# STERN NOTES, CHAPTER 5 (VERSION 1)

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## 1. The Brocot array

The Brocot array, which goes back to 1861, is historically at least as prominent as the Stern sequence. It can be construed as a double diatomic array. The basic idea was discussed at the beginning of these notes: if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in the $r$-th row, then the *mediant*, $\frac{a+c}{b+d}$, is inserted beween them in the $r+1$-st row. The full array starts with $(\frac{0}{1}, \frac{1}{0})$:

$$
\begin{array}{ccccccccc}
 & & & & \frac{0}{1} \quad \frac{1}{0} & & & & \\
 & & & \frac{0}{1} & \frac{1}{1} & \frac{1}{0} & & & \\
 & & \frac{0}{1} & \frac{1}{2} & \frac{1}{1} & \frac{2}{1} & \frac{1}{0} & & \\
\frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} & \frac{3}{2} & \frac{2}{1} & \frac{3}{1} & \frac{1}{0} \\
 & & & & \cdots & & & &
\end{array} \tag{1}
$$

It is clear, using the earlier notation, that the $r$-th row (starting with $r=0$) has $2^r$ elements, and that the $k$-th element (starting with $k=0$) is

$$\frac{Z(r,k;0,1)}{Z(r,k;0,1)} = \frac{s(k)}{s(2^r-k)}. \tag{2}$$

Considering the symmetry of the array and the dubiousness of "$\frac{1}{0}$", it is customary to write only the first half of this picture, thus starting in effect with $(\frac{0}{1}, \frac{1}{1})$, and call it the *Brocot array*:

$$
\begin{array}{ccccccccc}
 & & & & \frac{0}{1} \quad \frac{1}{1} & & & & \\
 & & & \frac{0}{1} & \frac{1}{2} & \frac{1}{1} & & & \\
 & & \frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} & & \\
\frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1} \\
 & & & & \cdots & & & &
\end{array} \tag{3}
$$

So, in the same labeling, the $k$-th entry in the $r$-th row is

$$\frac{Z(r,k;0,1)}{Z(r,k;1,1)} = \frac{s(k)}{s(2^{r+1}-k)} = \frac{s(k)}{s(2^r+k)}. \tag{4}$$

(This last expression is possible for $0 < k < 2^r$, because then $(2^{r+1}-k)' = 2^r + k$; it is true for $k = 0, 2^r$ because $s(2^r) = s(2^{r+1})$.)

**Lemma 1.** *The entries of the $r$-th row of the Brocot array are increasing.*

*Proof.* We prove something more, that

$$\frac{s(k+1)}{s(2^r+k+1)} - \frac{s(k)}{s(2^r+k)} = \frac{1}{s(2^r+k)s(2^r+k+1)}. \tag{5}$$

Indeed, this is true in the first row; $\frac{1}{1} - \frac{0}{1} = \frac{1}{1}$. Assuming it is true in the $r$-th row, with entries $\frac{a}{b}, \frac{c}{d}$, we note that

$$\begin{aligned} \frac{a+c}{b+d} - \frac{a}{b} &= \frac{b(a+c)-a(b+d)}{b(b+d)} = \frac{bc-ad}{b(b+d)} = \frac{1}{b(b+d)}, \\ \frac{c}{d} - \frac{a+c}{b+d} &= \frac{c(b+d)-d(a+c)}{d(b+d)} = \frac{bc-ad}{d(b+d)} = \frac{1}{d(b+d)}. \end{aligned} \tag{6}$$

Thus the result holds by induction. □

(We also note that $s(k+1)s(2^r+k) - s(k)s(2^r+k+1) = 1$, which suggests that the next problem set will contain an examination of $s(m)s(n+1) - s(m+1)s(m)$.)

This property can be used to show that every rational in $[0,1]$ appears in the Brocot array, but we can also use Theorem 11 from last week: If $k$ is odd, $2^{r_0} < k < 2^{r_0+1}$ and $r \ge r_0 + 1$, then

$$\frac{s(2^r+k)}{s(k)} = \frac{s((\overleftarrow{2^r+k})')}{s((\overleftarrow{2^r+k})'+1)}. \tag{7}$$

Let's work backwards. Suppose $0 < \frac{p}{q} < 1$ and suppose that the quotient in (7) is $\frac{q}{p}$. We already know that there exists a unique odd $n$ so that $s(n) = q$ and $s(n+1) = p$. Suppose $2^r < n < 2^{r+1}$, and write $n = 2^r + \ell$, so that $n' = 2^{r+1} - \ell$. We have

$$n = (\overleftarrow{2^r+k})' \iff 2^{r+1} - \ell = \overleftarrow{2^r+k} \iff \overleftarrow{2^{r+1}-\ell} = 2^r + k. \tag{8}$$

Thus, if we keep the same $r$, and define $k$ by

$$k = (\overleftarrow{2^{r+1}-\ell}) - 2^r, \tag{9}$$

then we have

$$\frac{p}{q} = \frac{s(k)}{s(2^r+k)} = \frac{s(n+1)}{s(n)}. \tag{10}$$

It is possible to derive a more explicit formula for $k$ from this information, but it is actually faster and more instructive to do it from scratch.

The *Minkowski ?-function* is defined on $[0,1]$, but for now we define it on the entries of the Brocot array by:

$$?\left(\frac{s(k)}{s(2^r+k)}\right) := \frac{k}{2^r}. \tag{11}$$

We first note that $?(x)$ is well-defined, because $s(2k) = s(k)$, $s(2^{r+1}+2k) = s(2^r+k)$ and $\frac{2k}{2^{r+1}} = \frac{k}{2^r}$. It is also strictly increasing on each row of the Brocot array, and so on the rationals. It will be helpful to also consider the inverse function $?^{-1}(x)$

$$?^{-1}\left(\frac{k}{2^r}\right) = \frac{s(k)}{s(2^r+k)}. \tag{12}$$

In order to develop the closed form, we consider the continued fractions of $?^{-1}(\frac{k}{2^r})$ for small values of $r$.

$$\frac{1}{2} \to \frac{1}{2^1};$$

$$\frac{1}{3} \to \frac{1}{2^2}, \quad \cfrac{1}{1+\cfrac{1}{2}} \to \frac{3}{2^2}; \tag{13}$$

$$\frac{1}{4} \to \frac{1}{2^3}, \quad \cfrac{1}{2+\cfrac{1}{2}} \to \frac{3}{2^3}, \quad \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}} \to \frac{5}{2^3}, \quad \cfrac{1}{1+\cfrac{1}{3}} \to \frac{7}{2^3}.$$

There are several patterns implicit in this table, which we now prove. The first lemma may well appear earlier in the notes.

**Lemma 2.** *If $0 < k < 2^r$ is odd, $s(k) = p$ and $s(2^r+k) = q$, then $s(2^{r+1}+k) = p+q$.*

*Proof.* We have seen *ad nauseum* that $s(2^r n + k)$ is linear in $r$ for $r \ge \lceil \log_2 k \rceil$, with coefficient $s(n)s(k)$. Just take $n = 1$. □

**Lemma 3.**

$$?\left(\frac{p}{q}\right) = \frac{k}{2^r} \implies ?\left(\frac{p}{p+q}\right) = \frac{k}{2^{r+1}} \quad \text{and} \quad ?\left(\frac{q}{p+q}\right) = \frac{2^{r+1}-k}{2^{r+1}}. \tag{14}$$

*Proof.* We assume that $s(k) = p$ and $s(2^r+k) = q$. As noted above, $s(2^{r+1}+k) = p+q$. Since $2^r < 2^r+k < 2^{r+1}$, $(2^r+k)' = 2^{r+1}-k$ and $s(2^{r+1}-k) = q$; since $2^{r+1} < 2^{r+1}+k < 2^{r+2}$, $(2^{r+1}+k)' = 2^{r+2}-k$ and $s(2^{r+2}-k) = p+q$. Thus, we have

$$\frac{s(k)}{s(2^{r+1}+k)} = \frac{p}{p+q}, \qquad \frac{s(2^{r+1}-k)}{s(2^{r+1}+(2^{r+1}-k))} = \frac{q}{p+q}. \tag{15}$$

□

On taking $x = \frac{p}{q}$ and noting that $?(\frac{1}{2}) = \frac{1}{2}$, it follows by induction from this lemma that for $x \in (0,1) \cap \mathbb{Q}$,

$$?(x) + ?(1-x) = 1, \qquad ?\left(\frac{x}{x+1}\right) = \frac{?(x)}{2}, \qquad ?\left(\frac{1}{x+1}\right) = 1 - \frac{?(x)}{2}. \tag{16}$$

It is now convenient to introduce some notation. Let $a = (a_1, \dots, a_w) \in \mathbb{N}^w$ for some $w \ge 1$ and let

$$[a] = [0, a_1, \dots, a_w] := 0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_w}}} \ . \tag{17}$$

We also define

$$\|a\| = \sum_{j=1}^{w} a_j, \tag{18}$$

and observe that if $\|b\| = r + 1$, then either $(b_1, \dots, b_w) = (1 + a_1, \dots, a_w)$ or $(b_1, \dots, b_w) = (1, a_2, \dots, a_w)$ , where $\|a\| = r$, depending whether or not $a_1 > 1$.

**Lemma 4.**

$$[0, a_1, \dots, a_w] = \frac{p}{q} \implies [0, 1+a_1, \dots, a_w] = \frac{p}{p+q}, \quad [0, 1, a_1, \dots, a_w] = \frac{q}{p+q} \ . \tag{19}$$

*Proof.* Since

$$\frac{q}{p} = a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_w}} \ , \tag{20}$$

we have immediately that

$$[0, 1 + a_1, \dots, a_w] = \frac{1}{1 + \frac{q}{p}}, \quad \text{and} \quad [0, 1, a_1, \dots, a_w] = \frac{1}{1 + \frac{p}{q}} \ . \tag{21}$$

□

**Theorem 5.**

$$?([0, a_1, \dots, a_w]) = \frac{1}{2^{a_1 - 1}} - \frac{1}{2^{a_1 + a_2 - 1}} + - \cdots + \frac{(-1)^{w+1}}{2^{a_1 + a_2 + \cdots + a_w - 1}} \ . \tag{22}$$

*Proof.* We first observe that there is no ambiguity in the definition arising from

$$[0, a_1, \dots, a_{w-1}, a_w, 1] = [0, a_1, \dots, a_{w-1}, a_w + 1], \tag{23}$$

inasmuch as

$$\frac{(-1)^{w+1}}{2^{a_1 + a_2 + \cdots + a_w - 1}} + \frac{(-1)^{w+2}}{2^{a_1 + a_2 + \cdots + a_w + 1 - 1}} = \frac{(-1)^{w+1}}{2^{a_1 + a_2 + \cdots + a_w + 1 - 1}}. \tag{24}$$

To make the right-hand side look more familiar, observe that

$$\frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + - \cdots + \frac{(-1)^{w+1}}{2^{a_1+a_2+\cdots+a_w-1}} = \frac{k}{2^{\|a\|-1}} \tag{25}$$

for some odd $k$. We induct on $\|a\|$. The cases in which $\|a\| \leq 3$ are $[1]$, $[2] = [1, 1]$, $[3] = [2, 1]$ and $[1, 2] = [1, 1, 1]$. Referring to the previous table, we do indeed have

$$\begin{gathered} ?([1]) = \frac{1}{2^{1-1}}; \qquad ?([2]) = ?\left(\frac{1}{2}\right) = \frac{1}{2^{2-1}}; \\ ?([3]) = ?\left(\frac{1}{3}\right) = \frac{1}{2^{3-1}}, \qquad ?([1, 2]) = ?\left(\frac{2}{3}\right) = \frac{1}{2^{1-1}} - \frac{1}{2^{1+2-1}} = \frac{3}{4}. \end{gathered} \tag{26}$$

Suppose $\|b\| = r + 1$. There are two cases. If $b_1 > 1$, then $b = (1 + a_1, \ldots, a_w)$ with $\|a\| = r$. Write $[0, a_1, \ldots, a_w] = \frac{p}{q}$ and $?(\frac{p}{q}) = \frac{k}{2^r}$. By Lemmas 3 and 4 and the inductive hypothesis,

$$\begin{gathered} ?([0, 1 + a_1, \ldots, a_w]) = ?\left(\frac{p}{p+q}\right) = \frac{k}{2^{r+1}} = \\ \frac{1}{2} \cdot \left(\frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + - \cdots + \frac{(-1)^{w+1}}{2^{a_1+a_2+\cdots+a_w-1}}\right) \\ = \frac{1}{2^{1+a_1-1}} - \frac{1}{2^{1+a_1+a_2-1}} + - \cdots + \frac{(-1)^{w+1}}{2^{1+a_1+a_2+\cdots+a_w-1}}. \end{gathered} \tag{27}$$

If $b_1 = 1$, then $b = (1, a_1, \ldots, a_w)$ with $\|a\| = r$. Using the same notation and arguments as above,

$$\begin{gathered} ?([0, 1, a_1, \ldots, a_w]) = ?\left(\frac{q}{p+q}\right) = \frac{2^{r+1} - k}{2^{r+1}} = 1 - \frac{1}{2} \cdot \frac{k}{2^r} = \\ \frac{1}{2^{1-1}} - \frac{1}{2} \cdot \left(\frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + - \cdots + \frac{(-1)^{w+1}}{2^{a_1+a_2+\cdots+a_w-1}}\right) \\ = \frac{1}{2^{1-1}} - \frac{1}{2^{1+a_1-1}} + - \cdots + \frac{(-1)^{w+2}}{2^{1+a_1+a_2+\cdots+a_w-1}}. \end{gathered} \tag{28}$$

This completes the proof. □

It is worth noting that this proof essentially establishes the formula for $?^{-1}$, and that it does *not* try to compute the continued fraction expansion of the mediant from its component parts, which would have been the first, natural step. We could have derived (32) from the previous formula for the Stern sequence, although the appearance of $n'$ means that there would be cases depending on whether the initial and terminal denominators are 1. (This case breakdown does not occur in the above derivation, but see below.) One can also use this approach to derive a closed formula for the Stern sequence, but it seems longer than what we did.

We can always specify that $w$ is even. In this case, the binary expression of $\frac{k}{2^r}$ might depends on whether $a_1 > 1$ or not. If $a_1 > 1$, then this expression contains

$a_1 - 1$ 0's, followed by $a_2$ 1's, $a_3$ 0's, ending up with $a_w$ 1's. If $a_1 = 1$, then we have $a_2$ 1's, $a_3$ 0's, ending up with $a_w$ 1's.

If $a^{(r,1)}, \dots a^{(r,2^{r-1})}$ are the continued fraction denominators which appear corresponding to $\frac{s(k)}{s(2^r+k)}$ for odd $k$ on the $r$-th row, then $\|a^{(r,j)}\| = r+1$, and the corresponding denominators on the $(r+1)$-st row are, first, the ones from the $r$-th row, in order, with $a_1$ incremented by one, and then, the ones from the $r$-th row, in reverse order, with 1 appended to the left. Thus, for $r \le 3$, we have:

$$\begin{array}{c} [1] \\ [2] \quad [1,1] \\ [3] \quad [2,1] \quad [1,1,1] \quad [1,2] \\ [4] \quad [3,1] \quad [2,1,1] \quad [2,2], \quad [1,1,2], \quad [1,1,1,1], \quad [1,2,1], \quad [1,3] \\ \cdots \end{array} \tag{29}$$

## 2. Some finite computations

It is worth noting that the computation of continued fractions with repeating denominators is closely linked to linear recurrences. For example, suppose we consider the continued fraction with $n$ identical denominators equal to $a$.

$$a + \cfrac{1}{a + \cfrac{1}{\cdots + \cfrac{1}{a}}} = \frac{p_n(a,\dots,a)}{p_{n-1}(a,\dots,a)} . \tag{30}$$

Let $x_n(a) = p_n(a,\dots,a)$. Then:

$$x_0(a) = 1, \quad x_1(a) = a, \quad x_n(a) = a x_{n-1}(a) + x_{n-2}(a). \tag{31}$$

The characteristic equation and its roots are

$$\lambda^2 - a\lambda - 1 = 0 \implies \lambda_+ = \frac{a + \sqrt{a^2+4}}{2}, \qquad \lambda_- = \frac{a - \sqrt{a^2+4}}{2} . \tag{32}$$

We know that $x_n(a) = c_1\lambda_+^n + c_2\lambda_-^n$, and since $x_{-1}(a) = 0$, at least consistently with the initial conditions, it is not hard to show that

$$x_n(a) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\sqrt{a^2+4}}. \tag{33}$$

This reminds one that $x_n(1) = F_{n+1}$. If $a \in \mathbb{N}$, then $\lambda_+ > 1$ and $\lambda_+\lambda_- = -1$ imply that $\lambda_-^m \to 0$ as $m \to \infty$ so that as $n \to \infty$,

$$x_n(a) = a + \cfrac{1}{a + \cfrac{1}{\cdots + \cfrac{1}{a}}} = \frac{x_n(a)}{x_{n-1}(a)} \to \lambda_+ = \frac{a + \sqrt{a^2+4}}{2} . \tag{34}$$

Inasmuch as

$$x_n(a) = a + \frac{1}{x_{n-1}(a)}, \tag{35}$$

if we knew *a priori* that $\lim x_n(a) = \theta$, for some $\theta$, we would have $\theta = a + \frac{1}{\theta}$, which implies that $\theta^2 - a\theta - 1 = 0$. This is true, as part of a more general calculation in the next section.

In terms of the Minkowski ?-function, we have

$$\begin{aligned} ?\left(\frac{x_{n-1}(a)}{x_n(a)}\right) &= ?([0, a, \dots, a]) = \frac{1}{2^{a-1}} - \frac{1}{2^{2a-1}} + - \cdots + \frac{(-1)^{n+1}}{2^{na-1}} \\ &= \frac{1}{2^a+1} \cdot \frac{2^{na} - (-1)^n}{2^{na-1}} = \frac{2}{2^a+1} + \frac{(-1)^{n+1}}{(2^a+1)2^{na-1}} . \end{aligned} \tag{36}$$

For example, taking $a = 1$, we have

$$?\left(\frac{F_n}{F_{n+1}}\right) = \frac{2}{3} + \frac{(-1)^{n+1}}{3 \cdot 2^{n-1}}. \tag{37}$$

These will be useful formulas when we look at $?'(x)$. It is also not too difficult to compute explicitly

$$a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \dots}}} = \frac{p_n(a, b, a, b, \dots)}{p_{n-1}(b, a, b, a, \dots)} . \tag{38}$$

We consider the numerator first, defining $y_n(a, b)$ by:

$$\begin{gathered} y_0(a, b) = 1, \quad y_1(a, b) = a, \\ y_{2n}(a, b) = by_{2n-1}(a, b) + y_{2n-2}(a, b), \quad y_{2n+1}(a, b) = ay_{2n}(a, b) + y_{2n-1}(a, b). \end{gathered} \tag{39}$$

This doesn't look very promising, but if we suppress the argument for brevity, we can note that

$$\begin{aligned} y_{2n} &= y_{2n}, \\ y_{2n+1} &= ay_{2n} + y_{2n-1}, \\ y_{2n+2} &= by_{2n+1} + y_{2n} = (ab+1)y_{2n} + by_{2n-1}, \\ y_{2n+3} &= ay_{2n+2} + y_{2n+1} = (a^2b + 2a)y_{2n} + (ab+1)y_{2n-1}, \\ y_{2n+4} &= by_{2n+3} + y_{2n+2} = (a^2b^2 + 3ab + 1)y_{2n} + (ab^2 + 2b)y_{2n-1}. \end{aligned} \tag{40}$$

This looks even less promising; miraculously though,

$$y_{2n+4} = (ab+2)y_{2n+2} - y_{2n}, \tag{41}$$

and since this equation is symmetric in $\{a, b\}$, it would also have applied if we had started at $y_{2n+1}$, with $a$ and $b$ reversed. Therefore, we can say that for $r \ge 4$,

$$y_r = (ab + 2)y_{r-2} - y_{r-4} \tag{42}$$

and in this way obtain explicit formulas. It is not accidental that this might be helpful in some problems on the second homework. Finally, if the limit in (38) is $\theta$, then it is reasonable to expect that

$$\theta = a + \frac{1}{b + \frac{1}{\theta}} \implies b\theta^2 - ab\theta - a = 0, \tag{43}$$

and $\theta$ would be the positive root of this quadratic. (If $a = b$, this reduces to the previous equation, as it should.)

## 3. Biting the bullet

We would like to extend the definition of $?(x)$ to the entire interval $[0, 1]$, which means we will have to consider infinite continued fractions. This means that we really should more standard terminology. We shall say that

$$[x_0, x_1, \dots x_n] := x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\dots + \cfrac{1}{x_n}}}}\ , \tag{44}$$

under the convention that $x_i \in \mathbb{Z}$, with $x_i \ge 1$ for $i \ge 1$. We already know that

$$[x_0, x_1, \dots x_n] = \frac{p_{n+1}(x_0, \dots, x_n)}{p_n(x_1, \dots, x_n)}\ . \tag{45}$$

We should have said explicitly, but haven't yet, that we can also write

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\dots + \cfrac{1}{x_n}}}} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\dots + \cfrac{1}{x_k + \cfrac{1}{[x_{k+1}, \dots, x_n]}}}} \tag{46}$$

for any $k$, $0 < k < n$, with the acknowledgement that the last "denominator" is not an integer.

**Lemma 6.** *If $x_i \ge 1$, then $p_n(x_1, \dots, x_n) \ge F_{n+1}$.*

*Proof.* Since $p_n$ is a polynomial with non-negative coefficients, we can only decrease it by replacing each $x_i$ with 1: $p_n(x_1, \dots, x_n) \ge p_n(1^n) = F_{n+1}$. □

**Lemma 7.** *Let integers $x_i$ be given as above and let $\xi_n = [x_0, x_1, \dots x_n]$. Then $(\xi_n)$ is a convergent sequence.*

*Proof.* For $n \geq 2$, by the very important Theorem 9 (Notes IV),

$$\xi_n - \xi_{n-1} = \frac{p_{n+1}(x_0, \ldots, x_n)}{p_n(x_1, \ldots, x_n)} - \frac{p_n(x_0, \ldots, x_{n-1})}{p_{n-1}(x_1, \ldots, x_{n-1})} = \frac{(-1)^{n+1}}{p_{n-1}(x_1, \ldots, x_{n-1})p_n(x_1, \ldots, x_n)}. \tag{47}$$

In view of the last lemma, this implies that for $n \geq 2$,

$$|\xi_n - \xi_{n-1}| \leq \frac{1}{F_{n-1}F_n} \leq \frac{1}{2^{n-1}}, \tag{48}$$

the last inequality coming from an easy induction ($\phi^2 = 1 + \phi \approx 2.618 > 2$.) It follows that $(\xi_n)$ is Cauchy, and hence is convergent. ☐

We have already discussed the continued fraction representations of rational numbers. We now talk about irrationals. Given real $t \notin \mathbb{Q}$, define

$$G(t) := \frac{1}{t - \lfloor t \rfloor} \ . \tag{49}$$

Since $t \notin \mathbb{Q}$, $t > \lfloor t \rfloor$, so that $G(t) \in (1, \infty)$ is well-defined, and since $t = \lfloor t \rfloor + \frac{1}{G(t)}$, we see that $G(t) \notin \mathbb{Q}$ as well. We also see that

$$t = \lfloor t \rfloor + \frac{1}{G(t)} = \lfloor t \rfloor + \cfrac{1}{\lfloor G(t) \rfloor + \cfrac{1}{G(G(t))}}, \qquad \text{etc.} \tag{50}$$

Let $G_n(t)$ denote the $n$-th iterate of $G$ ($G_1 = G,\ G_n = G \circ G_{n-1}$) and let $x_n = x_n(t) := \lfloor G_n(t) \rfloor$. Then the preceding discussion shows the existence of a family of *exact* formulas:

$$t = x_0(t) + \cfrac{1}{x_1(t) + \cfrac{1}{\cdots + \cfrac{\cdots}{x_{n-1}(t) + \cfrac{1}{G_n(t)}}}}. \tag{51}$$

Again, to be precise, keep in mind that $G_n(t)$ is not an integer, but there should be no confusion. We need some short lemmas.

**Lemma 8.** *The function $(-1)^n[x_0, x_1, \ldots, x_n]$ is increasing in real $x_n$.*

*Proof.* Clearly, $x_0$ is increasing in $x_0$ and $x_0 + \frac{1}{x_1}$ is decreasing in $x_1$. Since

$$[x_0, \ldots, x_n] = x_0 + \frac{1}{[x_1, \ldots, x_n]}, \tag{52}$$

the result is immediate by induction. ☐

**Lemma 9.** *Provided* $x_j \ge 1$ *for all* $j$*, we have*

$$[x_0, \dots, x_{2k}] < [x_0, \dots, x_{2k+2}] < [x_0, \dots, x_{2k+3}] < [x_0, \dots, x_{2k+1}]. \tag{53}$$

*Proof.* We have

$$\begin{aligned} [x_0, \dots, x_{2k}] &< [x_0, \dots, x_{2k} + [x_{2k+1}, \dots, x_{2k+r}]^{-1}] = [x_0, \dots, x_{2k+r}], \\ [x_0, \dots, x_{2k+1}] &> [x_0, \dots, x_{2k+1} + [x_{2k+2}, \dots, x_{2k+1+r}]^{-1}] = [x_0, \dots, x_{2k+1+r}]. \end{aligned} \tag{54}$$

□

We now prove the existence of infinite continued fractions.

**Theorem 10.** *If* $t \notin \mathbb{Q}$*, then*

$$t = \lim_{n\to\infty} [x_0(t), \dots, x_n(t)]. \tag{55}$$

*Proof.* Let $\xi_n = \xi_n(t) := [x_0(t), \dots, x_n(t)]$, then

$$\begin{aligned} &t - \xi_n \\ &= x_0 + \cfrac{1}{x_1 + \cfrac{1}{\cdots + \cfrac{\cdots}{x_{n-1} + \cfrac{1}{g_n(t)}}}} \quad - \quad x_0 + \cfrac{1}{x_1 + \cfrac{1}{\cdots + \cfrac{\cdots}{x_{n-1} + \cfrac{1}{x_n}}}}. \end{aligned} \tag{56}$$

Since $x_n < g_n(t) < 1 + x_n$, it follows from monotonicity that

$$\begin{aligned} |t - \xi_n| &< \big| [x_0, \dots, x_n] - [x_0, \dots, 1 + x_n] \big| \\ &= \left| \frac{p_{n+1}(x_0, \dots, x_n)}{p_n(x_1, \dots, x_n)} - \frac{p_{n+1}(x_0, \dots, 1 + x_n)}{p_n(x_1, \dots, 1 + x_n)} \right|. \end{aligned} \tag{57}$$

Let us write $a = p_n(x_0, \dots, x_{n-1})$, $b = p_{n-1}(x_0, \dots, x_{n-2})$, $c = p_{n-1}(x_1, \cdots, x_{n-1})$ and $d = p_{n-2}(x_1, \cdots, x_{n-2})$. As previously noted, $|bc - ad| = 1$. so this inequality becomes

$$\begin{aligned} |t - \xi_n| &< \left| \frac{x_n a + b}{x_n c + d} - \frac{(1 + x_n)a + b}{(1 + x_n)c + d} \right| \\ &= \frac{|ad - bc|}{(x_n c + d)((1 + x_n)c + d)} < \frac{1}{F_n^2}. \end{aligned} \tag{58}$$

It follows that $\xi_n \to t$ and, indeed, that

$$\xi_0(t) < \xi_2(t) < \xi_4(t) \cdots < t < \cdots < \xi_3(t) < \xi_1(t). \tag{59}$$

□

It is customary to say that the $\xi_n(t)$'s are the *convergents* of $t$.

**Theorem 11.** *The continued fraction representation of an irrational number is unique and infinite; conversely, every infinite continued fraction represents an irrational.*

*Proof.* Suppose $t = [x_0, x_1, \dots] = [y_0, y_1, \dots]$ with $x_j, y_j \in \mathbb{Z}$, $x_i, y_i \geq 1$ for $i \geq 1$. We prove by induction that $x_j = y_j$. Indeed, since

$$x_0 < [x_0, x_1, \dots] < 1 + x_0, \qquad y_0 < [y_0, y_1, \dots] < 1 + y_0, \tag{60}$$

we have $x_0 = y_0 = m_0$. But now,

$$\frac{1}{t - m_0} = [x_1, x_2, \dots] = [y_1, y_2, \dots], \tag{61}$$

and we may repeat the argument. Suppose on the other hand that $u = [x_0, x_1, \dots] = [y_0, y_1, \dots, y_n]$. Repeating the previous argument $n$ times leads us to $[x_n, x_{n+1}, \dots] = [y_n]$, and $x_n < [x_n, x_{n+1}, \dots] < 1 + x_n$ is not an integer, a contradiction. □

We now return to some topics touched on in the earlier notes. An infinite continued fraction is *purely periodic* of period $d \geq 1$ if all its denominators are repeating blocks of length $d$:

$$[\overline{x_0, \dots, x_{d-1}}] := [x_0, \dots, x_{d-1}, x_0, \dots, x_{d-1}, \dots]. \tag{62}$$

Let $\theta_k = \theta_k(x_0, \dots, x_{d-1})$ denote the finite continued fraction with $kd$ denominators, comprising $k$ complete cycles of the pattern $(x_0, \dots, x_{d-1})$, or, more formally, $\theta_1 = [x_0, x_1, \dots .x_{d-1}]$ and

$$\theta_k = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\cdots + \cfrac{\cdots}{x_{d-1} + \cfrac{1}{\theta_{k-1}}}}}. \tag{63}$$

If $d = 1$ and $x_0 = a$, we have the familiar

$$\theta_k = a + \frac{1}{\theta_{k-1}} = \frac{x_k(a)}{x_{k-1}(a)}\ , \tag{64}$$

and if $d \geq 2$,

$$\theta_k = \frac{p_{d+1}(x_0, \dots, x_{d-1}, \theta_{k-1})}{p_d(x_1, \dots, x_{d-1}, \theta_{k-1})} = \frac{\theta_{k-1} p_d(x_0, \dots, x_{d-1}) + p_{d-1}(x_0, \dots, x_{d-2})}{\theta_{k-1} p_{d-1}(x_1, \dots, x_{d-1}) + p_{d-2}(x_1, \dots, x_{d-2})}\ . \tag{65}$$

If $d = 1$, this is consistent with $p_0 = 1, p_{-1} = 0$, so we won't distinguish that case.

Observe that $(\theta_k)$ is the subsequence of the $kd$-th entries in a convergent sequence and so is also convergent, to, say, $\theta = \theta_{(x_0, \dots, x_{d-1})}$. By continuity of polynomials,

$$\theta = \frac{p_d(x_0, \dots, x_{d-1}) \cdot \theta + p_{d-1}(x_0, \dots, x_{d-2})}{p_{d-1}(x_1, \dots, x_{d-1}) \cdot \theta + p_{d-2}(x_1, \dots, x_{d-2})} := \frac{A_x \theta + B_x}{C_x \theta + D_x}\ , \tag{66}$$

and so $\theta$ is a root of the quadratic:

$$C_x T^2 + (D_x - A_x) T - B_x. \tag{67}$$

If $x_0 \geq 1$, then $B_x > 0$, so that there are two roots, one positive and one negative. Clearly, $\theta > 0$. Thus, every purely periodic continued fraction is a quadratic irrational; $\theta \in \mathbb{Q}(\sqrt{(D-A)^2 + 4BC})$. The uniqueness of continued fractions rules out the possibility that $\theta \in \mathbb{Q}$.

*Example.* Suppose $d = 1$, so that

$$\theta_k == \frac{x_k(a)}{x_{k-1}(a)} \; . \tag{68}$$

Plugging into the previous expression with $x = (a)$, we have $A_x = a, B_x = C_x = 1, D_x = 0$, and the quadratic above is $T^2 - aT - 1$, as we have already seen. If $d = 2$, then $A = ab + 1, B = b, C = a, D = 1$, and we get the quadratic seen in (43).

An infinite continued fraction is *periodic* if it is periodic after an initial string. We write

$$[y_0, \dots, y_e, \overline{x_0, \dots, x_{d-1}}] := [y_0, \dots, y_e, x_0, \dots, x_{d-1}, x_0, \dots, x_{d-1}, \dots]. \tag{69}$$

If $\theta = [\overline{x_0, \dots, x_{d-1}}]$ as before, and $\psi_k$ denotes the continued fraction with the initial string, followed by $k$ repeated blocks, then as before

$$\psi_k = \frac{p_{e+1}(y_0, \dots, y_{e-1}, \theta_k)}{p_e(y_1, \dots, y_{e-1}, \theta_k)} = \frac{\theta_k p_e(y_0, \dots, y_{d-1}) + p_{e-1}(y_0, \dots, y_{e-2})}{\theta_k p_{e-1}(y_1, \dots, y_{e-1}) + p_{e-2}(e_1, \dots, e_{d-2})}. \tag{70}$$

If $e = 1$, then we simply define

$$[y_0, \overline{x_0, \dots, x_{d-1}}] = y_0 + \frac{1}{\theta} \tag{71}$$

In any event, the convergence of continued fractions implies that $(\psi_k)$ is convergent, to $\psi$, say, and the convergence of $(\theta_k)$ implies

$$\psi = \frac{\theta p_e(y_0, \dots, y_{d-1}) + p_{e-1}(y_0, \dots, y_{e-2})}{\theta p_{e-1}(y_1, \dots, y_{e-1}) + p_{e-2}(e_1, \dots, e_{d-2})}. \tag{72}$$

Since each of the continuants is an integer, we see that so $\psi \in \mathbb{Q}(\sqrt{(D-A)^2 + 4BC})$ as well. Remarkably enough, the converse is true:

**Theorem 12.** *If $t = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$ is irrational, then the continued fraction expansion of $t$ is periodic.*

This will be proved in the next section.

## 4. Lagrange's Theorem

Lagrange's Theorem is the result alluded to at the end of the last section: if $u$ is a quadratic irrational, then the continued fraction expansion of $u$ is periodic. The first thing we need to do is write quadratic irrationals in a peculiar form.

**Lemma 13.** *Suppose $u \in \mathbb{Q}(\sqrt{n}) \setminus \mathbb{Q}$, where $n \in \mathbb{N}$ is not a square. Then there exist (non-unique) integers $m, d, q$ so that*

$$u = \frac{m + \sqrt{d}}{q}, \tag{73}$$

*where $m, d, q \in \mathbb{Z}$, $q \neq 0$, $d > 0$ is not a square and so that $q \mid d - m^2$.*

*Proof.* By hypothesis, we have $u = \alpha + \beta\sqrt{n}$, with $\alpha, \beta \in \mathbb{Q}$ and $\beta \neq 0$. After taking a common denominator, there exist integers $c_j$, with $c_2, c_3 \neq 0$ so that

$$u = \frac{c_1 + c_2\sqrt{n}}{c_3} = \frac{-c_1 + -c_2\sqrt{n}}{-c_3} \ . \tag{74}$$

We may thus assume without loss of generality that $c_2 > 0$ and so that

$$u = \frac{c_1 + \sqrt{nc_2^2}}{c_3} = \frac{c_1|c_3| + \sqrt{nc_2^2c_3^2}}{c_3|c_3|} \ . \tag{75}$$

If we let $m = c_1|c_3|$, $d = nc_2^2c_3^2$ and $q = c_3|c_3|$, then $m, d, q \in \mathbb{Z}$, $q \neq 0$, $d > 0$ is not a square and

$$d - m^2 = c_3^2(nc_2^2 - c_1)^2 = \pm q(nc_2^2 - c_1)^2. \tag{76}$$

We remark that this representation is not unique: if $r \in \mathbb{N}$, then another valid representation will hold under the substitution $(m, q, d) \to (rm, rq, r^2d)$. □

**Theorem 14** (Lagrange's Theorem)**.** *A quadratic irrational $u$ has a periodic continued fraction.*

*Proof.* By Lemma 1, we may write

$$u = \frac{m + \sqrt{d}}{q} = [a_0, a_1, \dots]. \tag{77}$$

Recall from the last section that

$$\xi_n(u) = \frac{p_{n+1}(a_0, \dots, a_n)}{p_n(a_1, \dots, a_n)} := \frac{p_{n+1}}{q_{n+1}} \to u. \tag{78}$$

(Note that where $p_{n+1}, q_{n+1} \in \mathbb{N}$.) Also, there is an *exact* expression

$$u = [a_0, \dots, a_{n-1}, G_n(u)] = \frac{p_{n+1}(a_0, \dots, a_{n-1}, G_n(u))}{p_n(a_1, \dots, a_{n-1}, G_n(u))} = \frac{G_n(u)p_n + p_{n-1}}{G_n(u)q_n + q_{n-1}}. \tag{79}$$

We claim that we can always write

$$G_n(u) = \frac{m_n + \sqrt{d}}{q_n}, \tag{80}$$

where $m_n, q_n \in \mathbb{Z}$, $q_n \neq 0$ and $q_n \mid d - m_n^2$. This is certainly true for $n = 0$ by hypothesis, taking $m_0 = m$ and $q_0 = q$, as $u = G_n(u)$. Supposing the claim holds for $n$, let $a_n = x_n(u) = \lfloor G_n(u) \rfloor$. Then we have

$$G_{n+1}(u) = \frac{1}{G_n(u) - a_n} = \frac{q_n}{m_n - a_n q_n + \sqrt{d}}$$

$$= \frac{q_n}{m_n - a_n q_n + \sqrt{d}} \cdot \frac{-(m_n - a_n q_n) + \sqrt{d}}{-(m_n - a_n q_n) + \sqrt{d}} \tag{81}$$

$$= \frac{-(m_n - a_n q_n) + \sqrt{d}}{(d - (m_n - a_n q_n)^2)/q_n}\ .$$

It is clear that $m_{n+1} := -(m_n - a_n q_n) \in \mathbb{Z}$. If $q_{n+1}$ is the last denominator above, then

$$q_n q_{n+1} = d - m_{n+1}^2 = d - (m_n - a_n q_n)^2 = (d - m_n^2) + q_n(2a_n m_n - a_n^2 q_n). \tag{82}$$

Since $q_n \mid d - m_n^2$, $q_{n+1} \in \mathbb{Z}$; further, $q_{n+1}$ divides $d - m_{n+1}^2$, as required, completing the proof of the claim.

We can solve for $G_n(u)$ in terms of $u$, using (7):

$$G_n(u) = -\frac{p_{n-1} - u q_{n-1}}{p_n - u q_n} = -\frac{q_n}{q_{n-1}} \cdot \frac{\xi_{n-2}(u) - u}{\xi_{n-1}(u) - u}. \tag{83}$$

We now invoke some algebraic number theory: conjugate is

$$\bar{u} = \frac{m - \sqrt{d}}{q}. \tag{84}$$

Take the conjugate of both sides in (11), observing that $p_i, q_i \in \mathbb{Z}$ (so $\xi_i(u) \in \mathbb{Q}$), and keeping alert to the fact that, in general $\overline{G_n(u)} \neq G_n(\bar{u})$. Then

$$\overline{G_n(u)} = -\frac{q_n}{q_{n-1}} \cdot \frac{\xi_{n-2}(u) - \bar{u}}{\xi_{n-1}(u) - \bar{u}}. \tag{85}$$

Since $\xi_n(u) \to u$ and since $u \neq \bar{u}$,

$$\frac{\xi_{n-2}(u) - \bar{u}}{\xi_{n-1}(u) - \bar{u}} \to \frac{u - \bar{u}}{u - \bar{u}} = 1. \tag{86}$$

Because $q_n, q_{n-1} > 0$, we may conclude that there exists $N$ so that for $n \geq N$,

$$\overline{G_n(u)} = \frac{m_n - \sqrt{d}}{q_n} < 0. \tag{87}$$

Since $G_n(u) \geq 1$ for $n \geq 1$ by definition, we have

$$G_n(u) - \overline{G_n(u)} = \frac{2\sqrt{d}}{q_n} > 0 \tag{88}$$

for $n \geq N$, hence $q_n > 0$ for $n \geq N$. But now recall from (10) that

$$q_n q_{n+1} + m_{n+1}^2 = d \implies q_n,\ m_{n+1}^2 < d. \tag{89}$$

This implies that there are only finitely many possibilities for $(q_n, m_n)$ for $n \geq N$ and so there exists $M \geq N, r > 0$ so that $(q_M, m_M) = (q_{M+r}, m_{M+r})$. It follows that $G_M(u) = G_{M+r}(u)$, and so, for $n \geq M$, $G_n(u) = G_{n+r}(u)$, implying that $a_n = a_{n+r}$. That is, the continued fraction expression for $u$ is periodic with period $r$. □

*Example.* Let

$$u = G_0(u) = \frac{1+\sqrt{2}}{3} = \frac{3+\sqrt{18}}{9} \approx .805. \tag{90}$$

Then $a_0 = 0$ and

$$G_1(u) = \frac{9}{3+\sqrt{18}} = -3+\sqrt{18} \approx 1.242. \tag{91}$$

We see that $a_1 = 1$ and

$$G_2(u) = \frac{1}{-3+\sqrt{18}-1} = \frac{1}{-4+\sqrt{18}} = \frac{4+\sqrt{18}}{2} \approx 4.121. \tag{92}$$

Therefore, $a_2 = 4$ and

$$G_3(u) = \frac{1}{\frac{4+\sqrt{18}}{2} - 4} = \frac{2}{-4+\sqrt{18}} = 4+\sqrt{18} \approx 8.243. \tag{93}$$

We're almost done, because $a_3 = 8$ and

$$G_4(u) = \frac{1}{4+\sqrt{18}-8} = \frac{1}{-4+\sqrt{18}} = G_2(u). \tag{94}$$

It follows that $u = [0, 1, \overline{4, 8}]$. We can verify this by our earlier calculation. We have previously shown that, if $\theta = [\overline{4, 8}]$, then

$$8\theta^2 - 32\theta - 4 = 0 \implies \theta = \frac{4+3\sqrt{2}}{2}. \tag{95}$$

It follows that

$$x = 0 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{8 + \cfrac{1}{\cdots}}}} = \frac{1}{1+\frac{1}{\theta}} = \frac{1}{1+\frac{2}{4+3\sqrt{2}}} = \frac{4+3\sqrt{2}}{6+3\sqrt{2}} = u. \tag{96}$$

Just for completeness, we observe that

$$\frac{1+\sqrt{2}}{3} = 0 + \cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{4+\sqrt{18}}}} \implies \frac{1-\sqrt{2}}{3} = 0 + \cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{4-\sqrt{18}}}}. \tag{97}$$

The actual continued fraction representation for $\bar{u}$ is $[-1, 1, 6, \overline{4, 8}]$, a formula whose resemblance to the earlier one might make want you to explore more deeply in the subject of continued fractions.

We also state a pretty neat theorem that we will not use directly. A proof can be found in your favorite high-quality number theory textbook:

**Theorem 15.** *The real quadratic irrational $u$ has a purely periodic continued fraction expansion if and only if $u > 1$ and $-1 < \bar{u} < 0$.*

(Note that for $\theta = [\overline{4, 8}]$, $\theta \approx 4.121$ and $\bar{\theta} \approx -.121$.) More generally, for $\theta = [\overline{a, b}]$, $\theta$ and $\bar{\theta}$ are the positive and negative roots of the equation

$$p_{a,b}(T) = bT^2 - abT - a = 0, \tag{98}$$

and $p_{a,b}(-1) = b + a(b - 1) > 0 > -a = p_{a,b}(0)$, so $\bar{\theta} \in (-1, 0)$, and $p_{a,b}(1) = -b(a - 1) - a < 0$, so $\theta > 1$. All bets are off if you want to play with periodic continued fractions whose denominators are not in $\mathbb{N}$.

**Corollary 16.** *If $m \in \mathbb{N}$ is not a perfect square and $a_0 = \lfloor \sqrt{m} \rfloor$, then $\sqrt{m} = [a_0, \overline{a_1, \dots, a_{d-1}, 2a_0}]$.*

*Proof.* Observe that $a_0^2 + 1 \le m \le (a_0 + 1)^2 - 1$. If we let $u = a_0 + \sqrt{m}$, then $u \ge 2a_0 > 1$ and $\bar{u} = a_0 - \sqrt{m}$. Thus

$$0 > a_0 - \sqrt{a_0^2 + 1} \ge \bar{u} > a_0 - (a_0 + 1) = -1, \tag{99}$$

verifying the hypotheses of Theorem 3. $\square$

In a few cases, the non-periodic continued fraction representations of non-quadratic irrationals is known. Here is a very brief sampling:

$$\begin{aligned} e &= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \dots], \\ \frac{e-1}{e+1} &= [0, 2, 6, 10, 14, 18, 22, 26, 30, 34 \dots], \\ \tan(1) &= [1, 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, \dots], \\ \pi &= [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \dots]. \end{aligned} \tag{100}$$

If you can find the pattern for $\pi$, wave to me from the stage as you pick up your Fields Medal.

## 5. A little analysis

We will come across the same situation more than once, so I'd like to formalize it.

**Theorem 17.** *Suppose $X$ is a dense subset of the real interval $[a, b]$, $Y$ is a dense subset of the real interval $[c, d]$ and $f$ is a strictly increasing bijection of $X$ onto $Y$. Then $f$ extends to a unique continuous function $F$ from $[a, b]$ to $[c, d]$; this function $F$ is also strictly increasing.*

*Proof.* We first remark that if $a \in X$, then $f(a) = c$. This is because $f(X) = Y \subseteq [f(a), d]$ and $Y$ is dense in $[c, d]$. A similar argument shows that if $b \in X$, then $f(b) = d$.

We now define $F(x)$ for $u \in [a, b]$. If $a \notin X$, $F(a) = c$ and if $b \notin X$, $F(b) = d$. If $u \in (a, b)$,

$$F_-(u) = \sup\{f(x) : x \in X \cap [a, u]\}; \qquad F_+(u) = \inf\{f(x) : x \in X \cap [u, b]\}. \tag{101}$$

The sup and inf are finite, because $c \le f(a), f(a') \le d$ for $a, a' \in X$. If $x \in X$, then clearly $F_-(x) = F_+(x) = f(x)$, so $F$ extends $f$. Suppose $u \notin X$. Observe that if $x_0 < u < x_1$ for $x_i \in X$, then $f(x_0) < f(x_1)$, hence $f(x_0) \le F_+(u)$ and $F_-(u) \le f(x_1)$. Taking sups or infs, we see that $F_-(u) \le F_+(u)$. In fact, we show that $F_-(u) = F_+(u)$. Suppose not, then by the denseness of $Y = f(X)$, there exists $z \in X$ so that $f(z)$ is in the open interval $(F_-(u), F_+(u))$. But $f(z) > F_-(x)$ so $x > u$ and $f(z) < F_+(x)$ so $x < u$, a contradiction.

We now define

$$F(u) = F_-(u) = F_+(u). \tag{102}$$

We must prove that $F$ is strictly increasing, continuous and unique with these conditions. First observe that $F$ is non-decreasing, because $a \le u < v \le b$ implies that

$$\{f(x) : x \in X \cap [a, u]\} \subseteq \{f(x) : x \in X \cap [a, v]\}. \tag{103}$$

This set inclusion implies that $F_+(u) \le F_+(v)$. Since $X$ is dense in $[a, b]$, there exist $x_i \in X$ so that $u < x_0 < \frac{u+v}{2} < x_1 < v$. Since $f$ is strictly increasing, it follows that

$$F(u) \le F(x_0) = f(x_0) < f(x_1) = F(x_1) \le F(v), \tag{104}$$

so $F$ is strictly increasing.

The continuity of $F$ on $(a, b)$ can be proved from the definition. Suppose that $F(u) = y$ and let $\epsilon > 0$ be small enough that $(y - \epsilon, y + \epsilon) \subseteq (c, d)$. As before, there exist $x_i \in X$ so that $y - \epsilon < f(x_0) < y < f(x_1) < y + \epsilon$. Since $F$ is strictly increasing, $x_0 < u < x_1$. Let $\delta = \min\{u - x_0, x_1 - u\}$. Then $|u - v| < \delta$ implies that $x_0 < v < x_1$, so $f(x_0) < F(v) < f(x_1)$, and so $|F(u) - F(v)| < \epsilon$.

Finally, suppose $G$ also extends $F$ and is continuous, and suppose $F(u) \ne G(u)$ for some $u \in [a, b]$, say $F(u) > G(u)$. Then by continuity, $F(x) > G(x)$ on some neighborhood of $u$ intersected with $[a, b]$. But this neighborhood will contain $x \in X$, and $F(x) = G(x)$ by hypothesis, a final contradiction. □

**Theorem 18.** *The Minkowski ?-function extends to a strictly increasing function on* $[0, 1]$ *defined on irrational arguments by*

$$?([0, a_0, a_1, a_2, \dots]) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{a_1 + \cdots + a_m - 1}}. \tag{105}$$

*This function maps the rationals in* $[0, 1]$ *to the dyadic rationals in* $[0, 1]$ *and the quadratic irrationals in* $[0, 1]$ *to the non-dyadic rationals in* $[0, 1]$.

*Proof.* First observe that since $a_i > 1$, the series above converges by the ratio test. Let $[a,b] = [c,d] = [0,1]$ in Theorem 5, let $X = \mathbb{Q} \cap [0,1]$ and $Y = \mathbb{Q}_2 \cap [0,1]$, where $\mathbb{Q}_2$ denotes the dyadic rationals. Then $X$ and $Y$ are both dense on $[0,1]$ and $?$ has already been shown to be strictly increasing on $X$. If $t = [0, a_0, a_1, \dots]$, then $t = \lim \xi_n(t)$ and $?(\xi_n(t))$ is the $n$-th partial sum of the series, and since $?$ extends to a continuous function on $[0,1]$, which we shall also call $?$,

$$?(t) = \lim_{n\to\infty} \xi_n(t) = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{(-1)^{m-1}}{2^{a_1+\cdots+a_m-1}} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{a_1+\cdots+a_m-1}}. \tag{106}$$

To show periodicity, we first remark that

$$[0, b_0, \dots, b_{e-1}, \overline{a_0, \dots, a_{d-1}}] = [0, b_0, \dots, b_{e-1}, \overline{a_0, \dots, a_{d-1}, a_0, \dots, a_{d-1}}], \tag{107}$$

so we may assume that any periodic continued fraction has even period. If $u$ is a quadratic irrational and $D = \sum a_j$ is the sum of the denominators in the period and $2s$ is the length of the period, then there exists $N = 2sn_0$ so that for $m \ge N$, if $m = (2s)n + r$, $0 \le r < 2s$, then

$$a_1 + \cdots + a_m - 1 = T_r + (n - n_0)D \tag{108}$$

for some integers $T_r$, determined by the non-purely periodic part and $a_0, \dots, a_r$. It follows that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{a_1+\cdots+a_m-1}} = \sum_{m=1}^{2sn_0-1} \frac{(-1)^{m-1}}{2^{a_1+\cdots+a_m-1}} + \sum_{r=0}^{2s-1} \sum_{k=0}^{\infty} \frac{(-1)^r}{2^{T_r+kD}}. \tag{109}$$

Since

$$\sum_{k=0}^{\infty} \frac{1}{2^{T_r+kD}} = \frac{1}{2^{T_r}} \cdot \frac{2^D}{2^D - 1}, \tag{110}$$

$?(t)$ is a finite sum of dyadic rationals and rationals, and so is rational.

Conversely, suppose $v \in (0,1) \cap \mathbb{Q}$ is not dyadic. We may write

$$v = \frac{p}{q} = \frac{1}{2^n}\frac{p}{q'} = \frac{1}{2^n}\left(c + \frac{p'}{q'}\right), \tag{111}$$

where $q'$ is odd, $0 \le c < 2^n \in \mathbb{N}$ and $0 < p' < q'$. Since $q'$ is odd, there exists $r$ so that $2^r \equiv 1 \pmod{q'}$, and so

$$v = \frac{m}{2^n} + \frac{t}{2^r - 1}. \tag{112}$$

As before, we can write both $t$ and $c$ in the form

$$2^{b_1} - 2^{b_2} + - \cdots + 2^{b_{2k-1}} - 2^{b_{2k}} \tag{113}$$

with $b_1 > b_2 > \dots b_{2k}$, and since

$$\frac{1}{2^r - 1} = \sum_{j=1}^{\infty} \frac{1}{2^{rj}}, \tag{114}$$

we obtain a representation

$$v = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{a_1+\cdots+a_m-1}} \tag{115}$$

in which the $a_k$'s are eventually periodic. □

**Corollary 19.** *If* $u \in [0,1]$, *then*

$$\begin{gathered} ?(u)+?(1-u)=1, \qquad ?\left(\frac{u}{u+1}\right)=\frac{?(u)}{2}, \qquad ?\left(\frac{1}{u+1}\right)=1-\frac{?(u)}{2} \\ ?\left(\frac{v}{1-kv}\right)=2^k \cdot ?(v), \quad 0 \le v \le \frac{1}{k+1}. \end{gathered} \tag{116}$$

*Proof.* The equations on the top row hold when $u = x$ is rational. If $u \notin \mathbb{Q}$, write $u = \lim x_n$ with $x_n \to u$. Since ? is continuous and $\phi_j(t) = 1-t, \frac{t}{t+1}, \frac{1}{t+1}$ are continuous, simply replace $u$ with $x_n$ and take the limit.

For the bottom, first observe that if $0 \le v \le \frac{1}{2}$, then $v = \frac{u}{u+1}$ for some $u \in [0,1]$, and $u = \frac{v}{1-v}$. If $T(z) = \frac{z}{z+1}$, then the $k$-th iterate is $T_k(z) = \frac{z}{kz+1}$, so that

$$?\left(\frac{u}{ku+1}\right) = \frac{?(u)}{2^k} \implies ?\left(\frac{v}{1-kv}\right) = 2^k \cdot ?(v), \tag{117}$$

provided $\frac{v}{1-kv} \in [0,1]$; that is, $v \in [0, \frac{1}{k+1}]$. □

*Example.* We have seen above that $\frac{1+\sqrt{2}}{3} = [0,1,\overline{4,8}]$. It follows that

$$\begin{aligned} ?\left(\frac{1+\sqrt{2}}{3}\right) &= \frac{1}{2^{1-1}} - \frac{1}{2^{1+4-1}} + \frac{1}{2^{1+4+8-1}} - \frac{1}{2^{1+4+8+4-1}} + - \dots \\ &= 1 - \frac{1}{2^4} + \frac{1}{2^{12}} - \frac{1}{2^{16}} = \left(1-\frac{1}{2^4}\right) \cdot \frac{2^{12}}{2^{12}-1} = \frac{15 \cdot 4096}{16 \cdot 4095} = \frac{256}{273}. \end{aligned} \tag{118}$$

*Example.* We have already seen in Notes, V(37), that

$$?\left(\frac{F_n}{F_{n+1}}\right) = \frac{2}{3} + \frac{(-1)^{n+1}}{3 \cdot 2^{n-1}}. \tag{119}$$

By taking the limit, we see that

$$?\left(\frac{\sqrt{5}-1}{2}\right) = \frac{2}{3}. \tag{120}$$

More generally, since

$$\theta_a = \frac{a+\sqrt{a^2+4}}{2} = [\overline{a}], \tag{121}$$

by taking the limit in Notes, V(36) we see that for $a \in \mathbb{N}$,

$$?\left(\frac{\sqrt{4+a^2}-a}{2}\right) = \frac{2}{2^a+1}. \tag{122}$$

It is useful to note a shortcut. define $\gamma_a$ so that

$$?(\gamma_a) = \frac{1}{2^a+1} \implies ?\left(\frac{\gamma_a}{1-a\gamma_a}\right) = \frac{2^a}{2^a+1} = ?(1-\gamma a). \tag{123}$$

It follows that $\gamma_a$ is a root of the equation

$$\frac{X}{1-aX} = 1 - X \implies aX^2 - (a+2)X + 1 = 0 \implies \gamma_a = \frac{a+2-\sqrt{a^2+4}}{2a}. \tag{124}$$

(Since the polynomial has a root between 1 and 2, $\gamma_a$ must be the smaller root.) Since

$$\frac{\gamma_a}{1-\gamma_a} = \theta_a, \tag{125}$$

this is consistent with the earlier calculation.

*Remark.* We remark that there is an implicit algorithm at work here. If $\frac{p}{q}$ is a non-dyadic rational, keep doubling until you get past $\frac{1}{2}$. Then take $1-x$ to get below $\frac{1}{2}$, keep doubling again, etc. These fractions all have denominator $q$ and lie in $(0,1)$, hence eventually you get repetition. By solving the resulting equation, you can compute $?^{-1}(\frac{p}{q})$ without having to compute dyadic expansions and continued fractions explicitly. (This information is implicitly contained in the ordering of doubling and folding back.) This ought to be a theorem. Later.

*Example.* A more complicated object is $\theta_{a,b} = [\overline{a,b}]$. We already know that

$$\theta_{a,b} = \frac{ab+\sqrt{a^2b^2+4ab}}{2b} \implies \theta_{a,b}^{-1} = \frac{-ab+\sqrt{a^2b^2+4ab}}{2a}, \tag{126}$$

The computation of $?(\theta_{a,b})$ is easy because the period has even length. We have

$$?\left(\frac{-ab+\sqrt{a^2b^2+4ab}}{2a}\right) = \sum_{j=0}^{\infty}\left(\frac{1}{2^{(j+1)a+jb-1}} - \frac{1}{2^{(j+1)a+(j+1)b-1}}\right) = \frac{2(2^b-1)}{2^{a+b}-1}. \tag{127}$$

*Example.* It is a little-appreciated fact that every integer $\le 10$ can be written in the form $2^a(2^b \pm 1)$. It is also little-appreciated that the inverse to the ?-function is also a continuous and strictly-increasing function. Based on the results on the previous pages, we can compute $?^{-1}(\frac{a}{b})$ for every proper fraction in $(0,1)$ with $b \le 10$. For $b = 2, 4, 8$, see Notes V(13). In view of (33) and (36) above, we find immediately that

$$?\left(\frac{3-\sqrt{5}}{2}\right) = \frac{1}{3}, \qquad ?\left(\frac{5-\sqrt{5}}{10}\right) = \frac{1}{6}, \qquad ?\left(\frac{5+\sqrt{5}}{10}\right) = \frac{5}{6}. \tag{128}$$

Taking $a = 2$ in (38), we get

$$?(\sqrt{2}-1) = \frac{2}{5} \implies ?(2-\sqrt{2}) = \frac{3}{5}, \quad ?\left(\frac{2-\sqrt{2}}{2}\right) = \frac{1}{5}, \quad ?\left(\frac{\sqrt{2}}{2}\right) = \frac{4}{5}. \tag{129}$$

Working backwards from the right-hand-side of (55), with $(a, b) = (2, 1)$ and $(1, 2)$, we see after some simplification that

$$?\left(\frac{-1+\sqrt{3}}{2}\right) = \frac{2}{7}, \qquad ?\left(-1+\sqrt{3}\right) = \frac{6}{7}. \tag{130}$$

More use of Corollary 7 implies that

$$?\left(\frac{3-\sqrt{3}}{2}\right) = \frac{5}{7}, \quad ?\left(2-\sqrt{3}\right) = \frac{1}{7}, \quad ?\left(\frac{\sqrt{3}}{3}\right) = \frac{4}{7}, \quad ?\left(\frac{3-\sqrt{3}}{3}\right) = \frac{3}{7}. \tag{131}$$

Here's a preview from the next homework set: compute $?^{-1}(\frac{a}{b})$ for reduced fractions in $(0, 1)$ with $b = 9$ and $b = 10$.

## 6. Return to the first section

It turns out that it will be useful to have a proof of Theorem 3, and more.

*Proof of Theorem 3.* Suppose $u = [\overline{a_0, \dots, a_{n-1}}]$. By repeating the block if necessary, we may assume that $n \ge 3$. Then we've seen in the earlier notes that

$$u = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_{n-1} + \cfrac{1}{u}}}} = \frac{p_{n+1}(a_0, \dots, a_{n-1}, u)}{p_n(a_1, \dots, a_{n-1}, u)} \tag{132}$$
$$= \frac{u \cdot p_n(a_0, \dots, a_{n-1}) + p_{n-1}(a_0, \dots, a_{n-2})}{u \cdot p_{n-1}(a_1, \dots, a_{n-1}) + p_{n-2}(a_1, \dots, a_{n-2})}.$$

If, as before, we write this as

$$u = \frac{Au + B}{Cu + D}, \tag{133}$$

then we've seen that $u$ is a (positive) root of the quadratic

$$f(T) = CT^2 + (D - A)T - B \implies u = \frac{A - D + \sqrt{(D - A)^2 + 4BC}}{2C}, \tag{134}$$

and the other root of $f$ must be $\bar{u}$. We have $u > a_0 \ge 1$. Note that $f(0) = -B < 0$ and $f(-1) = (A - B) + (C - D)$, and

$$\begin{aligned} A - B &= p_n(a_0, \dots, a_{n-1}) - p_{n-1}(a_0, \dots, a_{n-2}) \\ &= (a_{n-1} - 1)p_{n-1}(a_0, \dots, a_{n-2}) + p_{n-2}(a_0, \dots, a_{n-3}) > 0. \end{aligned} \tag{135}$$

Similarly, $C - D = (a_{n-1} - 1)p_{n-2}(a_1, \dots, a_{n-2}) + p_{n-3}(a_1, \dots, a_{n-3}) \ge 0$. Thus, $-1 < \bar{u} < 0$. (Alternatively, one can show that $u\bar{u} = \frac{-B}{C} \in (-1, 0)$.)

The converse is somewhat harder. We suppose that $u > 1$ and $\bar{u} \in (-1, 0)$. Since $u$ is a quadratic irrational, its continued fraction expansion is periodic. Thus there exist $n_0, r$ so that $G_{n_0}(u) = G_{n_0+r}(u)$, and so that $G_n(u) = G_{n+r}(u)$ for $n \ge n_0$. Assume without loss of generality that $n_0$ is minimal with this property. Our goal is to show that $n_0 = 0$.

To do this, we first wish to show that for all $n \ge 0$, we have $G_n(u) > 1$ and $\overline{G_n(u)} \in (-1, 0)$. The first is easy to establish, because, $G_0(u) = u > 1$ by hypothesis, and $G_n(u) > 1$ for $n \ge 1$ by construction. For the second, again, $\overline{G_n(u)} = \bar{u} \in (-1, 0)$ by hypothesis. We have

$$G_n(u) = a_n + \frac{1}{G_{n+1}(u)} \tag{136}$$

and this implies that

$$\overline{G_n(u)} = a_n + \frac{1}{\overline{G_{n+1}(u)}} \implies \overline{G_{n+1}(u)} = \frac{1}{\overline{G_n(u)} - a_n} \; . \tag{137}$$

By hypothesis, $\overline{G_n(u)} \in (-1, 0)$ and $a_n \geq 1$, hence

$$\overline{G_n(u)} - a_n < -1 \implies \overline{G_{n+1}(u)} = \frac{1}{\overline{G_n(u)}} - a_n \in (-1, 0). \tag{138}$$

This completes the induction. Observe that we have now shown that

$$-a_n - \frac{1}{\overline{G_{n+1}(u)}} = -\overline{G_n(u)} \in (0, 1), \tag{139}$$

and since $a_n$ is an integer, this means that

$$a_n = \left\lfloor -\frac{1}{\overline{G_{n+1}(u)}} \right\rfloor. \tag{140}$$

Suppose that $G_{n_0}(u) = G_{n_0+r}(u)$ and $n_0 \geq 1$. It follows from this last equation that $a_{n_0-1} = a_{n_0+r-1}$. But now, applying (64) with $n = n_0 - 1$ and $n = n_0 + r - 1$, we see that $G_{n_0-1}(u) = G_{n_0+r-1}(u)$, contradicting the minimality of $n_0$. Thus, $u$ is purely periodic. □

The next result is interesting in its own right.

**Theorem 20.**

$$u = [\overline{a_0, \ldots, a_{n-1}}] \implies -\bar{u}^{-1} = [\overline{a_{n-1}, \ldots, a_0}]. \tag{141}$$

*Proof.* Keep the notation of the last theorem. Then $u, \bar{u}$ are the roots of $f(T) = 0$. By the same reasoning, if $v = [\overline{a_{n-1}, \ldots, a_0}]$, then

$$v = \frac{v \cdot p_n(a_{n-1}, \ldots, a_0) + p_{n-1}(a_{n-1}, \ldots, a_1)}{v \cdot p_{n-1}(a_{n-2}, \ldots, a_0) + p_{n-2}(a_{n-2}, \ldots, a_1)} \; . \tag{142}$$

But by the reversability of the arguments of continuants, this means that $v$ and $\bar{v}$ are the roots of the equation

$$g(T) = BT^2 + (D - A)T - C = -T^2 f(-T^{-1}). \tag{143}$$

Thus $\{v, \bar{v}\} = \{-u^{-1}, -\bar{u}^{-1}\}$. Since $u, v > 1$, $\bar{u}, \bar{v} \in (-1, 0)$, it follows that $v = -\bar{u}^{-1}$. □

We now improve Corollary 4.

**Corollary 21.** *If $m \in \mathbb{N}$ is not a perfect square, and $a_0 = \lfloor \sqrt{m} \rfloor$, then $\sqrt{m} = [a_0, \overline{a_1, \ldots, a_{d-1}, 2a_0}]$, where $a_k = a_{d-k}$ for $1 \leq k \leq d - 1$.*

*Proof.* We have already proved that $u = a_0 + \sqrt{m}$ is purely periodic, and since $\lfloor u \rfloor = 2a_0$, we have

$$u = [\overline{2a_0, a_1, \dots, a_{d-1}}] \tag{144}$$

for some denominators $a_j$. We now take away the first denominator and observe that

$$u = 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_{d-1} + \cfrac{1}{2a_0 + \cfrac{1}{\dots}}}}} = 2a_0 + \frac{1}{[\overline{a_1, \dots, a_{d-1}, 2a_0}]}. \tag{145}$$

Thus

$$w := [\overline{a_1, \dots, a_{d-1}, 2a_0}] = \frac{1}{u - 2a_0} = \frac{1}{\sqrt{m} - a_0}. \tag{146}$$

It follows from Theorem 8 that

$$\begin{aligned} v := [\overline{2a_0, a_{d-1}, \dots, a_1}] &= -\bar{w}^{-1} = -\cfrac{1}{\cfrac{1}{-\sqrt{m} - a_0}} \\ &= a_0 + \sqrt{m} = [\overline{2a_0, a_1, \dots, a_{d-1}}] = u. \end{aligned} \tag{147}$$

Since continued fractions are unique, we conclude that $a_k = a_{d-k}$ for $1 \le k \le d - 1$. □

## 7. Bonus leftovers

This accidental section consists of a few items that should have been in section two. The first is a definition of the Minkowski ?-function which is seemingly independent of anything connected with the Stern sequence.

**Corollary 22.** *The Minkowski ?-function is the unique function $F$ satisfying the following properties:*

(1) *$F$ is continuous on $[0,1]$;*
(2) *$F(0)=0$, $F(1)=1$;*
(3) *$F(x)+F(1-x)=1$ for all $x\in[0,1]$;*
(4) *$F(\frac{x}{x+1})=\frac{F(x)}{2}$ for all $x\in[0,1]$ .*

*Proof.* Properties (2), (3) and (4) iterate to showing that $F(x)=?(x)$ for any $x\in\mathbb{Q}$, and so by (1), $F$ must be the unique continuous extension. □

*Example.* The approach in the last remark on p.8 is less useful than perhaps it seems. What it amounts to is a way of quickly determining the repeating portion of the binary representation of $y=\frac{p}{q}$, and then translating that into finding $?^{-1}(y)$. If $q=2^v m$ for odd $m$, then the number of steps is $v$ plus the smallest $k$ so that $2^k\equiv\pm1 \pmod{m}$.

Recall that if $2p<q$, then

$$?\left(\frac{p}{q}\right)=y \iff ?\left(\frac{q-p}{q}\right)=1-y,\ ?\left(\frac{p}{q-p}\right)=2y,\ ?\left(\frac{q-2p}{q-p}\right)=1-2y. \tag{148}$$

We apply this repeatedly to one not-so-easy illustrative example. Suppose

$$?(x)=\frac{13}{44} \iff ?(1-x)=\frac{31}{44}. \tag{149}$$

Since the first of these is less than $\frac{1}{2}$, we "use" it:

$$?\left(\frac{x}{1-x}\right)=\frac{13}{22} \iff ?\left(\frac{1-2x}{1-x}\right)=\frac{9}{22}. \tag{150}$$

Now $\frac{9}{22}<\frac{1}{2}$, so

$$?\left(\frac{1-2x}{x}\right)=\frac{9}{11} \iff ?\left(\frac{3x-1}{x}\right)=\frac{2}{11}. \tag{151}$$

We can already see from this that $x\in(\frac{1}{3},\frac{1}{2})$. Here, $\frac{2}{11}<\frac{1}{2}$, so

$$?\left(\frac{3x-1}{1-2x}\right)=\frac{4}{11} \iff ?\left(\frac{2-5x}{1-2x}\right)=\frac{7}{11}. \tag{152}$$

Repeating,

$$?\left(\frac{3x-1}{2-5x}\right)=\frac{8}{11} \iff ?\left(\frac{3-8x}{2-5x}\right)=\frac{3}{11}. \tag{153}$$

Don't give up!

$$?\left(\frac{3-8x}{3x-1}\right)=\frac{6}{11} \iff ?\left(\frac{11x-4}{3x-1}\right)=\frac{5}{11}. \tag{154}$$

We now know that $x \in (\frac{4}{11}, \frac{3}{8})$. Almost there!

$$?\left(\frac{11x-4}{3-8x}\right)=\frac{10}{11} \iff ?\left(\frac{7-19x}{3-8x}\right)=\frac{1}{11}. \tag{155}$$

(Thus, $\frac{4}{11} = .3636.. < x < .3684.. = \frac{7}{19}$.) Finally:

$$?\left(\frac{7-19x}{11x-4}\right)=\frac{2}{11} \iff ?\left(\frac{30x-11}{11x-4}\right)=\frac{9}{11}. \tag{156}$$

At last, a match!

$$?\left(\frac{3x-1}{x}\right)=?\left(\frac{7-19x}{11x-4}\right)=\frac{2}{11} \implies \frac{3x-1}{x}=\frac{5-11x}{7x-3}. \tag{157}$$

This gives a quadratic, which is good:

$$\begin{aligned}&(3x-1)(7x-3)=x(5-11x) \implies 52x^2-30x+4=0\\ &\implies x=\frac{15+\sqrt{17}}{52}\approx .36775, \quad x=\frac{15-\sqrt{17}}{52}\approx .20917.\end{aligned} \tag{158}$$

There are two roots, but they imply that

$$\frac{3x-1}{x}=\frac{\pm 17-3}{4} \implies ?\left(\frac{\pm 17-3}{4}\right)=\frac{2}{11}. \tag{159}$$

We must choose the "+" sign; it's also the only one that is in the correct range (otherwise the formulas above are not accurate.) We conclude that

$$?\left(\frac{15+\sqrt{17}}{52}\right)=\frac{13}{44}. \tag{160}$$

Interestingly, it turns out that

$$?\left(\frac{15-\sqrt{17}}{52}\right)=\frac{3}{44}. \tag{161}$$

To confirm these formulas, we make some Mathematica calculations:

$$\frac{15+\sqrt{17}}{52}=[0,2,1,2,\overline{1,1,3}] \tag{162}$$

A calculation similar to that found in the proof of Theorem 6 shows that

$$\begin{aligned}\frac{13}{44}&=\frac{1}{4}\cdot\frac{52}{44}=\frac{1}{4}\left(1+\frac{2}{11}\right)=\frac{1}{4}+\frac{1}{4}\cdot\frac{186}{1023}\\ &=\frac{1}{2^2}+\frac{1}{2^2}\cdot\frac{2^8-2^7+2^6-2^3+2^1}{2^{10}-1}=[.01\overline{0010111010}]_2 .\end{aligned} \tag{163}$$

These match up. It is worth noting as well that one obtains the following result from a more careful examination of this algorithm. If $q$ is odd and $a \equiv \pm 2^k b \pmod{q}$, then $?^{-1}(\frac{a}{q})$ and $?^{-1}(\frac{b}{q})$ belong to the same quadratic extension of $\mathbb{Q}$. The smallest odd number $q$ for which there exist $a, b$ *not* satisfying this condition is $q = 17$. It's not hard to show that

$$?\left(\frac{3-\sqrt{5}}{4}\right) = \frac{1}{17}, \qquad ?\left(\frac{4-\sqrt{10}}{3}\right) = \frac{3}{17}. \tag{164}$$

Finally, we sketch the proof of a theorem, which I have given a somewhat disparaging name. It is quite striking but is really a trivial extension of what's already known. I don't remember having seen it before but would not be surprised to learn it was 100 years old.

**Theorem 23** (Low hanging fruit)**.** *Suppose*

$$?(x) = \frac{2p}{q}. \tag{165}$$

*Then $p$ and $q$ are both odd if and only if $x$ is a quadratic irrational and $\bar{x} < -1$.*

*Proof.* Let $u = 1/x \in (1, \infty)$. We have seen that $u$ has a purely periodic continued fraction if and only if $\bar{u} \in (-1, 0)$, and $xu = 1$ implies $\bar{x}\bar{u} = 1$, hence $\bar{x} < -1$. We now want to show that $?(x) = \frac{2p}{q}$ with odd $p, q$ if and only if $u$ is purely periodic.

First suppose $u = [\overline{a_0, \dots, a_{d-1}}]$. As remarked earlier, by repeating the period, we may assume without loss of generality that $d$ is even. Let

$$T_j = \sum_{i=0}^{j} a_i, \qquad 0 \le j \le d-1. \tag{166}$$

By the periodicity of the $a_i$'s, if $n = dk + j$, $0 \le j \le d-1$, then

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{dk+j} a_i = kT_{d-1} + T_j, \tag{167}$$

and from (33),

$$\begin{gathered} ?(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{a_0+\cdots+a_n-1}} = \\ \sum_{j=0}^{d-1} \frac{(-1)^j}{2^{T_j-1}} \sum_{k=0}^{\infty} \frac{1}{2^{kT_{d-1}}} = \left(\sum_{j=0}^{d-1} \frac{(-1)^j}{2^{T_j-1}}\right)\left(\frac{2^{T_{d-1}}}{2^{T_{d-1}}-1}\right). \end{gathered} \tag{168}$$

The proof is complete upon the identification

$$p = \sum_{j=0}^{d-1} (-1)^j 2^{T_{d-1}-T_j}, \qquad q = 2^{T_{d-1}} - 1. \tag{169}$$

(Note that $T_{d-1} > T_j$ for $j < d-1$, hence $p$ is odd.)

Conversely, suppose $?(x) = \frac{2p}{q}$, where $p$ and $q$ are odd. Since $q \mid 2^T - 1$ for some $T$, we may write

$$\frac{2p}{q} = \frac{2m}{2^T - 1} \tag{170}$$

for some odd $m$. We can now write

$$m = \sum_{j=0}^{d-1} (-1)^j 2^{T-T_j} \tag{171}$$

where $d$ is even and the $T_j$'s are strictly increasing, with $T_0 \geq 1$. Since $m$ is odd, $T_{d-1} = T$. The argument of the last paragraph reverses, and we find that $x$ is purely periodic. This proves that $\bar{x} < -1$, completing the proof. □

*Remark.* Checking against the examples on p.9, we see that for 1/3, 2/3, 1/6, 5/6,

$$\frac{-\sqrt{5}-1}{2} < -1, \quad \frac{3+\sqrt{5}}{2} > 1, \quad \frac{5 \pm \sqrt{5}}{10} > 0. \tag{172}$$

In checking $?^{-1}(\frac{a}{b})$ with $b = 5, 7$, we see easily that only $\frac{2}{5}, \frac{2}{7}, \frac{6}{7}$ are the only ones fitting the hypothesis of this theorem, and their images are the only ones whose conjugates are $< -1$.

Moreover, we've seen that for $\theta_{ab} = [\overline{a, b}]$,

$$?(\theta_{a,b}^{-1}) = \frac{2(2^b - 1)}{2^{a+b} - 1}; \tag{173}$$

fortunately,

$$\overline{\theta_{a,b}^{-1}} = \frac{-ab - \sqrt{a^2b^2 + 4ab}}{2a} < \frac{-2ab}{2a} \leq -1. \tag{174}$$

## 8. The differentiability of $?(x)$

First, an important correction – the error was noted by Jason Benda after class. I had made a mistake in an earlier draft and not completely replaced it in the final. You should read (85) and (86) as

$$?\left(\frac{3x-1}{x}\right) = ?\left(\frac{7-19x}{11x-4}\right) = \frac{2}{11} \implies \frac{3x-1}{x} = \frac{7-19x}{11x-4}. \tag{175}$$

$$(3x-1)(11x-4) = x(7-19x) \implies 52x^2 - 30x + 4 = 0. \tag{176}$$

Before talking about the differentiability of $?'(x)$, we need a more general consideration. Let $\mathcal{F}$ denote the set of continuous, strictly increasing functions from $[0,1]$ to itself. Observe that $f \in \mathcal{F}$ implies that the inverse $f^{-1} \in \mathcal{F}$ and that both ? and ¿ are in $\mathcal{F}$. (Another nod to Jason for finding the way for me to write the inverse.) As is well-known, if $f \in \mathcal{F}$, then $f'$ exists almost everywhere on $[0,1]$.

For $f \in \mathcal{F}$, We define

$$\begin{aligned} D(f,0) &:= \left\{ x_0 : \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0 \right\}, \\ D(f,\infty) &:= \left\{ x_0 : \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \infty \right\}. \end{aligned} \tag{177}$$

Obviously, $D(f,0)$ is the set of points at which $f' = 0$; but $D(f,\infty)$ is not just the set of points at which $f'$ does not exist. It is possible for the difference quotient to oscillate; we are interested in the set where it goes to $\infty$. Inasmuch as

$$x \to x_0 \iff f(x) \to f(x_0), \qquad y \to y_0 \iff f^{-1}(y) \to f^{-1}(y_0), \tag{178}$$

we have the immediate lemma.

**Lemma 24.** *If $f \in \mathcal{F}$, then*

$$\begin{aligned} x_0 \in D(f,0) &\iff f(x_0) \in D(f^{-1},\infty); \\ x_0 \in D(f,\infty) &\iff f(x_0) \in D(f^{-1},0). \end{aligned} \tag{179}$$

*Proof.* Simply observe that if $y = f(x)$, then

$$\frac{f(x)-f(x_0)}{x-x_0} = \left(\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\right)^{-1}. \tag{180}$$

□

Because $f'$ exists ae, we must have $\mu(D(f,\infty)) = 0$. However, it does not follow that $f(D(f,\infty)) = D(f^{-1},0)$ also has measure 0. This would imply by symmetry that $\mu(D(f,0)) = 0$ as well. But this is false in general. (For example, the Cantor function maps the Cantor set to $[0,1]$.) In fact, as Salem proves in his paper, $\mu(D(?,0)) = 1$.

Let us say that if $v_m \uparrow x$, $u_m \downarrow x$ and there exists $\lambda > 0$ so that $x - v_{m+1} > \lambda(x - v_m)$ and $u_{m+1} - x > \lambda(u_m - x)$, then $(u_m)$ and $(v_m)$ *slowly approach* $x$.

**Lemma 25.** *Suppose* $(u_m)$ *and* $(v_m)$ *slowly approach* $x$ *and suppose* $f \in \mathcal{F}$. *Then*

$$
\begin{aligned}
\lim_{m\to\infty} \frac{f(u_m) - f(x)}{u_m - x} = \lim_{m\to\infty} \frac{f(v_m) - f(x)}{v_m - x} = 0 \implies x \in D(f,0); \\
\lim_{m\to\infty} \frac{f(u_m) - f(x)}{u_m - x} = \lim_{m\to\infty} \frac{f(v_m) - f(x)}{v_m - x} = \infty \implies x \in D(f,\infty).
\end{aligned} \tag{181}
$$

*Proof.* Observe that if $u > x$ is sufficiently close to $x$, then $u \in (x_{m+1}, x_m)$ for some $m$ and, by the monotonicity of $f$,

$$
\begin{aligned}
\frac{f(u) - f(x)}{u - x} < \frac{f(u_m) - f(x)}{u_{m+1} - x} < \frac{1}{\lambda} \cdot \frac{f(u_m) - f(x)}{u_m - x}, \\
\frac{f(u) - f(x)}{u - x} > \frac{f(u_{m+1}) - f(x)}{u_m - x} > \lambda \cdot \frac{f(u_{m+1}) - f(x)}{u_{m+1} - x}.
\end{aligned} \tag{182}
$$

Similar inequalities apply if $v < x$. We see that if $y$ is sufficiently close to $x$, then the "slow" sequential limits to 0 or $\infty$ imply continuous limits. It is easy to give counterexamples when the difference quotient is finite and non-zero, or when the sequences don't slowly approach $x$. The reverse implication always holds of course. □

**Lemma 26.** *(i) If* $x_0 \in D(?,0)$, *then* $1 - x_0$, $\frac{x_0}{1+x_0} \in D(?,0)$; *if* $x_0 \in D(?,\infty)$, *then* $1 - x_0$, $\frac{x_0}{1+x_0} \in D(?,\infty)$.

*(ii) If* $x_0 \in D(¿\,,0)$, *then* $1 - x_0$, $\frac{1}{2}x_0 \in D(¿\,,0)$; *if* $x_0 \in D(¿\,,\infty)$, *then* $1 - x_0$, $\frac{1}{2}x_0 \in D(¿\,,\infty)$.

*Proof.* Suppose $x \to x_0$. Then $1 - x \to 1 - x_0$ and

$$
\begin{aligned}
\lim_{x\to x_0} \frac{?(1-x) - ?(1-x_0)}{(1-x) - (1-x_0)} &= \lim_{x\to x_0} \frac{(1-?(x)) - (1-?(x_0))}{(1-x) - (1-x_0)} \\
&= \lim_{x\to x_0} \frac{?(x_0) - ?(x)}{x_0 - x} = \lim_{x\to x_0} \frac{?(x) - ?(x_0)}{x - x_0}.
\end{aligned} \tag{183}
$$

Similarly, if $x \to x_0$, then $\frac{x}{1+x} \to \frac{x_0}{1+x_0}$ (since $x \in [0,1]$). Observe that

$$
\begin{aligned}
?\left(\frac{x}{1+x}\right) - ?\left(\frac{x_0}{1+x_0}\right) = \frac{?(x) - ?(x_0)}{2}; \\
\frac{x}{1+x} - \frac{x_0}{1+x_0} = \frac{(x - x_0)}{(1+x)(1+x_0)}.
\end{aligned} \tag{184}
$$

Thus, upon taking the difference quotient, we find that

$$
\begin{aligned}
\lim_{x\to x_0} \frac{?(\frac{x}{1+x}) - ?(\frac{x_0}{1+x_0})}{\frac{x}{1+x} - \frac{x_0}{1+x_0}} &= \lim_{x\to x_0} \frac{(1+x)(1+x_0)}{2} \cdot \frac{?(x) - ?(x_0)}{x - x_0} \\
&= \frac{(1+x_0)^2}{2} \lim_{x\to x_0} \frac{?(x) - ?(x_0)}{x - x_0}.
\end{aligned} \tag{185}
$$

The second set of implications follow from Lemma 12. □

**Theorem 27.** *If $u \in (0,1)\cap\mathbb{Q}$, then $u \in D(?,0)$; that is, $?'(u) = 0$. If $v = \frac{p}{2^n} \in (0,1)$, then $v \in D(¿\,,\infty)$.*

*Proof.* We first consider $u = \frac{1}{2}$. Let

$$\begin{aligned} v_m &= \frac{1}{[0,2,m]} = \frac{m}{2m+1} \implies ?(v_m) = \frac{1}{2} - \frac{1}{2^{m+1}}, \\ u_m = 1 - v_m &= \frac{1}{[0,1,1,m]} = \frac{m+1}{2m+1} \implies ?(u_m) = \frac{1}{2} + \frac{1}{2^{m+1}}. \end{aligned} \tag{186}$$

Observe that $u_{m+1} - \frac{1}{2} = \frac{2m+1}{2m+3}(u_m - \frac{1}{2})$ and $\frac{1}{2} - v_{m+1} = \frac{2m+1}{2m+3}(\frac{1}{2} - v_m)$, so $(u_m)$ and $(v_m)$ slowly approach $\frac{1}{2}$. Further,

$$\frac{?(u_m)-?(\frac{1}{2})}{u_m - \frac{1}{2}} = \frac{?(v_m)-?(\frac{1}{2})}{v_m - \frac{1}{2}} = \frac{4m+2}{2^{m+1}} \to 0. \tag{187}$$

It follows that $\frac{1}{2} \in D(?,0)$.

As we have seen earlier in these notes, every rational number can be constructed from $\frac{1}{2}$ by repeated application of the maps $x \to 1-x$ and $x \to \frac{x}{1+x}$, and this completes the proof by Lemma 14. Alternatively, we have $\frac{1}{2} \in D(¿\,,\infty)$, and every dyadic rational is derived from $\frac{1}{2}$ by repeated application of $x \to 1-x$ and $x \to \frac{x}{2}$. ☐

**Theorem 28.**

$$\frac{1}{\phi} = \frac{\sqrt{5}-1}{2} \in D(?,\infty) \implies \frac{2}{3} \in D(¿\,,0). \tag{188}$$

*Proof.* We have already shown in (47), (48) that

$$?\left(\frac{F_n}{F_{n+1}}\right) = \frac{2}{3} + \frac{(-1)^{n+1}}{3\cdot 2^{n-1}}, \qquad ?\left(\frac{1}{\phi}\right) = \frac{2}{3}. \tag{189}$$

We note that, like all finite continued fraction approximations, the sequence $F_n/F_{n+1}$ alternates above and below its limit:

$$\frac{F_0}{F_1} < \frac{F_2}{F_3} < \cdots < \frac{1}{\phi} < \cdots < \frac{F_3}{F_4} < \frac{F_1}{F_2}. \tag{190}$$

Recall the Binet formula for $F_n$, and the identities $\bar{\phi} = -\phi^{-1}$ and $\phi^2 + 1 = \sqrt{5}\phi$, so

$$\begin{aligned} z_n := \frac{F_n}{F_{n+1}} - \frac{1}{\phi} &= \frac{\frac{1}{\sqrt{5}}(\phi^n + (-1)^{n+1}\phi^{-n})}{\frac{1}{\sqrt{5}}(\phi^{n+1} + (-1)^{n+2}\phi^{-(n+1)})} - \frac{1}{\phi} \\ &= \frac{\phi^{2n+1} + (-1)^{n+1}\phi}{\phi^{2n+2} + (-1)^{n+2}} - \frac{1}{\phi} = \frac{(-1)^{n+1}\sqrt{5}}{\phi^{2n+2} + (-1)^{n+2}}. \end{aligned} \tag{191}$$

It follows that

$$\left|\frac{z_{n+1}}{z_n}\right| \to \frac{1}{\phi^2} > 0. \tag{192}$$

and so if

$$v_m = \frac{F_{2m}}{F_{2m+1}} \uparrow \frac{1}{\phi}, \qquad u_m = \frac{F_{2m+1}}{F_{2m+2}} \downarrow \frac{1}{\phi}, \tag{193}$$

then $(u_m)$ and $(v_m)$ slowly approach $\frac{1}{\phi}$. We also have

$$\frac{?\left(\frac{F_n}{F_{n+1}}\right) - ?\left(\frac{1}{\phi}\right)}{\frac{F_n}{F_{n+1}} - \frac{1}{\phi}} = \frac{\phi^{2n+2} + (-1)^{n+2}}{3\sqrt{5} \cdot 2^{n-1}} \to \infty, \tag{194}$$

since $\phi^2 = \frac{3+\sqrt{5}}{2} > 2$. The result now follows by Lemma 13. The divergence isn't particularly rapid. With $n = 10$, we have

$$\frac{?(\frac{55}{89}) - ?(\frac{1}{\phi})}{\frac{55}{89} - \frac{1}{\phi}} = \frac{\frac{341}{512} - \frac{2}{3}}{\frac{55}{89} - \frac{1}{\phi}} \approx \frac{-.000651}{-.000056} \approx 11.53. \tag{195}$$

□

**Corollary 29.** *If $x = [0, a_0, \dots, a_n, \overline{1}]$ for any $a_j \in \mathbb{N}$, then $x \in D(?, \infty)$. If $u = \frac{p}{3\cdot 2^r} \in (0,1)$ with $\gcd(p,3) = 1$, then $u \in D($¿ $, 0)$.*

*Proof.* This is a direct consequence of Lemma 14. The first part may be clearer than the second. If $u < \frac{1}{2}$, then $u = \frac{v}{2}$, where $v = \frac{p}{3\cdot 2^{r-1}}$. If $u > \frac{1}{2}$, then $u = 1 - \frac{v}{2}$, where $v = \frac{3\cdot 2^r - p}{3\cdot 2^{r-1}}$. □

**Corollary 30.** *The Minkowski ?-function is nowhere continuously differentiable.*

*Proof.* The rationals of the form $\frac{p}{3\cdot 2^r}$ are dense in $[0,1]$, hence so are the quadratic irrationals of the form $[0, a_0, \dots, a_n, \overline{1}]$. It follows that every open interval of $(0,1)$ contains points where $?' = 0$ and points where $?'$ is not differentiable. □

We sketch a stab at generalizing Theorem 16. Fix $a, b \in \mathbb{N}$ and let

$$\theta_{a,b} = \frac{1}{[\overline{a,b}]} = \frac{-ab + \sqrt{a^2b^2 + 4ab}}{2a}\ . \tag{196}$$

(This was discussed on p.9 of this supplement, see (54) and (55).) More specifically, define sequences $(u_m), (v_m)$ as above, with

$$\begin{aligned} u_0 = 1, \quad u_m &= \cfrac{1}{a + \cfrac{1}{b + u_{m-1}}} := \frac{p_{2m}}{q_{2m}}; \\ v_0 = \frac{1}{a}, \quad v_m &= \cfrac{1}{a + \cfrac{1}{b + v_{m-1}}} := \frac{p_{2m+1}}{q_{2m+1}}. \end{aligned} \tag{197}$$

Then, as before $v_m \uparrow \theta_{a,b}$ and $u_m \downarrow \theta_{a,b}$. Further, as these are consecutive approximants, we have

$$\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} \ . \tag{198}$$

As consecutive approximants are on alternate sides of their limit, we have

$$\left|\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}\right| < \left|\frac{p_n}{q_n} - \theta_{a,b}\right| < \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| \ . \tag{199}$$

It is (and shall be!) an exercise to show that there exists a constant $c_{a,b} > 0$ so that

$$q_n q_{n+1} = c_{a,b} \left(\frac{ab + 2 + \sqrt{a^2b^2 + 4ab}}{2}\right)^n (1 + o(1)). \tag{200}$$

(Hints: see equation (41) of these notes, V. You need to look at $q_{2k}q_{2k+1}$ and $q_{2k+1}q_{2k+2}$ separately. This formula is valid for the product; $q_{2k}$ and $q_{2k+1}$ have a different constant in the asymptotics. Note also that if $a = b$, then

$$\frac{ab + 2 + \sqrt{a^2b^2 + 4ab}}{2} = \frac{a^2 + 2 + a\sqrt{a^2 + 4}}{2} = \left(\frac{a + \sqrt{a^2 + 4}}{2}\right)^2, \tag{201}$$

so this result is consistent with Theorem 16. In any case, this is a fairly routine computation. It follows that $(u_m)$ and $(v_m)$ slowly approach $\theta_{a,b}$, and after a bit more work, that

$$\left|\frac{p_n}{q_n} - \theta_{a,b}\right| \sim c' \left(\frac{ab + 2 + \sqrt{a^2b^2 + 4ab}}{2}\right)^{-n}. \tag{202}$$

It is also easy to see that

$$\left|?\left(\frac{p_n}{q_n}\right) - ?(\theta_{a,b})\right| \sim 2^{-n(a+b)/2}. \tag{203}$$

Thus, $\theta_{a,b} \in D(?, \infty)$ if and only if

$$\frac{ab + 2 + \sqrt{a^2b^2 + 4ab}}{2} > 2^{(a+b)/2}. \tag{204}$$

In our earlier case with $a = b = 1$, this condition reverts to $\frac{3+\sqrt{5}}{2} > 2$. Because of the exponential growth on the right-hand side in $(a, b)$, it is easy to believe that this inequality is satisfied for only finitely many pairs $(a, b)$. Noting the symmetry in $(a, b)$, a Mathematica check shows that the inequality holds precisely when $1 \le a \le b \le 4$, together with $(2, 5)$ and $(3, 5)$. This gives several more countably infinite families of elements in $D(?, \infty)$ as well as irrational elements in $D(?, 0)$.

There is an extensive literature on the differentiability of $?(x)$, and it is not our intention to cover it all. Salem, in his paper, first cites the result from metric continued fraction theory that, for $x \in (0,1)$,

$$x = [a_0(x), a_1(x), \dots], \tag{205}$$

if we let

$$A = \{x : \limsup_{n\to\infty} a_n(x) = \infty\}, \tag{206}$$

then $\mu(A) = 1$; alternatively, the set of $x$ for which $a_n(x)$ is bounded has measure zero. He then proves that if $x \in A$ and if ? is differentiable at $x$, then $?'(x) = 0$. Since $?'$ exists a.e., it follows that $\mu(D(?,0)) = 1$.

There are two fairly recent papers by Paradis, Viader and Biblioni, preprints of which will be distributed. the main result of interest is a generalization of the foregoing. Let

$$T(x) := \limsup_{n\to\infty} \frac{a_0(x) + \cdots + a_n(x)}{n+1}. \tag{207}$$

If $?'(x)$ exists and $T(x) > k_0 \approx 5.31972...$, then $x \in D(?,0)$; if $?'(x)$ exists and $T(x) < k_1 \approx 1.38848...$, then $x \in D(?,\infty)$. Here,

$$2\log_2(1+k_0) = k_0, \qquad k_1 = 2\log_2\left(\frac{1+\sqrt{5}}{2}\right). \tag{208}$$

These conditions would resolve the behavior of $?'(\theta_{a,b})$ when $a+b \ge 11$ and $a+b \le 2$, under the condition that we knew that $?'$ existed.

I believe, but have not been able to find yet in the literature, or prove, that there exist points $x$ with the property that the difference quotients oscillate arbitrarily close to zero and arbitrarily large. To make such a point, we consider first consider a sequence of positive integers $n_k$ which grows very rapidly. (I don't know yet how rapidly.) We then consider

$$u_{2k} = [0, 1^{n_1}, n_2, \dots, n_{2k}]; u_{2k+1} = [0, 1^{n_1}, n_2, \dots, n_{2k+1}]; \tag{209}$$

The intuition is that in $u_{2k}$, the behavior is dominated by the large final denominator, which makes the difference quotient very small, but in $u_{2k+1}$, the behavior is dominated by the large number of 1's in the denominator, which makes the difference quotient very large.