n^2}$. The problem became well known when Jakob Bernoulli wrote about it in 1689. Jakob was the brother of Euler's teacher and mentor Johann Bernoulli, who probably showed the problem to Euler. By the 1730's, the problem had thwarted many of the day's best mathematicians, and it had achieved the same kind of mystique that Fermat's Last Theorem had before 1993.

It seems, though, that Euler did not come to the Basel problem directly. Indeed, his attack on the problem began with one of those forgotten problems of the early 18th Century, the interpolation of series.

When Euler arrived in St. Petersburg in 1727, Christian Goldbach was the Secretary of the Imperial Academy of Sciences. Goldbach became a kind of mentor for the young Euler. Euler initiated this relationship with a letter of 13 October 1729.[1] The letter gets right down to business:

> "Most Celebrated Sir: I have been thinking about the laws by which a series may be interpolated. … The most Celebrated Bernoulli suggested that I write to you."

Euler continues the letter by proclaiming, with only this little preamble, that the general term of the "series" that we now call the factorial numbers, 1, 2, 6, 24, 120, etc. is given by

$$\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{1-m}\cdot 3^m}{2+m}\cdot\frac{3^{1-m}\cdot 4^m}{3+m}\cdot\frac{4^{1-m}\cdot 5^m}{5+m}\ etc.$$

This is a remarkable infinite product. The modern eye immediately wants to do some cancellation among factors in the numerator to "reduce" this to

$$\frac{1}{1+m}\cdot\frac{2}{2+m}\cdot\frac{3}{3+m}\cdot\frac{4}{4+m}\ etc.$$

Then, we see that the value of m "shifts" the denominator, and when m is a positive integer, each of the factors in the denominator exactly cancels with a factor in the numerator, leaving exactly $m!$.

So, why didn't Euler use the simpler form? It turns out that Euler's form converges, albeit slowly, to $m!$ without any rearrangement of factors. The modern-looking form converges to zero. Even at this early date, Euler was only 22, he was aware of issues of convergence, even though he never completely understood them.

Euler then substitutes $m=\frac{1}{2}$ and tells us that

> The term with exponent $\frac{1}{2}$ is equal to this $\frac{1}{2}\sqrt{\sqrt{-1}\cdot l-1}$, which is equal to the side of the square equal to the circle with diameter =1.

Here Euler uses l to denote the natural logarithm and he does not use yet the symbol p . The mathematical community had not yet adopted a standard symbol for the value we now call p . In part this was because they still regarded p as a ratio, and they still sometimes maintained a distinction between ratios and numbers.

That Euler's radical equals $\frac{\sqrt{p}}{2}$ follows easily from knowing $e^{pi}=-1$ by taking logs of both sides, but the fact was not widely known at the time, and the notation would have been quite foreign. As we will see later, the expression $e^{pi}=-1$, often known as "the Euler identity" is due to Euler, but as we see here, the idea itself predates Euler. Some credit Roger Cotes (1682-1716) in 1714. [N, p. 162]

In the paper, E19, though not in the letter, Euler goes into more detail, substituting ½into the form above and getting

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[1] At the time, indeed until the Revolution of 1914, Russia used the old Julian calendar. The rest of Europe had converted to the Gregorian calendar still in use today. Hence, dates in Russia were eleven days behind the dates in the rest of Europe. October 13 in St. Petersburg was October 24 elsewhere. Following usual practice, we will give Julian dates when events occur in Russia, Gregorian dates elsewhere, and both if there may be confusion.

$$\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}\,etc.$$

which he compares to Wallis's formula $\frac{4}{p} = \frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot \ldots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot \ldots}$ to get the result he claimed.

Euler spends most of the remainder of the paper E19 developing integral formulas that interpolate products of arithmetic sequences. Since the sequence 1, 2, 3, 4, … is a special arithmetic sequence, these formulas were generalizing the factorial function, and their interpolations generalized the gamma function.

In the very last sections of the paper, Euler makes some speculative remarks. He is motivated by the idea that the *n*th derivative of a polynomial x^k can be written as $k(k-1)\ldots(k-n+1)x^{k-n} = \frac{k!}{(k-n)!}x^{k-n}$. Since Euler has generalized the factorial function, he can give a definition to the *n*th derivative, even if *n* is a fraction. In his words:

> "To round off this discussion, let me add something which certainly is more curious than useful. It is known that dx^n denotes the differential of *x* of order *n* and if *p* denotes any function of *x* and *dx* is taken to be constant then … the ratio of dx^p to dx^n can be expressed algebraically. … We now ask, if *n* is a fractional number, what the value of that ratio should be."

Before returning to Euler's letter to Goldbach, let us remind the reader that the harmonic progression is the sequence of reciprocals of the positive integers. It begins

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Its progression of partial sums would begin

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$$

Euler wants to do something with this sequence of partial sums that resembles what he did with the sequence of factorial numbers. He wants to interpolate the sequence, that is to give meaning to the sequence for values that fall between the whole numbers for which it is naturally defined. He tells Goldbach, without any substantiation, that "I have found that the general term, whose index is $-\frac{1}{2}$ is $-2l2$. The term whose exponent is $\frac{1}{2}$ is $2-2l2$" where "*l* 2 signifies the hyperbolic logarithm of two, which is =0,69314738056." In Euler's day, as in much of the world today, they used commas as decimal points. Euler says nothing more of this in the letter, but he treats the subject thoroughly, as we will see below, in the paper E20.

Six weeks later, Goldbach replies from Moscow with a letter dated December 1/12, 1729, in which he repeats Euler's form of the Gamma function and asks the sum of the "hypergeometric series" $1+1\cdot 2+1\cdot 2\cdot 3+1\cdot 2\cdot 3\cdot 4+\ldots$. As a final note, he writes

> P.S. A note to you is that Fermat has observed that all numbers with the formula $2^{2^x}+1$, that is 3, 5, 17, etc. are primes, but he himself was not able to prove this, and, as far as I know, nobody since him has proved it, either.

We will see more of this in E26, "Observations on some theorems of Fermat".

Euler replied to Goldbach from St. Petersburg on January 8/19, 1730 with an outline of some of the integral methods he was using to analyze the form for the Gamma function, steps he would detail in his paper E19. He closes with a note "I have been able to discover nothing about the observation noticed by Fermat."

Goldbach's reply from Moscow was written five months later on May 22/31, 1730. Goldbach echoes back one of Euler's integral forms for the Gamma function, and forwards a couple of other of Fermat's observations on number theory.

After this fourth letter, Euler and Goldbach exchange letters at a more rapid pace, but they do not return to these topics. It is clear from these letters that the mathematical insights and initiative were Euler's, but that Goldbach was a willing, if inept, partner in the correspondence.

Let us examine how Euler expanded his remarks on the interpolation of the partial sums of the harmonic series in E20.

He asks us to look at the integral $\int_0^1 \frac{1-x^n}{1-x}\,dx$, or, as Euler says " $\int \frac{1-x^n}{1-x}\,dx$, "and take it =0 if x=0, and then set x=1." This peculiar wording is how Euler says to take the integral from 0 to 1. It amounts to taking a particular antiderivative, the one chosen so that the constant of integration makes the value of the antiderivative equal to zero at the left hand endpoint. For integer values of *n*, the integrand expands to give $1+x+x^2+\ldots+x^{n-1}$. This finite series integrates to give $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}$, which, integrated from 0 to 1, gives the *n*th partial sum of the harmonic series.

The integral, though, is defined even if *n* is not an integer, and Euler takes this integral to be the interpolation of the series. In particular, if $n = ½$ the integrand is $\frac{1-\sqrt{x}}{1-x}$ or $\frac{1}{1+\sqrt{x}}$. Integrating this from 0 to 1 gives $2-2\ln 2$. This explains his unsubstantiated remark in his first letter to Goldbach.

Euler generalizes this idea to the case of series of fractions where the numerator is any geometric series and the denominator is any arithmetic series. He goes on to use partial fractions to extend it to the case where the denominator is any polynomial with distinct real roots. Partial fractions do not enable him to deal with complex or multiple roots. We note that the Basel Problem is a case with multiple roots.

To deal with multiple roots, Euler needs to invent another trick. Note that when Euler integrated $1+x+x^2+\ldots+x^{n-1}$ he got $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}$. Integrating again gives $\frac{x^2}{1\cdot 2}+\frac{x^3}{2\cdot 3}+\frac{x^4}{3\cdot 4}+\ldots+\frac{x^{n+1}}{n\cdot(n+1)}$, closely related to the series studied by Huygens and Leibniz. However, if he divides by *x* before integrating again, or, as Euler writes it, if he takes $\int \frac{dx}{x}\int\frac{1-x^n}{1-x}\,dx$ he gets $x+\frac{x}{4}+\frac{x^2}{9}+\frac{x^3}{16}+\ldots+\frac{x^n}{n^2}$. This series, if $x = 1$ and *n* goes to infinity, would be a solution to the Basel problem. The integral, however, is intractable and appears to the modern reader to be nonsense, as well. To make the integral make sense in modern notation, we would write $\int_0^1 \frac{dx}{x}\left(\int_0^x \frac{1-t^{n-1}}{1-t}\,dt\right)$. As we have seen, Euler did not yet have the notation to write it this way, but the technique of Euler's time, to choose the appropriate antiderivative at each step, then evaluate the antiderivative at the end of the calculation, gives his double integral a perfectly clear and unambiguous meaning to his contemporaries.

The integral is still intractable, but Euler devises some clever approximations to get the integral to six decimal places, 1.644924. Euler notes that to achieve such accuracy using direct calculation requires more than 1000 terms, so his estimate of the solution to the Basel problem was far more accurate than any available to his competitors. He later improved his estimate to 17 decimal places. Moreover, Euler was a genius at arithmetic, so he probably recognized this value as what will turn out to be the exact solution to the Basel problem, $\frac{p^2}{6}$. Euler didn't share this with the world, so he had a valuable advantage as he raced to solve the problem; he knew the answer.

Euler will assemble all his results on these topics into Chapter XVII "De interpolatione serierum" (On the interpolation of series) of Part II of his 1755 *Institutiones Calculi Differentialis*. [E212]

Let us move forward in time from 1729, when Euler wrote E19, and 1730, when he wrote E20, to 1735, when he wrote E41, his solution to the Basel problem and his first really important mathematical accomplishment. Euler gives four distinct solutions to the Basel problem, three in E41 and a fourth in E63, written in 1741. The "usual" solution is the third one, recounted excellently in [D]. Briefly, Euler notes that the function $\frac{\sin x}{x}$ has roots at $\pm p, \pm 2p, \pm 3p, \pm 4p$, *etc.* and makes the bold assumption that it is the same function as the infinite product, which has the same roots,

$$\left(1-\frac{x}{p}\right)\left(1+\frac{x}{p}\right)\left(1-\frac{x}{2p}\right)\left(1+\frac{x}{2p}\right)\left(1-\frac{x}{3p}\right)\left(1+\frac{x}{3p}\right)\ldots$$

This last, he rewrites as

$$\left(1-\frac{x^2}{p^2}\right)\left(1-\frac{x^2}{4p^2}\right)\left(1-\frac{x^2}{9p^2}\right)\ldots$$

then notes that the coefficient on x^2 in the expansion will be

$$-\frac{1}{p^2}-\frac{1}{4p^2}-\frac{1}{9p^2}-\ldots$$

Meanwhile, he notes that the Taylor series expansion of $\frac{\sin x}{x}$ at $x = 0$ is

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots$$

Matching coefficients, he gets that

$$-\frac{1}{p^2}-\frac{1}{4p^2}-\frac{1}{9p^2}-\ldots=\frac{-1}{3!}$$

so that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\frac{p^2}{6}$$

Euler gives considerably more detail, he uses p where we use p , and he does not use the factorial notation, so his exposition takes a good deal longer. He is often criticized for failing to justify his conclusion that $\frac{\sin x}{x}$ equals the infinite product just because they have the same roots. Modern critics note, for example, that $e^x\frac{\sin x}{x}$ also has the same roots, yet it is not equal to the given infinite product. Euler does revisit the question of infinite products and their roots in E63, written in 1741, where does a little bit to strengthen his case, but it still does not approach modern standards of analytical rigor.

Let us look at Euler's first and less elegant solution to the Basel Problem, as it is so seldom described elsewhere, and it gives us insight into how Euler came upon the more elegant solution we describe above.

Euler takes s to be the length of an arc on a circle of radius 1, and takes $y=\sin s$. He writes the Taylor series

$$y=s-\frac{s^3}{1\cdot 2\cdot 3}+\frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{s^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+etc.$$

Dividing by y and subtracting 1 transforms this into

$$0=1-\frac{s}{y}+\frac{s^3}{1\cdot 2\cdot 3y}-\frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5y}+etc.$$

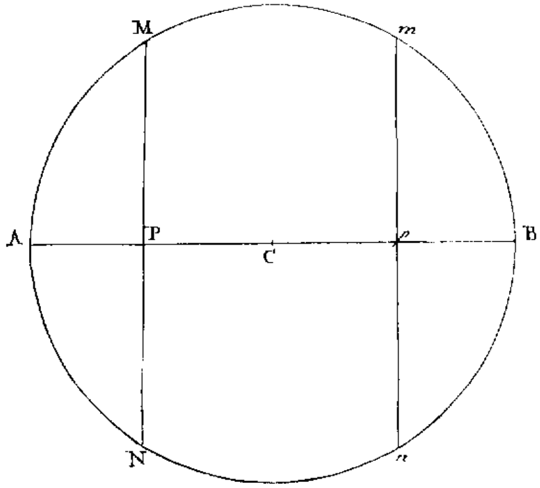
Euler factors this as if it were a polynomial to write

$$1-\frac{s}{y}+\frac{s^3}{1\cdot 2\cdot 3y}-\frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5y}+etc.=\left(1-\frac{s}{A}\right)\left(1-\frac{s}{B}\right)\left(1-\frac{s}{C}\right)\left(1-\frac{s}{D}\right)etc.$$

Matching the terms containing s to the first power gives

$$\frac{1}{y}=\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}+etc.$$

Euler notes that if he matches terms containing s^2, he gets zero on the left and "the sum of factors taken two at a time" (without repetition) from the terms of the series on the right. He describes terms containing s^3 up to s^5 similarly.



Now, in the figure above, Euler takes the arc length AM to be *s* and also takes AM = A, where A is the smallest positive arc with the same sine as *s*. That sine is y = PM. He also takes p to be the "semiperiphery of a circle of radius 1", that is to say p . Now, by the periodicity of the sine functions, the other arcs with the same sine as *s* are

$$A,\ p - A,\ 2p + A,\ 3p - A,\ 4p + A,\ 5p - A,\ 6p + A,\ etc.$$

and

$$-p - A,\ -2p + A,\ -3p - A,\ -4p + A,\ -5p - A\ etc.$$

This gives Euler values for *A*, *B*, *C*, etc. in his series and his infinite product above, so he can say that

$$\frac{1}{y} = \frac{1}{A} + \frac{1}{p-A} + \frac{1}{-p-A} + \frac{1}{2p+A} + \frac{1}{-2p+A} + \frac{1}{3p-A} + \frac{1}{-3p+A}$$

Matching coefficients again, he notes that the sum of the products taken two at a time will be zero, taken three at a time will be $\frac{-1}{1\cdot 2\cdot 3y}$, and so forth.

Euler recalls some facts about series that he uses quite often, but that are today seldom seen. If

$$a = a + b + c + d + e + f + etc.$$

is a series, and if b is "the sum of these terms taken two at a time" without repetition, that is

$$b = ab + ac + bc + ad + bc + ae + bd + af + be + cd + etc.$$

then

$$a^2 + b^2 + c^2 + d^2 + etc. = a^2 - 2b$$

Euler goes into a good deal of detail here, denoting by $a,\ b,\ g,\ d,\ etc.$ the sums of the terms taken one, two, three and four at a time, respectively, and denoting by *P, Q, R, S, etc.* the sums of the terms taken to the first, second, third and fourth powers respectively. Then he gets

$$P = a$$

$$Q = Pa - 2b$$

$$R = Qa - Pb + 3g$$

$$S = Ra - Qb + Pg - 4d,\ etc.$$

In Euler's series,

$$a = \frac{1}{y}$$
$$b = 0$$
$$g = \frac{-1}{1 \cdot 2 \cdot 3y}$$
$$d = 0$$
$$e = \frac{+1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y}$$
$$z = 0 \quad etc.$$

and so

$$P = \frac{1}{y}$$
$$Q = \frac{P}{y} = \frac{1}{y^2}$$
$$R = \frac{Q}{y} - \frac{1}{1 \cdot 2y}$$
$$S = \frac{R}{y} - \frac{P}{1 \cdot 2 \cdot 3y}$$

Up to now, Euler has been doing his development in some generality, without taking any particular value for the arc s. He now takes $q = \frac{p}{2}$, and sets $s = q$, so that $y = 1$. This makes $P = 1$, so the sum of the first powers of the terms of his series becomes

$$1 = \frac{1}{q} + \frac{1}{q} + \frac{-1}{3q} + \frac{-1}{3q} + \frac{1}{5q} + \frac{1}{5q} + \frac{-1}{7q} + \frac{-1}{7q} + etc.$$
$$= \frac{2}{q}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + etc.\right)$$

This says that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + etc. = \frac{p}{4}$$

a fact that was well known to Euler and that he attributes to Leibniz. Euler included this particular calculation to check that his methods had not led him too far astray.

Now, Euler turns to the sum of the squares of his sequence, or

$$\frac{1}{q^2} + \frac{1}{q^2} + \frac{1}{9q^2} + \frac{1}{9q^2} + \frac{1}{25q^2} + \frac{1}{25q^2} + etc. = \frac{2}{q^2}\left(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + etc.\right).$$

This sum equals Q, by definition. But $Q = \frac{1}{y^2}$ and $y = 1$, so $Q = 1$. Substitution gives

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + etc. = \frac{q^2}{2} = \frac{p^2}{8}.$$

That is, the sum of the reciprocals of the odd squares is $\frac{p^2}{8}$. Since every even square is a power of 4 times an odd square, Euler knows that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc.=\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+etc.\right)\cdot\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+etc.\right)$$
$$=\frac{p^2}{8}\cdot\frac{1}{1-\frac{1}{4}}$$
$$=\frac{p^2}{6}$$
and the Basel Problem is solved. In typical Euler form, Euler continues matching higher order coefficients and evaluates sums of reciprocals of higher even powers, as well as alternating reciprocals of higher odd powers of odd numbers.

Euler's second solution to the Basel problem is based on taking $y=\frac{\sqrt{3}}{2}$ in place of $y = 1$ and doing a calculation similar to the one above.

Euler seems to believe that there is something more to be done with this technique of infinite products. It is not quite clear whether he had doubts about the reliability of the technique, or he hoped that it could yield new results on a par with his solution to the Basel problem. In 1743, in one of his first mathematical papers written after he moved to Berlin, he wrote a twenty page paper, E61 with a twenty word title, "De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera in qua eaedem summations ex fonte maxime diverso derivantur" (On the sums of reciprocals of powers of natural numbers arising from the previous dissertation in which those summations are derived in a different way). This was based in part on E130, written in 1739 but, because of the immense publication delays in the St, Petersburg journals at the time, not published until 1750, "De seriebus quibusdam considerations" (On considerations about certain series). In both papers, he makes his calculations more clearly and in more generality. He succeeds in re-deriving known results, providing evidence that the techniques are reliable, but he finds no important new results, nor does he provide more rigor.

It is worth noting in passing that in E61, Euler does some of his series calculations using complex numbers. He has recently adopted the symbols e and p to denote their now familiar constants. In the course of his calculations in E61, he remarks that the series for

$$\sin s = s-\frac{s^3}{1\cdot 2\cdot 3}+\frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{s^7}{1\cdot 2\cdot 3\cdots 7}+etc.=\frac{e^{s\sqrt{-1}}-e^{-s\sqrt{-1}}}{2\sqrt{-1}}.$$

This is Euler's first use of this, and it is one of several mathematical facts now known as an "Euler formula."

Later in the same paragraph, he writes, "denoting by e that number whose logarithm is = 1," that

$$e^z=\left(1+\frac{z}{n}\right)^n$$

"where n is an infinite number." Euler did not have the tools of limits, so he often calculated with infinite and infinitesimal numbers.

Perhaps Euler sensed that his method of infinite products would continue to elicit criticism. In 1741 he wrote in French a fourth solution to the Basel problem. This paper, E63, "Demonstration de la somme de cette suite $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc.$" (Proof of the sum of this series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc.$) uses only elementary calculus tools, Taylor series and integration by parts, to obtain the result.

Euler takes $x=\sin s$, so $s=\arcsin x$ and $ds=\frac{dx}{\sqrt{1-xx}}$. Euler chooses to write s as an integral, so $s=\int\frac{dx}{\sqrt{1-xx}}$. Euler follows the practice of his time and is not very explicit about what the bounds of integration ought to be. Here he gives a hint that, in the case $x = 1$, the integral will give $\frac{p}{2}$ where "It is clear that I employ here the letter π to indicate the number of Ludolf of Kuelen, 3.14159265, etc." By this time, Euler has been using π to denote that constant for several years, but the convention will take many years more before it is universally adopted.

Now, Euler considers $sds = \frac{dx}{\sqrt{1-xx}}\int \frac{dx}{\sqrt{1-xx}}$. Integrating from 0 to 1 gives $\frac{pp}{8}$ on the left. This done, he turns his attention to the integral on the right.

Euler expands this integral for *s*, first using the generalized binomial theorem with $m = \frac{-1}{2}$, then integrating, to get

$$\int \frac{dx}{\sqrt{1-xx}} = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}x^9 + etc.$$

Now, multiplying by $ds = \frac{dx}{\sqrt{1-xx}}$ gives

$$s\ ds = \frac{x\ dx}{\sqrt{1-xx}} + \frac{1}{2\cdot 3}\frac{x^3\ dx}{\sqrt{1-xx}} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{x^5\ dx}{\sqrt{1-xx}} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{x^7\ dx}{\sqrt{1-xx}} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}\frac{x^9\ dx}{\sqrt{1-xx}} + etc.$$

Here, he says "One will see clearly, if he gives it proper reflection, that in general,"

$$\int \frac{x^{n+2}\,dx}{\sqrt{1-xx}} = \frac{n+1}{n+2}\int \frac{x^n\ dx}{\sqrt{1-xx}} - \frac{x^{n+1}}{n+2}\sqrt{1-xx}$$

By "proper reflection", Euler means "integration by parts." Here, the integral is to be taken from 0 to 1, and so the second term on the right drops out, leaving us with only

$$\int \frac{x^{n+2}\,dx}{\sqrt{1-xx}} = \frac{n+1}{n+2}\int \frac{x^n\ dx}{\sqrt{1-xx}}$$

With this, Euler can evaluate the integrals in each term of the integral of *s ds*:

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 - \sqrt{1-xx} = 1$$

$$\int \frac{x^3\,dx}{\sqrt{1-xx}} = \frac{2}{3}\int \frac{xdx}{\sqrt{1-xx}} = \frac{2}{3}$$

$$\int \frac{x^5\,dx}{\sqrt{1-xx}} = \frac{4}{5}\int \frac{x^3\,dx}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5}$$

$$\int \frac{x^7\,dx}{\sqrt{1-xx}} = \frac{6}{7}\int \frac{x^5\,dx}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

With this, Euler can integrate *s ds* to get

$$\frac{pp}{8} = \frac{ss}{2} = \int sds = \int \frac{xdx}{\sqrt{1-xx}} + \frac{1}{2\cdot 3}\int \frac{x^3\,dx}{\sqrt{1-xx}} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\int \frac{x^5\,dx}{\sqrt{1-xx}} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\int \frac{x^7\,dx}{\sqrt{1-xx}} + etc.$$

Substituting the values for the integrals that he found above gives

$$\frac{pp}{8} = 1 + \frac{1}{3\cdot 3} + \frac{1}{5\cdot 5} + \frac{1}{7\cdot 7} + etc.$$

Now, as in his first proof, this sum of the reciprocals of the odd squares gives the sum of the reciprocals of all squares to be

$$\frac{pp}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + etc.$$

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