STERN NOTES, CHAPTER 6 (FIRST DRAFT)

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1. The summatory function

The fact that there is such a simple formula as

$$\sum_{k=2^r}^{2^{r+1}} {}^{*} s(n) = 3^r \tag{1}$$

suggests that it might be interesting to see if there is an underlying measure on $[0, 1]$ induced by the Stern sequence. To this end, we define the function f on $[0, 1]$ by

$$f(\lambda) := \lim_{r \to \infty} \frac{1}{3^r} \cdot \sum_{k=2^r}^{\lfloor (1+\lambda)2^r \rfloor} {}^{*} s(n). \tag{2}$$

Except for the trivial values $f(0) = 0$, $f(\frac{1}{2}) = \frac{1}{2}$ (by symmetry) and $f(1) = 1$, there is no *a priori* reason that this limit should exist, although the pictures of the Stern sequence distributed earlier suggested a regular pattern.

We first consider dyadic rationals. Suppose $\lambda = \frac{k}{2^v}$, so that the upper limit in (2) has a simple expression for $r \geq v$:

$$f\left(\frac{k}{2^v}\right) = \lim_{r \to \infty} \frac{1}{3^r} \cdot \sum_{k=2^r}^{2^r + 2^{r-v}k} {}^{*} s(n) = \lim_{r \to \infty} \frac{1}{3^r} \cdot \sum_{j=0}^{k-1} \left(\sum_{k=2^r+2^{r-v}j}^{2^r+2^{r-v}(j+1))} {}^{*} s(n) \right). \tag{3}$$

Recalling our earlier notation

$$\Sigma(f; m, r) = \sum_{n=2^r m}^{2^r(m+1)} {}^{*} f(n), \tag{4}$$

this becomes

$$f\left(\frac{k}{2^v}\right) = \lim_{r \to \infty} \frac{1}{3^r} \cdot \sum_{j=0}^{k-1} \Sigma(s; 2^v + j, r - v). \tag{5}$$

Further, recall that we observed that

$$\begin{aligned}\Sigma(s;m,1) &= \frac{s(2m)}{2} + s(2m+1) + \frac{s(2m+2)}{2} \\ &= \frac{s(m)}{2} + s(m) + s(m+1) + \frac{s(m+1)}{2} \\ &= \frac{3}{2}\left(s(m)+s(m+1)\right) = 3\Sigma(s,m,0),\end{aligned} \tag{6}$$

hence $\Sigma(s;m,t) = 3^t\Sigma(s;m,0)$, and the limit in (5) is that of a *constant* sequence (at least for r sufficiently large). We find that for $r \ge v$,

$$\begin{aligned} f\left(\frac{k}{2^v}\right) &= \frac{3^{r-v}}{3^r}\cdot\sum_{j=0}^{k-1}\Sigma(s;2^v+j,0) = \frac{1}{3^v}\cdot{\sum_{n=2^v}^{2^v+k}}^{*} s(n) \\ &= \frac{1}{3^v}\cdot\sum_{n=2^v}^{2^v+k-1}\frac{s(n)+s(n+1)}{2} = \frac{1}{2\cdot 3^v}\cdot\sum_{n=2^v}^{2^v+k-1} s(2n+1).\end{aligned} \tag{7}$$

It's also useful to observe that for $k \ge 1$,

$$f\left(\frac{k}{2^v}\right) - f\left(\frac{k-1}{2^v}\right) = \frac{s(2^{v+1}+2k-1)}{2\cdot 3^v}, \tag{8}$$

and since $f(0) = 0$, this can serve as an alternate definition of f.

To compute this function, we need to look at the odd elements of the Stern sequence. In the table below, the first row is $s(3)$, the second is $s(5), s(7)$, etc.

$$\begin{array}{c} 2 \\ 3 \quad 3 \\ 4 \quad 5 \quad 5 \quad 4 \\ 5 \quad 7 \quad 8 \quad 7 \quad 7 \quad 8 \quad 7 \quad 5 \\ 6 \quad 9 \quad 11 \quad 10 \quad 11 \quad 13 \quad 12 \quad 9 \quad 9 \quad 12 \quad 13 \quad 11 \quad 10 \quad 11 \quad 9 \quad 6 \end{array} \tag{9}$$

Reading from the bottom row, $f(\frac{1}{16}) = \frac{6}{162}$, $f(\frac{2}{16}) = \frac{6+9}{162} = \frac{5}{54} = f(\frac{1}{8})$, $f(\frac{3}{16}) = \frac{6+9+11}{162}$ and so on. We also can easily see that

$$f\left(\frac{1}{2^v}\right) = \frac{v+2}{2\cdot 3^v}. \tag{10}$$

It is worth noting here that this table of the odd Stern sequence can be generated autonomously. We have $s(2^r+1) = s(2^{r+1}-1) = r+1$,

$$\begin{aligned} s(4n+1)+s(4n+3) &= (2s(n)+s(n+1)) + (s(n)+2s(n+1)) \\ &= 3\left(s(n)+s(n+1)\right) = 3s(2n+1)\end{aligned} \tag{11}$$

and

$$\begin{aligned} s(4n+5)-s(4n+3) &= (s(2n+2)+s(2n+3)) - (s(2n+1)+s(2n+2)) \\ &= s(2n+3)-s(2n+1).\end{aligned} \tag{12}$$

Thus, if you have a row of odd terms, then you can make the next row by starting appropriately and then following these rules. To be specific,

$$\begin{aligned} & s(2n+1)=a, \quad s(2n+3)=b, \quad s(4n+1)=c \\ \implies & s(4n+3)=3a-c, \quad s(4n+5)=s(4n+3)+(b-a). \end{aligned} \tag{13}$$

Theorem 1. *The function f defined above extends to a continuous, strictly increasing function F from $[0,1]$ to itself, satisfying $F(1-x)=1-F(x)$.*

Proof. We apply Theorem 5, from the Notes V, supplement. It is certainly the case that f is defined on a dense subset X of $[0,1]$, namely the dyadic rationals, and that f is strictly increasing on its image. We need only show that the image, Y, is dense in $[0,1]$. But it follows from (8) that

$$f\left(\frac{k}{2^v}\right)-f\left(\frac{k-1}{2^v}\right)\le\frac{F_{v+3}}{2\cdot 3^v}\approx\frac{\phi^3}{2\sqrt{5}}\left(\frac{\phi}{3}\right)^v, \tag{14}$$

which goes to 0 as $v\to\infty$. Thus, Y is indeed dense in $[0,1]$.

The mirror symmetry of the r-th row of the diatomic array implies that $f(1-x)=1-f(x)$ for any dyadic rational x, and this is inherited by F by continuity. □

In the notation of the last notes, $F\in\mathcal{F}$. There are now two natural questions. What is the formula for $F(\lambda)$ when λ is *not* a dyadic rational? What can be said about the differentiability of F? We can answer the first question more easily than the second.

We begin by rephrasing the definition of f. Let (r_j) be a strictly increasing sequence of positive integers and let $\lambda_0=0$ and

$$\lambda_m=\frac{1}{2^{r_1}}+\cdots+\frac{1}{2^{r_m}},\quad\text{for } m\ge 1, \tag{15}$$

so that $\lambda_m=\lambda_{m-1}+\frac{1}{2^{r_m}}$. We see from (8) that

$$f(\lambda_m)-f(\lambda_{m-1})=\frac{s(2^{r_m+1}(1+\lambda_m)-1)}{2\cdot 3^{r_m}}, \tag{16}$$

and since $f(0)=0$, it follows that

$$f(\lambda_m)=\sum_{j=1}^{m}\frac{s(2^{r_j+1}(1+\lambda_j)-1)}{2\cdot 3^{r_j}}\ . \tag{17}$$

Since every $x\in[0,1]$ is a limit of dyadic rationals x_m given as above, we now have our definition of F:

$$x_m=\sum_{j=1}^{m}\frac{1}{2^{r_j}},\quad x=\lim_{m\to\infty}x_m\implies F(x)=\sum_{j=1}^{\infty}\frac{s(2^{r_j+1}(1+\lambda_j)-1)}{2\cdot 3^{r_j}}\ . \tag{18}$$

We note that this gives a peculiar recurrence satisfied by F.

Theorem 2. *We have $F(x)-6F(\frac{x}{2})+9F(\frac{x}{4})=0$ for $x\in[0,1]$; that is, $3^nF(\frac{x}{2^n})$ is linear in n.*

Proof. Let x be given, with r_j and λ_j as above, then for $t = 1, 2$

$$\frac{x}{2^t} = \sum_{j=1}^{\infty} \frac{1}{2^{r_j+t}}, \quad \lambda_{m,t} = \frac{\lambda_m}{2^t} \implies F(x) - 6F\left(\frac{x}{2}\right) + 9F\left(\frac{x}{4}\right) = \sum_{j=1}^{\infty} \frac{W_j}{3^{r_j}}, \tag{19}$$

where

$$\begin{aligned} W_j &= s(2^{r_j+1}(1+\lambda_j) - 1) - 2s(2^{r_j+2}(1+\lambda_j/2) - 1) + s(2^{r_j+3}(1+\lambda_j/4) - 1) \\ &= s(2^{r_j+1} + 2^{r_j+1}\lambda_j - 1) - 2s(2^{r_j+2} + 2^{r_j+1}\lambda_j - 1) + s(2^{r_j+3} + 2^{r_j+1}\lambda_j - 1). \end{aligned} \tag{20}$$

But we have already seen that $s(2^n + k)$ is linear in n, when $2^n > k$ (as is the case here), and if g is any linear function, then $g(r_j + 1) - 2g(r_j + 2) + g(r_j + 2) = 0$. Thus, $W_j = 0$ for all j, completing the proof of the identity. If $a_n = 3^n F(\frac{x}{2^n})$, then $a_{n+2} - 2a_{n+1} + a_n = 0$, so a_n is linear in n. □

There is already enough information to prove the following theorem.

Theorem 3. *If x is a dyadic rational, then $F'(x) = 0$.*

Proof. We will use the "slowly approaching" lemmas of Notes, V, supplement, pp. 17–18. Let $x = \frac{k}{2^r}$, and for $m > r$, define

$$v_m = \frac{k}{2^r} - \frac{1}{2^m}, \qquad u_m = \frac{k}{2^r} + \frac{1}{2^m}. \tag{21}$$

Then $x - v_{m+1} = \frac{1}{2}(x - v_m)$ and $u_{m+1} - x = \frac{1}{2}(u_m - x)$, so Lemma 13 applies. Now observe that

$$\begin{aligned} F(x) - F(v_m) &= F\left(\frac{k}{2^r}\right) - F\left(\frac{k}{2^r} - \frac{1}{2^m}\right) = \frac{s(2^{m+1} + k2^{m-r+1} - 1)}{2 \cdot 3^m}, \\ F(u_m) - F(x) &= F\left(\frac{k}{2^r} + \frac{1}{2^m}\right) - F\left(\frac{k}{2^r}\right) = \frac{s(2^{m+1} + k2^{m-r+1} + 1)}{2 \cdot 3^m}. \end{aligned} \tag{22}$$

But

$$\begin{aligned} &s(2^{m+1} + k2^{m-r+1} \pm 1) = s(2^{m-r+1}(2^r + k) \pm 1) = \\ &s(2^{m-r+1} - 1)s(2^r + k) + s(1)s(2^r + k \pm 1) \\ &= (m - r + 1)s(2^r + k) + s(2^r + k \pm 1), \end{aligned} \tag{23}$$

so, for appropriate constants c_j we have

$$\frac{F(x) - F(v_m)}{x - v_m} = \frac{2^m(c_1 m + c_2)}{3^m} \to 0, \qquad \frac{F(u_m) - F(x)}{u_m - x} = \frac{2^m(c_1 m + c_3)}{3^m} \to 0, \tag{24}$$

completing the proof. □

We shall show later that, if $gcd(k, 3) = 1$, then $\frac{k}{3 \cdot 2^r} \in F(f, \infty)$. But in order to do this, we need a way to calculate F at non-dyadic rationals.

2. F AT NON-DYADIC RATIONALS

(Please disregard pp. 5 → 7 from the last handout; lots of typos!)

It follows from the geometric series that

$$\frac{1}{3} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots , \tag{25}$$

and so from the discussion above, we have

$$F\left(\frac{1}{3}\right) = \frac{s(2^3+1)}{2\cdot 3^2} + \frac{s(2^5+2^3+1)}{2\cdot 3^4} + \frac{s(2^7+2^5+2^3+1)}{2\cdot 3^6} + \cdots . \tag{26}$$

The numerator of the r-th term is $s(w_r)$, where

$$\begin{aligned} w_r = 2^{2r+1} + 2(2^{2r-2} + \cdots + 1) - 1 &= 2^{2r+1} + 2^{2r-1} + \cdots + 2^3 + 2^1 - 1 \\ &= 2^{2r+1} + \cdots + 2^3 + 1. \end{aligned} \tag{27}$$

By a stroke of luck (I can't plan things this well!), we have already calculated this expression: $w_r = n_{2r+1} - 2$ in the notation of equations (71)–(73) on p.11 of the Notes, IV (keeping the typos in mind), and

$$s(w_r) = 3F_{2r} + F_{2r-1}. \tag{28}$$

(We have another computation of this later in the section.) It follows that

$$F\left(\frac{1}{3}\right) = \sum_{r=1}^{\infty} \frac{3F_{2r} + F_{2r-1}}{2\cdot 3^{2r}} . \tag{29}$$

In one sense, this is a routine computation, if one breaks down the Fibonacci numbers via the Binet formula. It's also interesting to see how this can be done without using "$\sqrt{5}$" at all, but entirely via the even and odd parts of generating functions:

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \implies \sum_{n=0}^{\infty} (-1)^n F_n x^n = \frac{-x}{1+x-x^2} . \tag{30}$$

By adding and subtracting these equations, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} F_{2n} x^{2n} &= \frac{1}{2}\left(\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right) = \frac{x^2}{1-3x^2+x^4}; \\ \sum_{n=0}^{\infty} F_{2n+1} x^{2n+1} &= \frac{1}{2}\left(\frac{x}{1-x-x^2} + \frac{x}{1+x-x^2}\right) = \frac{x-x^3}{1-3x^2+x^4}. \end{aligned} \tag{31}$$

Since $F_0 = 0$, a mild amount of reindexing gives

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{F_{2n}}{3^{2n}} &= \frac{1/9}{1-1/3+1/81} = \frac{9}{55} \implies \sum_{r=1}^{\infty} \frac{3F_{2r}}{2\cdot 3^{2r}} = \frac{27}{110}; \\ \sum_{n=0}^{\infty} \frac{F_{2n+1}}{3^{2n+1}} &= \frac{1/3-1/27}{1-1/3+1/81} = \frac{24}{55} \implies \sum_{r=1}^{\infty} \frac{F_{2r-1}}{2\cdot 3^{2r}} = \sum_{r=0}^{\infty} \frac{F_{2r+1}}{2\cdot 3^{2r+2}} = \frac{8}{110}. \end{aligned} \tag{32}$$

We conclude that

$$F\left(\frac{1}{3}\right) = \frac{27}{110} + \frac{8}{110} = \frac{35}{110} = \frac{7}{22}. \tag{33}$$

You may be forgiven for not quite believing this. As a nod towards the skeptical, $2^{16} = 65536$, $\frac{4}{3} \cdot 2^{16} = 87381\frac{1}{3}$, and a Mathematica calculation shows that

$$\frac{1}{3^{16}} \cdot \sum_{n=65536}^{87381}{}^{*} s(n) - \frac{7}{22} = \frac{27391903}{86093442} - \frac{7}{22} = -\frac{8057}{473513931}, \tag{34}$$

if I've copied right. The difference is in the fifth decimal place, which is pretty close.

What does this mean? Since $F(\frac{1}{3}) = \frac{7}{22}$, we must have $F(\frac{2}{3}) = \frac{15}{22}$, so there is slightly more Stern "stuff" in the middle than at the ends. This is also shown by $F(\frac{1}{4}) = \frac{2}{9}$ and $F(\frac{3}{4}) = \frac{7}{9}$.

We generalize, and reaffirm these numbers. Suppose $k \geq 2$. Then

$$\frac{1}{2^k - 1} = \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}} + \cdots . \tag{35}$$

It follows that

$$F\left(\frac{1}{2^k - 1}\right) = \frac{s(2^{k+1} + 1)}{2 \cdot 3^k} + \frac{s(2^{2k+1} + 2^{k+1} + 1)}{2 \cdot 3^{2k1}} + \cdots . \tag{36}$$

We have made a very similar computation recently. If we take $m = 1$ in the "Bonus round" Theorem of HW2 solutions, and define $a_r = \frac{2^{kr}-1}{2^k-1} = 2^{k(r-1)} + \cdots + 2^k + 1$, then since $s(m+1) + s(2^k - m) = s(2) + s(2^k - 1) = k + 1$, we have that

$$s(a_0) = 0, \quad s(a_1) = 1, \qquad s(a_{r+2}) - (k+1)s(a_{r+1}) + s(a_r) = 0. \tag{37}$$

In this notation, if we let $c_j = 2^{jk+1} + \cdots + 2^{k+1} + 1 = 2a_{j+1} - 1$, then

$$s(c_j) = s(a_{j+1}) + s(a_{j+1} - 1) = s(a_{j+1}) + s(2^k a_j) = s(a_{j+1}) + s(a_j), \tag{38}$$

and so $s(c_{j+2}) - (k+1)s(c_{j+1}) + s(c_j) = 0$ as well. That is,

$$F\left(\frac{1}{2^k - 1}\right) = \sum_{j=1}^{\infty} \frac{s(c_j)}{2 \cdot 3^{jk}}, \tag{39}$$

and can now use the method of generating functions to evaluate the sum:

$$\begin{aligned} &\Phi(X) = \sum_{j=1}^{\infty} s(c_j) X^j \\ &\implies (1 - (k+1)X + X^2)\Phi(X) = s(c_1)X + (s(c_2) - (k+1)s(c_1))X^2. \end{aligned} \tag{40}$$

Now $s(c_1) = s(a_2) + s(a_1) = (k+1) + 1 = k+2$ and $s(c_2) = s(a_3) + s(a_2) = (k^2+2k) + (k+1) = k^2+3k+1$, so

$$\Phi(X) = \frac{(k+2)X - X^2}{1-(k+1)X+X^2} \tag{41}$$
$$\implies F\left(\frac{1}{2^k-1}\right) = \frac{1}{2}\cdot\Phi\left(\frac{1}{3^k}\right) = \frac{(k+2)3^k - 1}{2(3^{2k} - (k+1)3^k + 1)}.$$

As a check, for $k = 2$, this becomes

$$F\left(\frac{1}{3}\right) = \frac{36-1}{2(81-27+1)} = \frac{35}{110} = \frac{7}{22}. \tag{42}$$

The next step would be to calculate $F(\frac{2}{2^k-1})$, because then the recurrence in Theorem 3 will give us $F(\frac{2^j}{2^k-1})$ for all integral $j < k$, including negative j. We don't need this for $k = 2$, because $\frac{2}{3} = 1 - \frac{1}{3}$. Let

$$a_n = 3^n F\left(\frac{1}{2^n}\cdot\frac{2}{3}\right). \tag{43}$$

Theorem 3 shows that a_n is linear in n, and since $a_0 = F(\frac{2}{3}) = \frac{15}{22}$ and $a_1 = 3F(\frac{1}{3}) = \frac{21}{22}$, we have $a_n = \frac{15+6n}{22}$. Thus,

$$F\left(\frac{1}{6}\right) = \frac{1}{9}\cdot\frac{27}{22} = \frac{3}{22}, \quad F\left(\frac{1}{12}\right) = \frac{1}{27}\cdot\frac{33}{22} = \frac{1}{18}, \quad F\left(\frac{1}{3\cdot 2^r}\right) = \frac{7+2r}{22\cdot 3^r}. \tag{44}$$

We now wish to show that F is not differentiable at $\lambda = \frac{m}{3\cdot 2^v}$, provided $gcd(3,m) = 1$. In the interest of time, and interest, we shall only write out the details in one direction. Recall our notations from p.3 of these notes.

Lemma 4. *If*

$$\lim_{n\to\infty} \frac{2^{r_n}\cdot s(2^{r_n+1}(1+\lambda_n)-1)}{3^{r_n}} = \infty, \tag{45}$$

then F is not differentiable at x.

Proof. We have the following inequalities:

$$F(\lambda) - F(\lambda_{n-1}) \geq \frac{s(2^{r_n+1}(1+\lambda_n)-1)}{2\cdot 3^{r_n}} \tag{46}$$
$$x - \lambda_{n-1} < \frac{1}{2^{r_n}} + \frac{1}{2^{r_n+1}} + \frac{1}{2^{r_n+2}}\cdots = \frac{2}{2^{r_n}}.$$

If (45) holds, then it follows that

$$\frac{F(\lambda)-F(\lambda_{n-1})}{\lambda-\lambda_{n-1}} > \frac{2^{r_n}\cdot s(2^{r_n+1}(1+\lambda_n)-1)}{4\cdot 3^{r_n}} \to \infty \tag{47}$$

as $n \to \infty$, and hence F is not differentiable at λ. ☐

A more careful argument can show that the difference quotient goes to ∞ as λ is approached from the left. (Note that there is no guarantee here that λ_m converges slowly to λ, because the difference between the r_m's might be unbounded if λ is irrational.)

Theorem 5. *If* $\lambda = \frac{m}{3\cdot 2^v}$ *and* $gcd(3,m) = 1$, *then* F *is not differentiable at* λ.

Proof. First suppose $\lambda = \frac{k}{2^v} + \frac{1}{3\cdot 2^v}$. We have

$$\lambda = \frac{k}{2^v} + \frac{1}{2^{v+2}} + \frac{1}{2^{v+4}} + \cdots, \tag{48}$$

so

$$\lambda_j = \frac{2^{2j}k + 2^{2j-2} + \cdots + 1}{2^{v+2j}}. \tag{49}$$

If

$$z_j = 2^{v+2j+1} + 2^{2j+1}k + 2^{2j-1} + \cdots + 2^3 + 1, \tag{50}$$

then

$$F(\lambda) = F\left(\frac{k}{2^v}\right) + \sum_{j=1}^{\infty} \frac{s(z_j)}{2\cdot 3^{v+2j}}. \tag{51}$$

However, $z_j = [\cdots(10)^{j-2}1001]_2$, so $z_j \sim [\cdots 1^{2j-3}21]$, and so $s(z_j) \ge F_{2j}$ – a very crude bound! Applying the lemma, we have

$$\frac{2^{r_m} s(2^{r_m+1}(1+\lambda_m) - 1)}{3^{r_m}} \ge \frac{2^{v+2m}F_{2m}}{3^{v+2m}} = \left(\frac{2}{3}\right)^v \cdot \frac{2^{2m}F_{2m}}{3^{2m}}. \tag{52}$$

The numerator grows like $(1+\sqrt{5})^{2m}$, and since $1+\sqrt{5} > 3$, it follows that the quotient goes to ∞ and F is not differentiable at λ.

If $\lambda = \frac{k}{2^v} + \frac{2}{3\cdot 2^v}$, then $\lambda = 1 - \lambda'$, where $\lambda' = \frac{2^v-k-1}{2^v} + \frac{1}{3\cdot 2^v}$, and F is not differentiable at λ'. But since $F(1-x) = 1 - F(x)$, it follows that F is not differentiable at λ either. $\square$

We conclude this section with a more careful computation of F. Let $s(2^v + k) = a$ and $s(2^v + k + 1) = b$ and recall the definition of w_r from equation (27) of these notes. We have

$$\begin{aligned} z_1 = 2^3(2^v + k) + 1 &\implies s(z_1) = s(7)a + s(1)b = 3a + b, \\ z_2 = 2^5(2^v + k) + 2^3 + 1 &\implies s(z_2) = s(23)a + s(9)b = 7a + 4b, \\ z_3 = 2^7(2^v + k) + 2^5 + 2^3 + 1 &\implies s(z_3) = s(87)a + s(41)b = 18a + 11b. \end{aligned} \tag{53}$$

Much of this should look familiar: these are elements of the Lucas sequence (recall: $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \ge 2$, or $L_n = F_{n+1} + F_{n-1}$.)

There are many ways to prove this (I'm sure it follows from earlier notes), but here's a different way, using the basic recurrence. We see directly that $z_{n+1} = 4z_n + 5$, so $z_{n+2} = 16z_n + 21$ and $z_{n+3} = 64z_n + 85$. Thus,

$$\begin{aligned} s(z_{n+1}) &= 2s(z_n + 1) + s(z_n + 2), \\ s(z_{n+2}) &= 5s(z_n + 1) + 3s(z_n + 2), \\ s(z_{n+3}) &= 13s(z_n + 1) + 8s(z_n + 2) \\ \implies s(z_{n+3}) &- 3s(z_{n+2}) + s(z_{n+1}) = 0 \end{aligned} \tag{54}$$

Any sequence (x_n) which satisfies $x_{n+2} = x_{n+1}+x_n$ will also satisfy $x_{n+4} = 3x_{n+2}-x_n$, and so it follows from the last two equations that

$$s(z_j) = L_{2j}a + L_{2j-1}b. \tag{55}$$

The next step is to form the generating function

$$\begin{aligned} \Psi(X) &= \sum_{j=1}^{\infty} s(z_j)X^j \\ \implies (1 - 3X + X^2)\Psi(x) &= s(z_1)X + (s(z_2) - 3s(z_1))X^2 \\ \implies \Psi(X) &= \frac{(3a + b)X + (-2a + b)X^2}{1 - 3X + X^2} \\ \implies \sum_{j=1}^{\infty} \frac{s(z_j)}{3^{2j}} &= \frac{9(3a + b) + (-2a + b)}{81 - 27 + 1} = \frac{5a + 2b}{11}. \end{aligned} \tag{56}$$

Theorem 6. *Suppose $a = s(2^v + k)$ and $b = s(2^v + k + 1)$. Then*

$$\begin{aligned} F\left(\frac{k}{2^v} + \frac{1}{3 \cdot 2^v}\right) - F\left(\frac{k}{2^v}\right) &= \frac{5a + 2b}{22 \cdot 3^v}; \\ F\left(\frac{k}{2^v} + \frac{2}{3 \cdot 2^v}\right) - F\left(\frac{k}{2^v} + \frac{1}{3 \cdot 2^v}\right) &= \frac{4a + 4b}{22 \cdot 3^v}; \\ F\left(\frac{k+1}{2^v}\right) - F\left(\frac{k}{2^v} + \frac{2}{3 \cdot 2^v}\right) &= \frac{2a + 5b}{22 \cdot 3^v}. \end{aligned} \tag{57}$$

Proof. The first equation follows from (56), applied to (50). The third follows from symmetry by considering $F(1 - x) = 1 - F(x)$. Finally, the second follows from subtracting the sum of the first and the last from (8), when rewritten in the way it will appear in the second draft:

$$F\left(\frac{k+1}{2^v}\right) - F\left(\frac{k}{2^v}\right) = \frac{a + b}{2 \cdot 3^v}. \tag{58}$$

□

It seems surely possible to continue this work for any odd division of a dyadic interval, but we'll stop here.

We close this section with some numbers. We have already seen that when $[0, 1]$ is broken up into $\frac{1}{8}$-ths, the fraction of the Stern mass is divided in ratio $5 : 7 : 8 : 7 : 7 : 8 : 7 : 5$. Using Theorem 6, we can compute the mass when $[0, 1]$ is broken up into $\frac{1}{24}$-ths. By symmetry, we only pressent the first 12: is divided in ratio

$$13 : 20 : 22 : 26 : 28 : 23 : 25 : 32 : 31 : 29 : 28 : 20. \tag{59}$$

The denominator here is $11 \cdot 54 = 594$. Thus, $F(\frac{1}{24}) = \frac{13}{594}$, $F(\frac{2}{24}) = F(\frac{1}{12}) = \frac{33}{594} = \frac{1}{18}$, as we have seen, etc. And in the largest six 24-ths of the interval – $x \in [\frac{7}{24}, \frac{10}{24}] \cup [\frac{14}{24}, \frac{17}{24}]$, we find $\frac{2(32+31+29)}{594} \approx 31\%$ of the total mass. In other words, it's pretty well distributed, at least on this level of granularity.

3. More to do

We bid a farewell to F with these notes, but there is more work to do.

• The closed form for F is unsatisfactory in some ways, and it would be good to find a version in which the infinite sum did not involve unevaluated elements of the Stern sequence.

• It is almost certainly true that if $x \in \mathbb{Q}$, then $F(x) \in \mathbb{Q}$. This would follow from the eventual periodicity of the binary expansion of x, when combined with the (unproved) lemma that if (y_n) is a sequence given by the recurrence $y_{n+1} = 2^r y_n + a$, with $0 \le a < 2^r$, then $s(y_n)$ satsifies a second order recurrence. This can be proved using a version of the bonus round theorem, which will undoubtedly have a more dignified name in the second draft.

• I'll put the computation of $F(\frac{1}{5})$ on the homework. Once this is done, it is also possible, if tedious, to write a version of Theorem 6 for 5-ths.

• It is totally natural to define the measure μ_F on $[0, 1]$ by "differentiating' F:

$$\mu([0, x]) := F(x) = \int_0^x d\mu_F \tag{60}$$

It is also totally natural to make a pun out of a cliche and call this the Stern measure. I have not found (yet) anything really interesting which will let me say that the "Stern measure must be applied." The most natural questions about this measure would be to find its moments; that is

$$\int_0^1 t^k \, d\mu_F. \tag{61}$$

• A picture of F was given in the first day's handout. Some questions which come to mind there are these: what is the maximum of $f(x) - x$? where does it occur? what can be said about the places where $f(x) = x$, besides $x = 0, \frac{1}{2}, 1$?

4. $F(\frac{1}{5})$ AND MORE

We really ought to work out $F(\frac{1}{5})$. First observe that

$$\frac{1}{5} = \frac{3}{15} = \frac{2^1+2^0}{2^4-1} = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^{11}} + \frac{1}{2^{12}} + \cdots . \tag{62}$$

We define the sequence of consecutive dyadic approximations to $\frac{1}{5}$:

$$\lambda_0 = 0, \quad \lambda_1 = \frac{1}{2^3}, \quad \lambda_2 = \lambda_1 + \frac{1}{2^4}, \quad \lambda_3 = \lambda_2 + \frac{1}{2^7}, \quad \lambda_4 = \lambda_3 + \frac{1}{2^8} \dots ; \tag{63}$$

that is,

$$\lambda_{2m} = \lambda_{2m-1} + \frac{1}{2^{4m}}, \qquad \lambda_{2m+1} = \lambda_{2m} + \frac{1}{2^{4m+3}}. \tag{64}$$

As we've seen before

$$\begin{aligned} F\left(\frac{1}{5}\right) &= \sum_{k=1}^{\infty} F(\lambda_k) - F(\lambda_{k-1}) \\ &= \sum_{m=0}^{\infty} \frac{s(2^{4m+4}(1+\lambda_{2m+1})-1)}{2\cdot 3^{4m+3}} + \sum_{m=1}^{\infty} \frac{s(2^{4m+1}(1+\lambda_{2m})-1)}{2\cdot 3^{4m}}. \end{aligned} \tag{65}$$

Let

$$\alpha_m = 2^{4m+1}(1+\lambda_{2m}) - 1 = 2y_m - 1, \qquad \beta_m = 2^{4m+4}(1+\lambda_{2m+1}) - 1 = 2z_m - 1. \tag{66}$$

Then

$$\begin{aligned} y_m &= 2^{4m}(1+\lambda_{2m}) = 2^{4m}\left(1+\lambda_{2m-1}+\frac{1}{2^{4m}}\right) = 2z_{m-1}+1 \\ z_m &= 2^{4m+3}(1+\lambda_{2m+1}) = 2^{4m+3}\left(1+\lambda_{2m}+\frac{1}{2^{4m+3}}\right) = 8y_m+1. \end{aligned} \tag{67}$$

Thus, $y_{m+1} = 2z_m + 1 = 16y_m + 3$ (so $y_{m+2} = 256y_m + 51$), and $z_{m+1} = 8y_{m+1} + 1 = 16z_m + 9$ (so $z_{m+2} = 256z_m + 153$). We have, in a disturbingly familiar calculation:

$$\begin{aligned} s(y_{m+1}) &= 5s(y_m) + 2s(y_m+1), \qquad & s(y_{m+2}) &= 29s(y_m) + 12s(y_m+1), \\ s(z_{m+1}) &= 3s(z_m) + 4s(z_m+1), \qquad & s(z_{m+2}) &= 17s(z_m) + 24s(z_m+1). \end{aligned} \tag{68}$$

It follows that the sequences $\{s(y_m)\}$ and $\{s(z_m)\}$ each satisfy the recurrence

$$x_{m+2} - 6x_{m+1} + x_m = 0. \tag{69}$$

Observe as well that

$$\begin{aligned} s(\alpha_m) &= s(2y_m - 1) = s(y_m) + s(y_m - 1) = s(y_m) + s(2z_{m-1}), \\ s(\beta_m) &= s(2z_m - 1) = s(z_m) + s(z_m - 1) = s(z_m) + s(8y_m). \end{aligned} \tag{70}$$

Returning to the task at hand, we have

$$F\left(\frac{1}{5}\right) = \sum_{m=0}^{\infty} \frac{s(y_m)+s(z_m)}{2\cdot 3^{4m+3}} + \sum_{m=1}^{\infty} \frac{s(y_m)+s(z_{m-1})}{2\cdot 3^{4m}}. \tag{71}$$

Finally, let

$$\begin{aligned} \Phi(X) &= \sum_{m=0}^{\infty}(s(y_m)+s(z_m))X^m = 5 + 31X + 181X^2 + \dots \\ \Psi(X) &= \sum_{m=1}^{\infty}(s(y_m)+s(z_{m-1}))X^m = 11X + 65X^2 + 379X^3 + \dots . \end{aligned} \tag{72}$$

The coefficients of Φ and Ψ each satisfy the recurrence (69), and so when they are multiplied by $1-6X+X^2$ yield a polynomial:

$$\Phi(X) = \frac{5-X}{1-6X+X^2}, \qquad \Psi(X) = \frac{11X-X^2}{1-6X+X^2}. \tag{73}$$

A final computation gives

$$F\left(\frac{1}{5}\right) = \frac{\Phi(3^{-4})}{54} + \frac{\Psi(3^{-4})}{2} = \frac{87}{868} + \frac{445}{6076} = \frac{17}{98}. \tag{74}$$

I can assert with some confidence that someone some day will find a more direct way of computing $F(\frac{1}{5})$ without needing denominators as large as 6076. Perhaps this would follow from

$$\frac{1}{5} = \frac{1}{2^2} - \frac{1}{2^4} + \frac{1}{2^6} - + \dots . \tag{75}$$

In any event, some skepticism is appropriate, and we again check numerically: $\frac{6}{5}\cdot 2^{16} = 78643.2$, and another Mathematica calculation shows that

$$\frac{1}{3^{16}} \cdot \sum_{n=65536}^{78643}{}^{*} s(n) - \frac{17}{98} = \frac{7467080}{43046721} - \frac{17}{98} = -\frac{20417}{4218578658}, \tag{76}$$

if I've copied right. The difference again is in the fifth decimal place, which is pretty close, so this is probably right.

We have now by symmetry that $F(\frac{4}{5}) = 1 - F(\frac{1}{5}) = \frac{81}{98}$. Theorem 2 says that $F(\frac{4}{5}), 3F(\frac{2}{5})$ and $9F(\frac{1}{5})$ are in arithmetic progression, hence $3F(\frac{2}{5}) = \frac{117}{98}$ and so $F(\frac{2}{5}) = \frac{39}{98}$ and $F(\frac{3}{5}) = \frac{59}{98}$. Thus the Stern division by fifths is

$$17 : 22 : 20 : 22 : 17, \tag{77}$$

which is both regular and irregular enough to hint at some deeper structures.

More generally, and in a nod towards the second draft of these notes, let

$$x = \sum_{k=1}^{\infty} \frac{1}{2^{r_k}}, \quad \lambda_n = \sum_{k=1}^{n} \frac{1}{2^{r_k}}, \qquad \delta_1 = r_1, \quad \delta_k = r_k - r_{k-1}\ (k \ge 2). \tag{78}$$

Then as usual,

(79) $$F(x) = \sum_{n=1}^{\infty} \frac{s(2^{r_n+1}(1+\lambda_n)-1)}{2\cdot 3^{r_n}}.$$

Following the pattern for $\frac{1}{5}$, for $n \geq 1$, let

(80) $$2y_n - 1 = 2^{r_n+1}(1+\lambda_n) - 1.$$

It then follows that for $n \geq 2$,

(81) $$y_n = 2^{r_n}(1+\lambda_n) = 2^{\delta_n}2^{r_{n-1}}(1+\lambda_{n-1}+2^{-r_n}) = 2^{\delta_n}y_{n-1}+1.$$

Since $y_1 = 2^{r_1}+1$, this recurrence is also valid if we set $y_0 = 1$. Now, we have

(82) $$s(2y_n-1) = s(y_n)+s(y_n-1) = s(y_n)+s(2^{\delta_n}y_{n-1}) = s(y_n)+s(y_{n-1}),$$

and the expression for F simplifies:

(83) $$F(x) = \sum_{n=1}^{\infty} \frac{s(y_{n-1})+s(y_n))}{2\cdot 3^{r_n}}.$$

We have $s(y_0) = s(1) = 1$ and $s(y_1) = s(2^{r_1}+1) = \delta_1+1$. In general, if $x_1 = 2^a x_0 + 1$ and $x_2 = 2^b x_1 + 1 = 2^{a+b}x_0 + 2^b + 1$, then since

(84) $$s(2^{a+b}-2^b-1) = s(2^b(2^a-1)-1) = s(2^b-1)s(2^a-1)+s(1)s(2^a-2) = ab+a-1,$$

we have

(85) $$\begin{aligned} s(x_1) = as(x_0)+s(x_0+1), \quad s(x_2) &= (ab+a-1)s(x_0)+(b+1)s(x_0+1) \\ \implies s(x_2) &= (b+1)s(x_1)-s(x_0). \end{aligned}$$

It follows that the sequence $s(y_m)$ satisfies the recurrence

(86) $$s(y_m) = \delta_m s(y_{m-1}) - s(y_{m-2}), \qquad \text{for } m \geq 2.$$

If x is irrational, then the dyadic expression for x is not repeating, but if $x \in \mathbb{Q}$, then eventually the δ_m's are periodic, with period p say. It follows that each sequence $s(y_{pt+j})$ will satisfy a second-order recurrence of some kind and the method shown above will allow a computation of $F(x)$ via a number of generating functions. Also,

(87) $$\frac{s(y_m)}{s(y_{m-1})} = \delta_m - \frac{1}{\frac{s(y_{m-1})}{s(y_{m-2})}} = \ldots$$

Every positive integer can be written in this way:

(88) $$n = 2^{r_m}+\cdots+2^{r_1}+1 = 2^{r_1}(2^{r_m-r_1}+\cdots+2^{r_2-r_1})+1, \ldots$$

and so this gives a less canonical form for $s(n)$ as hte numerator of a (non-simple) continued fraction. If n has this form, then

(89) $$n \sim [1, r_m - r_{m-1} - 1, 1, \ldots, r_2 - r_1 - 1, 1, r_1 - 1, 1]_2$$

If $r_k = r_{k-1}+1$, then the 1's are consecutive. However, as we have observed, the continuant is smart enough to notice: see Notes, IV, Theorem 6.