

## BINARY FORMS WITH THREE DIFFERENT RELATIVE RANKS

BRUCE REZNICK AND NERIMAN TOKCAN

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ABSTRACT. Suppose  $f(x, y)$  is a binary form of degree  $d$  with coefficients in a field  $K \subseteq \mathbb{C}$ . The  $K$ -rank of  $f$  is the smallest number of  $d$ -th powers of linear forms over  $K$  of which  $f$  is a  $K$ -linear combination. We prove that for  $d \geq 5$ , there always exists a form of degree  $d$  with at least three different ranks over various fields. The  $K$ -rank of a form  $f$  (such as  $x^3y^2$ ) may depend on whether  $-1$  is a sum of two squares in  $K$ .

### 1. INTRODUCTION

Suppose  $f(x, y)$  is a binary form of degree  $d$  with coefficients in a field  $K \subseteq \mathbb{C}$ . The  $K$ -length or  $K$ -rank of  $f$ ,  $L_K(f)$ , is the smallest  $r$  for which there is a representation

$$(1.1) \quad f(x, y) = \sum_{j=1}^r \lambda_j (\alpha_j x + \beta_j y)^d$$

with  $\lambda_j, \alpha_j, \beta_j \in K$ . In case  $K = \mathbb{C}$  or  $\mathbb{R}$ , these are commonly called the *Waring rank* or *real Waring rank*. We shall say that two linear forms are *distinct* if they (or their  $d$ -th powers) are not proportional. A representation such as (1.1) is *honest* if the summands are pairwise distinct; that is, if  $\lambda_i \lambda_j (\alpha_i \beta_j - \alpha_j \beta_i) \neq 0$  whenever  $i \neq j$ . Any representation in which  $r = L_K(f)$  is necessarily honest.

Of course, if  $K \subseteq F \subseteq \mathbb{C}$ , then  $f \in F[x, y]$  as well, and one may consider  $L_F(f)$  to be the *relative rank* of  $f$  with respect to  $F$ . It is not hard to find forms with two different relative ranks; for example, suppose  $\gamma \notin \mathbb{Q}, \gamma^2 \in \mathbb{Q}$  and  $f(x, y) = (x + \gamma y)^d + (x - \gamma y)^d \in \mathbb{Q}[x, y]$ . Then  $L_F(f)$  is equal to 2 if  $\gamma \in F$  and  $d$  otherwise; see [10, Thm.4.6].

A form  $h \in \mathbb{C}[x, y]$  is *apolar* to  $f$  if  $h(D)f = h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})f(x, y) = 0$ . If (1.1) holds and is honest, then  $\prod(\beta_j x - \alpha_j y)$  is apolar to  $f$ . Sylvester showed ([11, 12]; see Theorem 2.1 below) that this is an if and only if condition: if  $\prod(\beta_j x - \alpha_j y)$  is square-free, and apolar to  $f$ , then there exist  $\lambda_j \in \mathbb{C}$  making (1.1) true. In [10], the first author showed that the same statement is true if  $\alpha_j, \beta_j, \lambda_j$  are restricted to be in any field  $F \subseteq \mathbb{C}$ . The computation of determining whether  $h$  is apolar to  $f$  is equivalent

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to the Sylvester algorithm. The set of all forms which are apolar to a given form  $f$  is called its *apolar ideal* and it is known, see [8, Thm.1.44(iv)], [1, Lemma 2.8] that this is generated by two relatively prime forms, the sum of whose degrees is  $d + 2$ . An upper bound on the relative rank is given by [10, Thm.4.10]: if  $f \in K[x, y]$ , then  $L_K(f) \leq \deg f$ .

Sylvester also proved ([13], see Corollary 2.3 below), a variation on Descartes’ Rule of Signs for univariate polynomials, which can be extended ([10]) to binary forms. Suppose  $f \in \mathbb{R}[x, y]$  is *hyperbolic*; that is,  $f$  splits over  $\mathbb{R}$  ( $f$  needn’t be square-free), but  $f$  is not a  $d$ -th power. Then  $L_{\mathbb{R}}(f) = \deg f$ . It was conjectured (and proved for  $d \leq 4$ ) in [10] that the converse is also true: if  $L_{\mathbb{R}}(f) = \deg f$ , then  $f$  is hyperbolic (and not a  $d$ -th power). This was proved by Causa and Re [6] and Comon and Ottaviani [7] when  $f$  is square-free, and very recently, unconditionally, by Blekherman and Sinn [1, Thm 2.2].

The first author showed in [10] that for  $\phi(x, y) = 3x^5 - 20x^3y^2 + 10xy^4$ , we have  $L_K(\phi) = 3$  if and only if  $\sqrt{-1} \in K$ ,  $L_K(\phi) = 4$  for  $K = \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-6})$  (at least) and  $L_{\mathbb{R}}(\phi) = 5$ . This example also shows (by taking  $K_1 = \mathbb{Q}(\sqrt{-2})$  and  $K_2 = \mathbb{Q}(\sqrt{-3})$ ) that  $L_{K_1}(f) = L_{K_2}(f) < L_{K_1 \cap K_2}(f)$  is possible. Furthermore ([10, Cor.5.1]), if  $f$  has  $k$  different ranks, then  $\deg f \geq 2k - 1$ ; so three different ranks cannot occur for forms of degree  $\leq 4$ .

The main result of this paper is that in all degrees  $d \geq 5$ , there exist binary forms of degree  $d$  with at least three different ranks over different fields (see Theorem 3.1). In particular, let  $\zeta_m$  denote a primitive  $m$ -th root of unity. We shall prove that if  $k \geq 3$  and  $p_{2k-1}(x, y) = x^{k-1}y^{k-1}(x - y)$ , then

$$L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k-1}) = k, \quad L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) = k + 1, \quad L_{\mathbb{R}}(p_{2k-1}) = 2k - 1 > k + 1.$$

Similarly, if  $k \geq 3$  and  $p_{2k}(x, y) = x^k y^k$ , then

$$L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k}) = k + 1, \quad L_{\mathbb{Q}(\zeta_k)}(p_{2k}) = k + 2, \quad L_{\mathbb{R}}(p_{2k}) = 2k > k + 2.$$

We are not aware of any binary form of any degree with more than three different ranks. We do not consider forms in more than two variables in this paper.

The relative rank can depend on algebraic properties of the underlying field. The *Stufe* of a non-real field  $F$ ,  $s(F)$ , is the smallest integer  $n$  such that  $-1$  can be written as a sum of  $n$  squares in  $F$ . It is already known that  $L_{\mathbb{C}}(x^3y^2) = 4$  (from [4, Prop.3.1]) and  $L_{\mathbb{R}}(x^3y^2) = 5$  (from [2, Prop.4.4]). We show in Theorem 4.1 that  $L_K(x^3y^2) = 4$  if and only if  $s(K) \leq 2$  and  $L_K(x^3y^2) = 5$  otherwise. (For more on the real rank of monomials, see [5].) We show in Theorem 4.2 that if  $m$  is a square-free positive integer and  $f(x, y) = \binom{6}{1}x^5y - \binom{6}{3}x^3y^3$ , then  $L_{\mathbb{Q}(\sqrt{-m})}(f) = 4$  if and only if  $s(\mathbb{Q}(\sqrt{-m})) = 2$  if and only if  $m \not\equiv 7 \pmod{8}$  (see [9, 14]), and  $L_{\mathbb{Q}(\sqrt{-7})}(f) = 5$ .

## 2. TOOLS

The following theorems are proved in [10] and, for  $K = \mathbb{C}$ , are due to Sylvester [11, 12] in 1851.

**Theorem 2.1** ([10, Thm.2.1, Cor.2.2]). *Suppose  $K \subseteq \mathbb{C}$  is a field,*

$$(2.1) \quad f(x, y) = \sum_{j=0}^d \binom{d}{j} a_j x^{d-j} y^j \in K[x, y]$$

and suppose  $r \leq d$  and

$$(2.2) \quad h(x, y) = \sum_{t=0}^r c_t x^{r-t} y^t = \prod_{j=1}^r (-\beta_j x + \alpha_j y)$$

is a product of pairwise distinct linear factors, with  $\alpha_j, \beta_j \in K$ . Then there exist  $\lambda_j \in K$  so that

$$(2.3) \quad f(x, y) = \sum_{j=1}^r \lambda_j (\alpha_j x + \beta_j y)^d$$

if and only if

$$(2.4) \quad \begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-r} & a_{d-r+1} & \cdots & a_d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

that is, if and only if

$$(2.5) \quad \sum_{t=0}^r a_{\ell+t} c_t = 0, \quad \ell = 0, 1, \dots, d - r.$$

We remark that if  $f$  and  $h$  are defined by (2.1) and (2.2), then

$$h(D)f = \sum_{m=0}^{d-r} \frac{d!}{(d-r-m)!m!} \left( \sum_{i=0}^r a_{i+m} c_i \right) x^{d-r-m} y^m.$$

Thus, Theorem 2.1 provides an algorithm for determining the forms of a given degree  $\leq \deg f$  which are apolar to  $f$ . If (2.4) holds and  $h$  is square-free, then we say that  $h$  is a *Sylvester form of degree  $r$*  for  $f$  over  $K$ . In other words,  $L_K(f) = r$  if and only if there is a Sylvester form for  $f$  over  $K$  of degree  $r$ .

The next theorem is a generalization of an 1864 theorem of Sylvester [13]; the original applied to real polynomials in one variable and was adapted to real binary forms in [10].

**Theorem 2.2** ([10, Thm.3.1,3.2]). *Suppose  $f(x, y)$  is a non-zero real form of degree  $d$  with  $\tau$  real linear factors (counting multiplicity),  $f$  is not the  $d$ -th power of a linear form and*

$$(2.6) \quad f(x, y) = \sum_{j=1}^r \lambda_j (\cos \theta_j x + \sin \theta_j y)^d,$$

where  $-\frac{\pi}{2} < \theta_1 < \dots < \theta_r \leq \frac{\pi}{2}$ ,  $r \geq 2$  and  $\lambda_j \neq 0$ . If there are  $\sigma$  sign changes in the tuple  $(\lambda_1, \lambda_2, \dots, \lambda_r, (-1)^d \lambda_1)$ , then  $\tau \leq \sigma$ . In particular,  $\tau \leq r$ .

**Corollary 2.3** ([10, Cor.4.11]). *If  $f \in \mathbb{R}[x, y]$  is a product of  $d$  real linear forms and not a  $d$ -th power, then  $L_{\mathbb{R}}(f) = d$ .*

**Corollary 2.4.** *If  $f \in \mathbb{R}[x, y]$  is a product of  $d$  real linear forms and not a  $d$ -th power, and  $g \in \mathbb{R}[x, y]$  is apolar to  $f$ , with  $\deg g < d$ , then  $g$  cannot be square-free.*

We shall also need the following result from [10].

**Theorem 2.5** ([10, Thm.4.10]). *If  $f \in K[x, y]$ , then  $L_K(f) \leq \deg f$ .*

The next tool is an exercise in a first course in algebraic number theory. We include the proof for completeness. (See [3, p. 158 (Lemma 3)] for a more incisive, but less elementary, proof.) Recall that  $\zeta_d = e^{\frac{2\pi i}{d}}$ .

**Theorem 2.6.** *Suppose  $m, n$  are integers. Then  $\zeta_m \in \mathbb{Q}(\zeta_n)$  if and only if  $m \mid n$  or  $n$  is odd and  $m \mid 2n$ .*

*Proof.* Note that  $\zeta_m = \zeta_{mt}^t$ . If  $n$  is odd and  $m$  divides  $2n$  but not  $n$ , then  $m = 2u$  and  $n = tu$  with odd  $t, u$ , so  $\zeta_m = \zeta_{2n}^t = -\zeta_{2n}^{t+tu} = -\zeta_n^{t(u+1)/2} \in \mathbb{Q}(\zeta_n)$ .

Conversely, let  $g = \gcd(m, n)$  so that  $m = gr, n = gs$ , where  $\gcd(r, s) = 1$ , and let  $q = grs = \text{lcm}(m, n)$ . Then  $\zeta_m = \zeta_q^s$  and  $\zeta_n = \zeta_q^r$ . Now choose integers  $e, f$  so that  $es + fr = 1$ . We have  $\zeta_m^e \zeta_n^f = \zeta_q^{es+fr} = \zeta_q$ . Since  $\zeta_m \in \mathbb{Q}(\zeta_n)$ , it follows that  $\zeta_q \in \mathbb{Q}(\zeta_n)$ , so  $\mathbb{Q}(\zeta_q) \subseteq \mathbb{Q}(\zeta_n)$ , but since  $n \mid q$ , the converse inclusion holds as well, and so  $\mathbb{Q}(\zeta_q) = \mathbb{Q}(\zeta_n)$ . This in turn implies that  $\Phi(n) = \Phi(q)$ . Since  $n \mid q$ , this implies that  $n = q$  (and  $gs = grs$ , so  $r = 1$  and  $m \mid n$ ) or  $n$  is odd and  $q = 2n$  (and  $grs = 2gs$ , so  $r = 2$  and  $m \mid 2n$ ).  $\square$

**Corollary 2.7.** *If  $m \geq 3$ , then  $\zeta_m \notin \mathbb{Q}(\zeta_{m\pm 1})$ .*

For our final result, we make a minor gloss on the work of Nagell [9]; see also the beautiful short proof of Szymiczek [14].

**Theorem 2.8.** *Suppose  $F = \mathbb{Q}(\sqrt{-m})$ , where  $m$  is a square-free positive integer. Then there exist solutions to either of the equations*

$$(2.7) \quad r^2 + s^2 = -1, \quad rs(r^2 - s^2) \neq 0, \quad r, s \in F,$$

$$(2.8) \quad t^2 + u^2 = -2, \quad tu(t^2 - u^2) \neq 0, \quad t, u \in F,$$

*if and only if  $m \not\equiv 7 \pmod{8}$ .*

*Proof.* First note that if (2.7) holds and  $(t, u) = (r + s, r - s)$ , then  $t^2 + u^2 = 2(r^2 + s^2) = -2$  and  $tu(t^2 - u^2) = 4rs(r^2 - s^2)$ , so (2.8) holds. This argument goes the other way with  $(r, s) = (\frac{t+u}{2}, \frac{t-u}{2})$ , and so it suffices to prove the theorem for (2.7).

Nagell [9] proves that  $s(\mathbb{Q}(\sqrt{-m})) = 2$  (that is, there is a solution to  $r^2 + s^2 = -1$  in  $\mathbb{Q}(\sqrt{-m})$ ) if and only if  $m \not\equiv 7 \pmod{8}$ , so all we need to do is consider the additional condition  $rs(r^2 - s^2) \neq 0$ . If  $r^2 + s^2 = -1$  and  $rs(r^2 - s^2) = 0$ , then up to permutation,  $(r, s) = (\pm i, 0)$  or  $(\pm \frac{\sqrt{-2}}{2}, \pm \frac{\sqrt{-2}}{2})$ . These solutions are relevant to  $\mathbb{Q}(\sqrt{-m})$  only when  $m = 1, 2$ , in which case the following alternatives suffice:

$$\mathbb{Q}(\sqrt{-1}) : \quad \left(\frac{3}{4}\right)^2 + \left(\frac{5i}{4}\right)^2 = -1, \quad \mathbb{Q}(\sqrt{-2}) : \quad 7^2 + (5\sqrt{-2})^2 = -1.$$

$\square$

### 3. THREE RANKS

Our general strategy is straightforward. Suppose  $d = 2k - 1$  is odd. Choosing  $r = k$ , we see that (2.4) is a  $k \times (k + 1)$  linear system, which in general has a unique solution. We consider a form  $f$  of degree  $2k - 1$  which is a product of real linear factors, so  $L_{\mathbb{R}}(f) = 2k - 1$ . We also choose  $K$  to be the field generated by the coefficients of this unique representation of  $f$  over  $\mathbb{C}$ , so  $L_K(f) = k$ ; necessarily,  $f \in K[x, y]$ . Finally, we somehow find a representation of rank between  $k$  and  $2k - 1$  over a non-real field which does not contain the rank  $k$  representation. If

$d = 2k$ , the same heuristic applies, but there will be, in general, infinitely many representations of rank  $k + 1$ . In certain cases though, each of these representations must contain a specific non-real root of unity  $\zeta$ .

**Theorem 3.1.** *If  $d \geq 5$ , then there exists a binary form  $p_d$  of degree  $d$  which takes at least three different ranks.*

*Proof.* Let  $p_{2k-1} = \binom{2k-1}{k}x^{k-1}y^{k-1}(x - y)$ , so that in (2.4),  $a_{k-1} = 1$ ,  $a_k = -1$  and  $a_i = 0$  otherwise. First, with  $r = k - 1$ , we see that the matrix from (2.4) is non-singular:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It follows that  $L_{\mathbb{C}}(p_{2k-1}) > k - 1$ . On taking  $r = k$ , (2.4) becomes:

$$(3.1) \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & -1 & \cdots & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Clearly, the only solution to (3.1) has  $c_i = c$  for all  $i$ , so that up to multiple,

$$h(x, y) = \sum_{t=0}^k x^{k-t}y^t = \frac{x^{k+1} - y^{k+1}}{x - y} = \prod_{j=1}^k (x - \zeta_{k+1}^j y),$$

and so  $L_K(p_{2k-1}) = k$  if and only if  $\zeta_{k+1} \in K$ ; in particular,  $L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k-1}) = k$ . Since  $p_{2k-1}$  is hyperbolic, it follows from Corollary 2.3 that  $L_{\mathbb{R}}(p_{2k-1}) = 2k - 1$ .

Now set  $r = k + 1$ , so that (2.4) becomes:

$$(3.2) \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The system (3.2) implies  $c_1 = \cdots = c_k$ , but places no conditions on  $c_0$  and  $c_{k+1}$ . In particular, we may choose  $c_0 = c_{k+1} = 0$  and  $c_1 = \cdots = c_k = 1$ , to get a Sylvester polynomial over  $\mathbb{Q}(\zeta_k)$ :

$$h(x, y) = \sum_{t=1}^k x^{k+1-t}y^t = xy \left( \frac{x^k - y^k}{x - y} \right) = xy \prod_{j=1}^{k-1} (x - \zeta_k^j y).$$

It follows that  $L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) \leq k + 1$ . Since  $\zeta_{k+1} \notin \mathbb{Q}(\zeta_k)$  by Corollary 2.7, it follows that  $L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) = k + 1$ .

Since

$$\begin{aligned} & c_0x^{k+1} + c_1(x^k y + \cdots + xy^k) + c_{k+1}y^{k+1} \\ &= (c_0x + (c_1 - c_0)y)(x^k + \cdots + y^k) + (c_{k+1} - c_1 + c_0)y^{k+1}, \end{aligned}$$

it is not hard to show that the apolar ideal of  $p_{2k-1}$  is generated by  $x^k + \cdots + y^k$  and  $y^{k+1}$ ; note that  $k + (k + 1) = (2k - 1) + 2$ . It seems to be a quite difficult

question to determine which fields  $K$  have the property that, for a suitable choice of  $c_i$ 's, this form is square-free and splits over  $K$ . We return to this type of question in the next section.

Suppose  $p_{2k}(x, y) = \binom{2k}{k} x^k y^k$ , so  $a_k = 1$  and  $a_i = 0$  otherwise in (2.4). (This example is also discussed in [10, Thm.5.5].) Taking  $r = k$ , we note that the matrix

$$(3.3) \quad \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

is non-singular, hence there are no representations of rank  $k$ . For  $r = k + 1$ ,

$$(3.4) \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies c_1 = \cdots = c_k = 0.$$

Thus every Sylvester form of degree  $k + 1$  has the shape  $h(x, y) = \alpha x^{k+1} - \beta y^{k+1}$  and the apolar ideal of  $p_{2k}$  is generated by  $x^{k+1}$  and  $y^{k+1}$ . If  $h$  has distinct factors, then  $\alpha\beta \neq 0$  and if  $u$  is one of the  $(k + 1)$ -st roots of  $\beta/\alpha$ , then

$$h(x, y) = \alpha \prod_{j=0}^k (x - \zeta_{k+1}^j u y).$$

If  $h$  splits over  $K$ , then  $u, \zeta_{k+1} u \in K$ , hence  $\zeta_{k+1} \in K$  and  $\mathbb{Q}(\zeta_{k+1}) \subseteq K$ . In particular, by taking  $\alpha = \beta = 1$ , we see that  $x^{k+1} - y^{k+1}$  is a Sylvester form for  $p_{2k}$  over  $\mathbb{Q}(\zeta_{k+1})$ , and so  $L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k}) = k + 1$ . Since  $x^k y^k$  is hyperbolic,  $L_{\mathbb{R}}(p_{2k}) = 2k$ .

Any expression of rank  $k + 2$  over  $K$  would have a Sylvester form of shape

$$(\alpha x + \beta y)x^{k+1} + (\gamma x + \delta y)y^{k+1}.$$

In particular,  $xy(x^k - y^k)$  splits over  $\mathbb{Q}(\zeta_k)$ , which does not contain  $\zeta_{k+1}$  and so we have  $L_{\mathbb{Q}(\zeta_k)}(p_{2k}) = k + 2$ . □

Here are explicit representations of  $p_5$ ,  $p_6$  and  $p_7$  as sums of powers of linear forms.

**Example 3.1.** For  $k = 3$ , the following two formulas may be directly verified (as usual,  $\omega = \zeta_3$  and  $i = \zeta_4$ ):

$$\begin{aligned} p_5(x, y) &= 10x^2y^2(x - y) \\ &= \frac{1}{4} \cdot ((-1 - i)(x + iy)^5 + 2(x - y)^5 + (-1 + i)(x - iy)^5) \in \mathbb{Q}(\zeta_4)[x, y] \\ &= x^5 - y^5 + \frac{1}{\omega - \omega^2} \cdot (\omega^2(x + \omega y)^5 - \omega(x + \omega^2 y)^5) \in \mathbb{Q}(\zeta_3)[x, y]. \end{aligned}$$

The expressions seem to get more complicated for larger values of  $k$ . For example,

$$\begin{aligned} &(1 + 2\zeta_5 + 3\zeta_5^2 - \zeta_5^3)p_7(x, y) \\ &= \zeta_5^4(x + \zeta_5 y)^7 - \zeta_5^2(1 + \zeta_5 + \zeta_5^2)(x + \zeta_5^2 y)^7 \\ &\quad + \zeta_5(1 + \zeta_5 + \zeta_5^2)(x + \zeta_5^3 y)^7 - \zeta_5(x + \zeta_5^4 y)^7. \end{aligned}$$

Here,  $1 + 2\zeta_5 + 3\zeta_5^2 - \zeta_5^3 = i\sqrt{\frac{5}{2}(5 + \sqrt{5})} \approx 4.25i$ .

**Example 3.2.** The representations of  $p_{2k}$  of rank  $k + 1$  are given in [10, Thm.5.5]. For  $k = 3$ , taking  $w = 1$  in [10, (5.6)], we obtain after some simplification,

$$\begin{aligned} p_6(x, y) &= 20x^3y^3 \\ &= \frac{1}{4} \cdot ((x + y)^6 + i(x + iy)^6 - (x - y)^6 - i(x - iy)^6) \in \mathbb{Q}(\zeta_4)[x, y] \\ &= \frac{1}{3} \cdot ((x + y)^6 + (x + \omega y)^6 + (x + \omega^2 y)^6 - 3x^6 - 3y^6) \in \mathbb{Q}(\zeta_3)[x, y]. \end{aligned}$$

The evident patterns shown above are easily proved, using the methods of [10].

#### 4. TWO MORE EXAMPLES

In this section, we give some additional examples, in which (2.4) is altered only slightly from  $p_5$  and  $p_6$ , but the results show a sensitivity to arithmetic conditions. Recall that  $s(K) \leq 2$  means that there exist  $r, s \in K$  so that  $r^2 + s^2 = -1$ .

**Theorem 4.1.** *Suppose  $f(x, y) = \binom{5}{2}x^3y^2$ . Then  $L_K(f) = 4$  iff  $s(K) \leq 2$ ; otherwise,  $L_K(f) = 5$ .*

*Proof.* We already know from [4] that  $L_{\mathbb{C}}(f) = 4$ , hence  $L_K(f) \geq 4$ . This can also be shown directly via Theorem 2.1. We omit the details.

Suppose now that  $L_K(f) = 4$ . Then,

$$(4.1) \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0$$

and  $h(x, y) = c_0x^4 + c_3xy^3 + c_4y^4$  is a Sylvester form for  $f$  over  $K$ . Thus, we are led to the question: for which choices of  $c_i$  and which fields  $K$  can such a square-free form split into distinct factors over  $K$ ?

If  $c_0 = 0$ , then  $h$  is not square-free, so we scale to take  $c_0 = 1$ . Then  $L_K(f) = 4$  if and only if there exist distinct  $r_i \in K$  so that

$$x^4 + c_3xy^3 + c_4y^4 = (x - r_1y)(x - r_2y)(x - r_3y)(x - r_4y);$$

that is, if and only if the Diophantine system

$$(4.2) \quad r_1 + r_2 + r_3 + r_4 = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = 0$$

has a solution in  $K$  with distinct  $r_i$ 's.

We solve (4.2), first ignoring the full restriction to distinct elements. If all  $r_i$ 's are equal, then they equal zero. Otherwise, we may assume without loss of generality that  $r_4 \neq r_3$ . Putting  $r_4 = -(r_1 + r_2 + r_3)$  into the second equation yields

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 + r_1r_2 + r_1r_3 + r_2r_3 &= 0 \\ \implies r_3 &= -\frac{r_1 + r_2}{2} \pm \frac{\sqrt{-3r_1^2 - 2r_1r_2 - 3r_2^2}}{2} \end{aligned}$$

Choose  $r_1, r_2 \in K$ . We see that  $r_3 \in K$  (and so  $r_4 \in K$ ) if and only if

$$-3r_1^2 - 2r_1r_2 - 3r_2^2 = -2(r_1 + r_2)^2 - (r_1 - r_2)^2 = w^2$$

is a non-zero square in  $K$ . (If  $w = 0$ , then  $r_1 + r_2 + 2r_3 = 0$ , so  $r_3 = r_4$ .) Let  $(X, Y, Z) = (w, r_1 - r_2, r_1 + r_2)$  be an ordered triple of elements in  $K$ . As in the

proof of Theorem 2.8, we have

$$-2Z^2 - Y^2 = X^2 \implies \left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right)^2 = -2 \implies \left(\frac{X+Y}{2Z}\right)^2 + \left(\frac{X-Y}{2Z}\right)^2 = -1.$$

Thus, if  $L_K(f) = 4$ , then  $s(K) \leq 2$ . The converse is almost immediate.

If (4.2) has repeated  $r_i$ 's, we may assume without loss of generality that  $r_1 = r_2$ , hence  $r_3, r_4 = r_1(-1 \pm \sqrt{-2})$ . The only fields in which this solution might occur contain  $\sqrt{-2}$ , so if we can find an alternate solution to (4.2) in  $\mathbb{Q}(\sqrt{-2})$ , we will be done. It may be checked that

$$\{r_1, r_2\} = \{5\sqrt{-2} \pm 6\}, \quad \{r_3, r_4\} = \{-5\sqrt{-2} \pm 8\}$$

is such an alternate solution to (4.2) with distinct  $r_i$ . □

Our final result presents another sextic with three different ranks.

**Theorem 4.2.** *Suppose  $f(x, y) = \binom{6}{1}x^5y - \binom{6}{3}x^3y^3 = 2x^3y(3x^2 - 10y^2)$ . Then  $L_K(f) = 4$  if and only if  $s(K) \leq 2$ . In particular, if  $m$  is a positive square-free integer, and  $m \not\equiv 7 \pmod{8}$ , then  $L_{\mathbb{Q}(\sqrt{-m})}(f) = 4$ . Further,  $L_{\mathbb{Q}(\sqrt{-7})}(f) = 5$  and  $L_{\mathbb{R}}(f) = 6$ .*

*Proof.* Again, taking (2.4) for  $r = 3$  gives a non-singular matrix

$$(4.3) \quad \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

so  $L_{\mathbb{C}}(f) > 3$ . Moving up one,

$$(4.4) \quad \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_0 = c_2, \quad c_1 = c_3 = 0,$$

so the possible Sylvester polynomials over  $K$  have the shape  $h(x, y) = c_0x^4 + c_0x^2y^2 + c_4y^4$ . If  $c_0c_4 = 0$ , then  $h$  is not square-free, so we may scale to  $c_0 = 1$ . Since  $h$  is an even polynomial, if  $x - ry$  is a factor with  $r \neq 0$  (since  $c_4 \neq 0$ ), then so is  $x + ry$ , hence if  $h$  splits over  $K$ , then there exist  $r, s \in K$  ( $r^2 \neq s^2 \neq 0$ ) so that

$$x^4 + x^2y^2 + c_4y^4 = (x^2 - r^2y^2)(x^2 - s^2y^2).$$

Thus,  $L_K(f) = 4$  if and only if  $K$  is a field in which the equation

$$(4.5) \quad r^2 + s^2 = -1$$

has a solution,  $r^2 \neq -\frac{1}{2}, 0, -1$ . As we have seen in the proof of Theorem 2.8, this is true precisely when  $s(K) \leq 2$ , so if  $K = \mathbb{Q}(\sqrt{-m})$ , precisely when  $m \not\equiv 7 \pmod{8}$ .

Since  $f$  is hyperbolic,  $L_{\mathbb{R}}(f) = 6$ . The previous paragraph shows that the apolar ideal for  $f$  is generated by  $x^4 + x^2y^2$  and  $y^4$ . We now wish to find at least one field  $K$  for which  $L_K(f) = 5$ . Since  $K$  must be non-real with  $s(K) > 2$ , we take  $K = \mathbb{Q}(\sqrt{-7})$  and look for a representation with relative rank 5. To this end, observe that  $y(x^4 + x^2y^2) - 2xy^4 = x^4y + x^2y^3 - 2xy^4 = xy(x - y) \left(x + \frac{1+\sqrt{-7}}{2}y\right) \left(x + \frac{1-\sqrt{-7}}{2}y\right)$  splits over  $\mathbb{Q}(\sqrt{-7})$ . □



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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801

*E-mail address:* reznick@illinois.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801

*Current address:* Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043

*E-mail address:* tokcan2@illinois.edu