Some non-minimal canonical representations of forms as a sum of powers

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0. Introduction and Acknowledgments

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I want to thank Lek-Heng Lim for inviting me to speak at this Minisymposium and I would like to thank you all for being in the audience.

1. Three representation theorems about cubic forms

Theorem (Reichstein)

A general cubic $p(x_1, \dots, x_n)$ has a unique representation as

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This constructive "completion of the cube" is a canonical form, since $\binom{n+2}{3} = n^2 + \binom{n}{3}$ and it yields p as a sum of $\approx \frac{1}{4}n^2$ cubes. This is about 50% larger than the true minimum which is $\approx \frac{1}{6}n^2$.

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$$35 = 5+5+5+5+5+3+3+3+1$$

This is parsimonious, even if it doesn't minimize the length.

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This simple-minded representation is also canonical, and it yields p as a sum of $\approx \frac{1}{2} n^2$ cubes, but it's very easy to compute. I can't believe it's not in the literature. If you've seen it, please save me from professional embarrassment and provide a reference!

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2. Basic Definitions

Let $H_d(\mathbb{C}^n)$ denote the set of forms $p(x_1, \dots, x_n)$ of degree d with coefficients in $\mathbb{C}$. The dimension of the vector space $H_d(\mathbb{C}^n)$ is $N(n, d) := \binom{n+d-1}{d}$. Let $\mathcal{I}(n, d)$ be the index set of monomials:

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Let $x^i = x_1^{i_1} \cdots x_n^{i_n}$ and $c(i) = \frac{d!}{\prod i_k!}$ denote the multinomial coefficient. If $p \in H_d(\mathbb{C}^n)$, then we can write

$$p(x_1, \dots, x_n) = \sum_{i \in \mathcal{I}(n,d)} c(i) a(p; i) x^i.$$

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Theorem

Suppose $F : \mathbb{C}^N \to \mathbb{C}^N$ *is a polynomial map; that is,*

$$F(t_1, \ldots, t_N) = (f_1(t_1, \ldots, t_N), \ldots, f_N(t_1, \ldots, t_N))$$

where each $f_j \in \mathbb{C}[t_1, \ldots, t_N]$. *Then either (i) or (ii) holds:*
(i) The N *polynomials* $\{f_j : 1 \le j \le N\}$ *are algebraically dependent and* $F(\mathbb{C}^N)$ *lies in some non-trivial* $\{P = 0\}$ *in* $\mathbb{C}^N$.
(ii) The N *polynomials* $\{f_j : 1 \le j \le N\}$ *are algebraically independent and* $F(\mathbb{C}^N)$ *is (at least) dense in* $\mathbb{C}^N$.
Furthermore, the second case occurs if and only there is a point $u \in \mathbb{C}^N$ *at which the Jacobian matrix* $\left[\frac{\partial f_i}{\partial t_j}(u)\right]$ *has full rank.*

2. Basic Definitions

When $N = N(n,d)$, we may interpret such an F as a map from $\mathbb{C}^N$ to $H_d(\mathbb{C}^n)$ by indexing $\mathcal{I}(n,d)$ as $\{i_j : 1 \le j \le N\}$ and making the interpretation in an abuse of notation that

$$F(t_1,\dots,t_N) = \sum_{j=1}^{N} f_j(t_1\dots,t_N)x^{i_j}.$$

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$$F(t_1,\dots,t_N) = \sum_{j=1}^{N} f_j(t_1\dots,t_N)x^{i_j}.$$

Definition

A **canonical form** for $H_d(\mathbb{C}^n)$ is any polynomial map F from $\mathbb{C}^N$ to $H_d(\mathbb{C}^n)$ so that almost every $p \in H_d(\mathbb{C}^n)$ is in the range of F.

Note that for any indexing of $\mathcal{I}(n,d)$,

$$F(\{t_j\})(x) = \sum_{j=1}^{N(n,d)} c(i_j)t_jx^{i_j}$$

is technically a canonical form.

2. Basic Definitions

There are two ways to show that F is a canonical form. One way is to use the Theorem and find a single point at which the Jacobian has full rank, or, equivalently, look for a particular representation $F(u)$ at which $\left\{\frac{\partial F}{\partial t_j}(u)\right\}$ spans $H_d(\mathbb{C}^n)$.

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This can often be done via apolarity, which there's no time for today. (See e.g. the Iowa beamer slides.) With the apolarity interpretation, this is known classically as the Lasker-Wakeford Theorem. A beautiful modern version is given in Ehrenborg-Rota.

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3. Quadratic forms

Every quadratic form $p \in H_2(\mathbb{C}^n)$ is a sum of n squares, but since the naive number of coefficients, $n \times n$, is $> N(n,2) = \frac{n(n+1)}{2}$, a sum of n squares is not, *per se*, a canonical form. However, the standard "upper triangular" representation is a canonical form.

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Then $\frac{\partial F}{\partial \alpha_{ij}} = 2L_i x_j$, and if we specialize to $L_i = x_i$, then the set of partials is literally $\{2x_ix_j : 1 \le i \le j \le n\}$, which spans $H_2(\mathbb{C}^n)$.

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A constructive proof is better, of course. Suppose $p \in H_2(\mathbb{C}^n)$ and $p(x) = \sum_i a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j$. Then

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If $p(1, 0, \ldots, 0) = a_{11} \neq 0$, which is generally true, define

$$q(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) - \frac{1}{a_{11}}\left(\sum_{j=1}^{n} a_{1j}x_j\right)^2.$$

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Now just iterate this, losing one variable at a time, to get the traditional upper triangular sum of squares.

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It's worth noting that *every* quadratic form in $H_2(\mathbb{C}^n)$ is a sum of n squares, and this can also be made algorithmic. Begin with

Theorem (Biermann's Theorem)

If $p \in H_d(\mathbb{C}^n)$ and $p(i) = 0$ for every $i \in \mathcal{I}(n, d)$, then $p = 0$.

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This gives a finite set of $N(n,2)$ points to check for quadratic forms. Here's the algorithm. Given $p \in H_2(\mathbb{C}^n)$, index $\mathcal{I}(n,2)$ as you wish and look at $p(i)$. If this is always zero, then $p = 0$ and there's nothing to prove. Otherwise, take the first i at which $p(i) \neq 0$, and make an invertible linear change of variables taking $i \mapsto (1,0,\ldots,0)$. Do the argument of the last slide, and get p as a square plus a quadratic form in $n-1$ variables. Iterate to get p as a sum of n squares.

4. Reichstein and canonically completing the cube

There is a wonderful non-trivial way to complete the cube, but almost nobody knows it. It appears in a paper by Boris Reichstein from 1987 which according to MathSciNet has had no citations. It is a truly beautiful theorem, though it was not transparently presented and was framed in the context of trilinear forms.

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It can be proved abstractly as a canonical form, but there is also a constructive proof, which I'll give. Here is some numerology. By Alexander-Hirschowitz, for $n \neq 5$, a general cubic form in n variables can be written as a sum of $\left\lceil \frac{1}{n} N(n,3) \right\rceil = \left\lceil \frac{1}{n}\binom{n+2}{3} \right\rceil = \left\lceil \frac{(n+1)(n+2)}{6} \right\rceil$ cubes. (For $n = 5$, you need $\left\lceil \frac{6 \cdot 7}{6} \right\rceil + 1$ cubes.)

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Reichstein's Theorem writes a general cubic form as

$$\sum_{k=1}^{n}(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^3 + q(x_3, \ldots, x_n).$$

This is a sum of $\sum_{0\le k\le n/2}(n-2k) = \frac{(n+1)^2}{4}$ cubes, which is, on average, about 50% larger than what is necessary.

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But $N(n,3) - N(n-2,3) = \frac{n^3+3n^2+2n}{6} - \frac{n^3-3n^2+2n}{6} = n^2$, so that the total number of coefficients is

$$\sum_{0\le k\le n/2}(n-2k)^2 = N(n,3),$$

showing that this is a potential canonical form.

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The validity can be verified by Lasker-Wakeford, specializing at $x_1, x_2, x_1 + kx_2 + x_k$ (for $k \geq 3$) for linear forms in $(x_1, \ldots, x_n)$, etc., but Reichstein's constructive proof is better.

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The proof requires a formerly well-known fact: A general pair of quadratic forms can be simultaneously diagonalized. That is, if general $f, g \in H_2(\mathbb{C}^n)$ are given, then there exist n linearly independent forms $L_i(x) = \sum_{j=1}^{n} \alpha_{i,j} x_k$ and $c_i \in \mathbb{C}$ so that

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This can be made constructive. If $rank(f) = n$ and the determinant of the symmetric matrix associated with the pencil $f - \lambda g$ has n distinct roots $\{c_i\}$, then each $f - c_i g$ is singular. Routine methods can then be used to find the L_i's.

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We now prove Reichstein's Theorem. Suppose $p \in H_3(\mathbb{C}^n)$. We can generally simultaneously diagonalize $\frac{\partial p}{\partial x_1}$ and $\frac{\partial p}{\partial x_2}$: there exist linearly independent $L_i(x) = \sum_{j=1}^n \alpha_{ij} x_j$ and $c_i \in \mathbb{C}$ so that

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Since mixed partials are equal, we obtain the equation

$$\sum_{i=1}^n 2\alpha_{i2} L_i = \sum_{i=1}^n 2c_i \alpha_{i1} L_i,$$

and since the L_i's are linearly independent, $\alpha_{i2} = c_i \alpha_{i1}$. (This is important!)

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$$\implies \frac{\partial q}{\partial x_1} = \frac{\partial p}{\partial x_1} - \sum_{i=1}^{n} \frac{3\alpha_{i1}}{3\alpha_{i1}} L_i^2 = 0,$$

$$\frac{\partial q}{\partial x_2} = \frac{\partial p}{\partial x_2} - \sum_{i=1}^{n} \frac{3\alpha_{i2}}{3\alpha_{i1}} L_i^2 = \frac{\partial p}{\partial x_2} - \sum_{i=1}^{n} c_i L_i^2 = 0$$

$$\implies q = q(x_3, \dots, x_n).$$

By iterating, we obtain Reichstein's form for cubics:

$$p(x_1, \dots, x_n) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=1}^{n-2i} \ell_{ij}^3(x_{1+2i}, \dots, x_n).$$

5. Slinky

Recall Slinky:

$$p(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} (\alpha_{\{i,j\},i} x_i + \cdots + \alpha_{\{i,j\},j} x_j)^3.$$

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Given $p \in H_3(\mathbb{C}^n)$, $\frac{\partial p}{\partial x_n}$ is a quadratic form, so we can generally complete the square in the upper triangular way:

$$\frac{\partial p}{\partial x_n} = \sum_{j=1}^{n} (\alpha_{jj} x_j + \cdots + \alpha_{jn} x_n)^2.$$

5. Slinky

Let

$$q(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) - \sum_{j=1}^{n} \tfrac{1}{3\alpha_{jn}} (\alpha_{jj} x_j + \cdots + \alpha_{jn} x_n)^3.$$

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$$q(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) - \sum_{j=1}^{n} \tfrac{1}{3\alpha_{jn}}(\alpha_{jj}x_j + \cdots + \alpha_{jn}x_n)^3.$$

Then

$$\frac{\partial q}{\partial x_n} = \frac{\partial p}{\partial x_n} - \frac{\partial p}{\partial x_n} = 0 \implies q = q(x_1, \ldots, x_{n-1}).$$

and repeat. We assume $\alpha_{jn} \neq 0$, etc., which is generally true. In this way, for each pair (i, j) with $1 \le i \le j \le n$, we get exactly one summand using only the x_k's with $i \le k \le j$.

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This last construction worked because in the upper diagonal sum of squares for quadratic forms, there is a variable, x_n, which appears in every summand. This is not the case for the cubic version, so there is no obvious way to bump it up to quartics.

5. Slinky

This last construction worked because in the upper diagonal sum of squares for quadratic forms, there is a variable, x_n, which appears in every summand. This is not the case for the cubic version, so there is no obvious way to bump it up to quartics.

The Reichstein form, on the other hand, **can** be generalized to quartics, in the same way, by integrating on the coefficient of x_n. One gets a general $p \in H_4(\mathbb{C}^n)$ as a sum of $\sum_{j=0}^{n} \frac{(n+1-j)^2}{4} \approx \frac{1}{12} n^3$ fourth powers, which is about twice the minimal number. But this quartic version has no universally-used variable, so it can't be bumped up to the fifth power.

There is another obstacle. Say that

$$p(x_1,\ldots,x_n) = \sum_{k=1}^{r}(\alpha_{k1}x_1+\cdots+\alpha_{kn}x_n)^d + q(x_1,\ldots,x_m).$$

is a "Reichstein-type" canonical form if $N(n,d) = rn + N(m,d)$. It turns out that if $n = 12$ and $d = 4$, there does **not** exist $m < 12$ so that 12 divides $\binom{15}{4} - \binom{m+3}{4}$, so number theory rules out universal Reichstein-type canonical forms for quartics in 12 variables.

6. Brief number theory interlude

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Let $A_d=\left\{n:0\le m<n \implies n\nmid\binom{n+d-1}{d}-\binom{m+d-1}{d}\right\}$.

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Let $A_d = \left\{ n : 0 \le m < n \implies n \not| \binom{n+d-1}{d} - \binom{m+d-1}{d} \right\}$.

If $3 \not| k$, then $n = 2^{2k} \cdot 3 \in A_4$; if $p \equiv 1 \pmod{144}$ is prime, then $12p \in A_4$. If p is prime, then $p \mid \binom{n+p-1}{p} - \binom{n}{p}$, hence A_p is empty for prime p. The smallest elements of $A_6, A_8, A_{10}, A_{12}, A_{14}$ and A_{15} are 10, 1792, 6, 242, 338 and 273 respectively. If A_9 or A_{16} are non-empty, then their smallest elements are at least 10^5.

7. Slowpoke

The last expression for cubic forms is not canonical: for any $p \in H_3(\mathbb{C}^n)$, there exists an invertible linear change of variables $y_j = \sum \lambda_{jk} x_k$ and n linear forms ℓ_j so that

$$p(x_1, \dots, x_n) = \sum_{j=1}^{n} \ell_j^3(x_1, \dots, x_n) + q(y_2, \dots, y_n).$$

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We need a lemma: for any integer m, there exist $m + 1$ linear forms $\ell_{j,m} = \ell_{j,m}(y_1, \dots, y_m)$ so that

$$\sum_{j=1}^{m+1} \ell_{j,m} = 0 \quad \text{and} \quad \sum_{j=1}^{m+1} \ell_{j,m}^2 = \sum_{k=1}^{m} y_k^2.$$

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The simplest proof is to set $\ell_{m+1,m} = -\sum_{j=1}^{m} \ell_{j,m}$ and then observe that the quadratic form $\sum_{j=1}^{m} t_j^2 + (\sum_{j=1}^{m} t_j)^2$ has full rank, and so can be written as a sum of m squares. Finally, invert the system.

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As an explicit solution, let $\alpha = \frac{-(m+1)+\sqrt{m+1}}{m(m+1)}$ and define

$$\ell_{j,m}(x_1,\dots,x_n) = x_j + \alpha \sum_{j=1}^{m} x_j, \qquad 1 \le j \le m,$$

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Now suppose $p \in H_3(\mathbb{C}^n)$. By Biermann's Theorem, there is a finite list to check to find a point u where $p(u) \neq 0$, and after an invertible linear change of variables, taking $\{x_j\} \mapsto \{u_j\}$, we may assume that

7. Slowpoke

$$p = u_1^3 + 3h_1(u_2, \dots, u_n)u_1^2 + 3h_2(u_2, \dots, u_n)u_1 + h_3(u_2, \dots, u_n),$$

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We then let $u_1 = y_1 - h_1(u_2, \dots, u_n)$ to clear the quadratic term :

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and do a standard diagonalization of $\tilde{h}_2$ as a quadratic form, with the accompanying change of variables, yielding:

$$p = y_1^3 + 3y_1(y_2^2 + \cdots + y_r^2) + k_3(y_2, \dots, y_n); \quad r \le n.$$

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Finally, observe that if

$$q = \frac{1}{r}\sum_{j=1}^{r}\left(y_1 + \sqrt{r}\cdot \ell_{j,r-1}(y_2,\dots,y_r)\right)^3,$$

then the lemma implies that $p - q$ is a cubic form in $(y_2,\dots,y_n)$, which is what we wanted.

8. Steampunk canonical forms

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In 1869, J. J. Sylvester (1814-1897) reflected on the discovery of some of his most famous research in 1851, done while he was working as an actuary:

"I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought — a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. That night we slept no more"

8. Steampunk canonical forms

Theorem (Sylvester)

Suppose $p(x,y) = \sum_{j=0}^{d} \binom{d}{j} a_j x^{d-j} y^j$ *and* $h(x,y) = \sum_{t=0}^{r} c_t x^{r-t} y^t = \prod_{j=1}^{r} (\beta_j x - \alpha_j y)$ *is a product of pairwise distinct linear factors. Then there exist* $\lambda_k \in \mathbb{C}$ *so that*

$$p(x,y) = \sum_{k=1}^{r} \lambda_k (\alpha_k x + \beta_k y)^d$$

if and only if

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-r} & a_{d-r+1} & \cdots & a_d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

8. Steampunk canonical forms

Theorem (Sylvester)

(i) A general binary form of degree $d = 2k - 1$ can be written as

$$\sum_{j=1}^{k} (\alpha_j x + \beta_j y)^{2k-1}.$$

(ii) For any non-zero linear form $\ell(x, y) = \alpha x + \beta y$, a general binary form of degree $d = 2k$ can be written as

$$\lambda \ell^{2k}(x, y) + \sum_{j=1}^{k} (\alpha_j x + \beta_j y)^{2k}.$$

for some $\lambda \in \mathbb{C}$.

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"$\lambda \ell^{2k}$" must be what Sylvester meant by "as far as yet made out".

8. Steampunk canonical forms

Sylvester defined the *catalecticant* to be the invariant of a binary form of even degree which vanishes when $\lambda = 0$. He apologized for introducing this term: "Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant." Sylvester was very interested in the technical aspects of poetry and a "catalectic" verse is one in which the last line is missing a foot.

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Owing to the action of the orthogonal group on sums of squares, another old canonical form for binary forms of even degree $2k$ is

$$p(x,y) = (\alpha_0 x^k + \alpha_1 x^{k-1} y + \cdots + \alpha_k y^k)^2 + (\beta_1 x^{k-1} y + \cdots + \beta_k y^k)^2$$

Because a general form of degree $2k$ has $2k$ distinct linear factors, this can be done in $\binom{2k-1}{k}$ different ways. If p is real and psd, then there are 2^{k-1} real representations.

9. New steampunk canonical forms

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Theorem

Suppose $d \geq 1$, $\ell_j(x,y) = \beta_j x + \gamma_j y$, $1 \leq j \leq m$, are fixed pairwise non-proportional linear forms and suppose $e_k \mid d$, $1 \leq k \leq r$ and $m + \sum_{k=1}^{r}(e_k + 1) = d + 1$. Then a general binary form of degree d can be written as

$$p(x,y) = \sum_{j=1}^{m} c_j \ell_j^d(x,y) + \sum_{k=1}^{r} f_k^{d/e_k}(x,y),$$

where $c_j \in \mathbb{C}$ and f_k is a form of degree e_k.

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where $c_j \in \mathbb{C}$ *and* f_k *is a form of degree* e_k.

This recovers Sylvester's canonical form, upon taking $r = \lfloor d/2 \rfloor$ and $e_k \equiv 1$, so that $m = 0$ if d is odd and $m = 1$ if d is even.

9. New steampunk canonical forms

If $r = 0$ and $m = d + 1$, this just gives a basis.
If $e_k \equiv 1$, then Sylvester's algorithm can be adapted to show uniqueness. These results may well be new, as are some canonical forms with mixed powers and some interesting enumerative questions.

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If $e_k \equiv 2$, an analogue to Sylvester's canonical forms occurs for general forms of even degree $d = 2k$: they are the sum of the k-th power of $\lfloor (d+1)/3 \rfloor$ quadratics plus a linear combination of any pre-specified $d - 3\lfloor (d+1)/3 \rfloor$ $2k$-th powers of linear forms. We don't have an algorithm for this. We want one. One problem is that it's easy to kill ℓ^d with a differential operator; $q^{d/2}$, not so much.

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If $d = 4$, $m = 0$, $e_1 = 2$ and $e_2 = 1$, a general binary quartic can be written as the sum of the square of a quadratic form and the fourth power of a linear form. (We have an algorithm for this which shows that it can be done in six different ways.)

9. New steampunk canonical forms

If $d = 6$, $m = 0$, $e_1 = 3$ and $e_2 = 2$, then $4 + 3 = 7$ implies that a general binary sextic form can be written as the sum of the square of a cubic form and the cube of a quadratic form. We don't have an algorithm for doing this and we $(\text{really})^2$ want one!

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I'll end with a proof that this is a canonical form. Suppose p is a sextic form and $F(\{t_j\})(x,y) = f^2(x,y) + g^3(x,y)$, where

$$f(x,y) = t_1x^3 + t_2x^2y + t_3xy^2 + t_4y^3,$$
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Then the partials with respect to the t_j's are:

$$2fx^3,\ 2fx^2y,\ 2fxy^2,\ 2fy^3; \quad 3g^2x^2,\ 3g^2xy,\ 3g^2y^2.$$

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If we specialize at $f = x^3, g = y^2$, then these partials become:

$$2x^6,\ 2x^5y,\ 2x^4y^2,\ 2x^3y^3;\quad 3x^2y^4,\ 3xy^5\ ,3y^6.$$

These trivially span $H_6(\mathbb{C}^2)$.

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If you think that "40" is obvious, a general sextic can be written as $g^3 + h_1^6 + h_2^6$, where $h_j(x, y) = \beta_{j1}x + \beta_{j2}y$. Numerical experiments show that the number of different $\{g^3, \{h_1^6, h_2^6\}\}$'s is 22.

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Numerical experiments on binary octics, written naively, crash the kernel of Mathematica. A general binary octic is the sum of three fourth powers of quadratics. I'd like to know a lot more about this theorem than I do.

9. New steampunk canonical forms

Many numerical experiments suggest that for a general sextic p, there are exactly 40 different $\{f^2, g^3\}$.
If you think that "40" is obvious, a general sextic can be written as $g^3 + h_1^6 + h_2^6$, where $h_j(x,y) = \beta_{j1}x + \beta_{j2}y$. Numerical experiments show that the number of different $\{g^3, \{h_1^6, h_2^6\}\}$'s is 22.
Numerical experiments on binary octics, written naively, crash the kernel of Mathematica. A general binary octic is the sum of three fourth powers of quadratics. I'd like to know a lot more about this theorem than I do.

Thank you for your patience.

10. Oh, I have some more time