Portugal. Math. (N.S.)
Vol. 78, Fasc. 2, 2021, 129–146
DOI 10.4171/PM/2063

Published by EMS Press

Sparse versions of the Cayley–Bacharach theorem

Laura Felicia Matusevich and Bruce Reznick

Abstract. We give combinatorial generalizations of the Cayley–Bacharach theorem and induced map.

Mathematics Subject Classification (primary; secondary): 51N35, 52B20; 14N10, 13P15
Keywords: Cayley–Bacharach, resultants

1. Introduction

The *Cayley–Bacharach theorem* states that, given two cubic curves in the projective plane that meet in nine points, any other cubic that passes through eight of the points, contains also the ninth. As with many attributions in mathematics, it is known that the Cayley–Bacharach theorem is originally due to Chasles; the article [EGH96] contains a thorough historical account of this result, including the roles of Cayley and Bacharach, as well as many geometric generalizations.

The Cayley–Bacharach theorem provides a map assigning a point in the plane to eight given points. More precisely, given eight points in the affine plane $\mathbb{C}^2$, the set of cubic polynomials in two variables that vanish at these points is a two-dimensional vector space, at least if no three of the points are on a line, and no six lie on a conic. If $F(x,y)$, $G(x,y)$ form a basis of this vector space, Bézout's theorem implies that F and G have nine common zeros (in the projective plane) counting multiplicity. For most choices of eight points, all the multiplicities equal 1, so a ninth point is determined. Again, for most choices of eight points, this ninth point lies in the affine plane. This is a natural map from a dense subset of $(\mathbb{C}^2)^8$ to $\mathbb{C}^2$. We call this the *extra point map*, Υ.

The map Υ is rational, and explicit formulas for it can be found in the article [RRS15]. We give a different proof of rationality, as well as alternative formulas, in Section 2. While our formulas are less beautiful and more complicated than those in [RRS15], and no one in their right mind would use them in a practical context, they do have one very positive attribute: they can be naturally generalized.

The generalizations we are interested in are of the following form:

> A choice of N generic points in $\mathbb{C}^d$ gives rise to d hypersurfaces satisfying given constraints, and it turns out that those hypersurfaces meet in exactly $N+1$ points.

The constraints we consider are combinatorial in nature: we fix the monomials that appear in the defining equations of the corresponding hypersurfaces. These support sets must be carefully chosen; in Section 5, we call them *Chasles configurations* and *Chasles structures*. While Chasles' work has received significant recognition (his name is on the Eiffel Tower), we thought it appropriate to name our generalizations in his honor.

Our main result, Theorem 5.3, states that the extra point map arising in this more general situation is still rational. The proof is essentially the same as our proof that Υ is rational (see Theorem 2.1), and consequently also produces explicit formulas.

The key ingredient we use to compute Υ and its generalizations is the notion of *resultant*. The well-known resultant of two univariate polynomials f and g is a polynomial in the coefficients of f and g that vanishes precisely when f and g have a common factor. Resultants are a very important tool when solving polynomial equations, due to their fundamental role in elimination theory. This has spurred much interest in resultants, and especially in explicit formulas for resultants. We make use of the following fact (that is made precise in Theorem 4.11):

> The product of the coordinates of the roots of a system of polynomial equations is a rational function of the coefficients of the system, that can be expressed in terms of resultants.

This result has been known since the late 1990s; see [K97, CDS98, R97] and also [PS93]. (In this article, we use the version from [DS15].) Its relevance is that it allows us to express the coordinates of the point we are interested in, in terms of the coordinates of the points we are given and the coefficients of the polynomials that define the hypersurfaces containing those points. Those coefficients can also be expressed as rational functions of the coordinates of the given points, since we have fixed the monomials that appear in those polynomials.

We remark that, while we use combinatorial language in this article, our results can be recast in geometric terms. If X is an algebraic variety, L is a line bundle, and N is an integer, the triple (X, L, N) is said to satisfy a Cayley–Bacharach theorem if a set S of N generic points gives rise to a linear series

$V \subseteq H^0(X, L)$ of sections vanishing at S whose common zeros are a set of exactly $N+1$ points. In this work, we concentrate on the case when X is a projective toric variety (Chasles configurations) or more generally when several toric line bundles are involved (Chasles structures). We are very grateful to the anonymous referee who provided us with this formulation.

Outline. In Section 2, we prove that Υ is rational by giving an explicit formula in terms of Sylvester resultants. Section 3 explains our motivation for considering Υ in the first place. Section 4 is a technical section containing results needed to generalize the Cayley–Bacharach theorem. The paper becomes readable again in Section 5, where we introduce our generalizations, and Section 6 contains infinitely many examples.

2. The extra point map is rational

We start by showing that the map arising from the Cayley–Bacharach theorem is rational.

Theorem 2.1. *Let Υ be the map that assigns, to eight generic points in the plane, a ninth point determined by the Cayley–Bacharach theorem. The map Υ is rational.*

Proof. We show that Υ is rational by giving an explicit formula. We denote the eight given points by $\rho_i = (a_i, b_i) \in (\mathbb{C}^*)^2$, $1 \le i \le 8$. By the genericity assumption, we may assume that two linearly independent cubic polynomials vanishing on $\rho_1, \dots, \rho_8$ are of the form

$$F(x,y) = x^3 + \kappa_{21}x^2y + \kappa_{12}xy^2 + \kappa_{20}x^2 + \kappa_{02}y^2 + \kappa_{11}xy + \kappa_{10}x + \kappa_{01}y + \kappa_{00},$$
$$G(x,y) = y^3 + \lambda_{21}x^2y + \lambda_{12}xy^2 + \lambda_{20}x^2 + \lambda_{02}y^2 + \lambda_{11}xy + \lambda_{10}x + \lambda_{01}y + \lambda_{00}.$$

Let M be the matrix whose rows are $[a_i^2b_i, a_ib_i^2, a_i^2, b_i^2, a_ib_i, a_i, b_i, 1]$ for $i = 1, \dots, 8$. Then $[\kappa_{21}, \kappa_{12}, \kappa_{20}, \kappa_{02}, \kappa_{11}, \kappa_{10}, \kappa_{01}, \kappa_{00}]^t$ is a solution of

$$Mv = [-a_1^3, \dots, -a_8^3]^t,$$

and $[\lambda_{21}, \lambda_{12}, \lambda_{20}, \lambda_{02}, \lambda_{11}, \lambda_{10}, \lambda_{01}, \lambda_{00}]^t$ is a solution of

$$Mv = [-b_1^3, \dots, -b_8^3]^t.$$

Again, as $\rho_1, \dots, \rho_8$ are generic, we may assume that $\det(M) \neq 0$, and consequently we may explicitly write the coefficients κ_{ij} and λ_{ij} as ratios of polynomials in a_i, b_j using Cramer's rule. For instance,

$$\kappa_{00} = \frac{\det \begin{bmatrix} a_1^2 b_1 & a_1 b_1^2 & a_1^2 & b_1^2 & a_1 b_1 & a_1 & b_1 & -a_1^3 \\ a_2^2 b_2 & a_2 b_2^2 & a_2^2 & b_2^2 & a_2 b_2 & a_2 & b_2 & -a_2^3 \\ a_3^2 b_3 & a_3 b_3^2 & a_3^2 & b_3^2 & a_3 b_3 & a_3 & b_3 & -a_3^3 \\ a_4^2 b_4 & a_4 b_4^2 & a_4^2 & b_4^2 & a_4 b_4 & a_4 & b_4 & -a_4^3 \\ a_5^2 b_5 & a_5 b_5^2 & a_5^2 & b_5^2 & a_5 b_5 & a_5 & b_5 & -a_5^3 \\ a_6^2 b_6 & a_6 b_6^2 & a_6^2 & b_6^2 & a_6 b_6 & a_6 & b_6 & -a_6^3 \\ a_7^2 b_7 & a_7 b_7^2 & a_7^2 & b_7^2 & a_7 b_7 & a_7 & b_7 & -a_7^3 \\ a_8^2 b_8 & a_8 b_8^2 & a_8^2 & b_8^2 & a_8 b_8 & a_8 & b_8 & -a_8^3 \end{bmatrix}}{\det M}.$$

Now consider F and G as polynomials in the variable x, with coefficients that are polynomials in y, and take the resultant to eliminate the variable x. The roots of this resultant (as a polynomial in y) are the y-coordinates of the nine solutions of $F = G = 0$. Consequently, the coefficient of y^0 in this resultant is the product of those nine y-coordinates. By taking resultant with respect to y, we can also obtain the product of the nine x-coordinates of the solutions of $F = G = 0$.

The above calculation can be performed explicitly using a computer algebra system. We used Macaulay2 [M2] to compute the coefficients we are interested in, which are given below.

The zeroth coefficient for the resultant of F and G with respect to y is:

$$\begin{aligned} R_x = {} & \kappa_{02}^3\lambda_{00}^2 - \kappa_{02}^2\kappa_{01}\lambda_{01}\lambda_{00} + \kappa_{02}\kappa_{01}^2\lambda_{02}\lambda_{00} + \kappa_{02}^2\kappa_{00}\lambda_{01}^2 - 2\kappa_{02}^2\kappa_{00}\lambda_{02}\lambda_{00} \\ & - \kappa_{02}\kappa_{01}\kappa_{00}\lambda_{02}\lambda_{01} + \kappa_{02}\kappa_{00}^2\lambda_{02}^2 - \kappa_{01}^3\lambda_{00} + 3\kappa_{02}\kappa_{01}\kappa_{00}\lambda_{00} + \kappa_{01}^2\kappa_{00}\lambda_{01} \\ & - 2\kappa_{02}\kappa_{00}^2\lambda_{01} - \kappa_{01}\kappa_{00}^2\lambda_{02} + \kappa_{00}^3. \end{aligned}$$

The zeroth coefficient for the resultant of F and G with respect to x (exchange λ's and κ's) is:

$$\begin{aligned} R_y = {} & \kappa_{00}^2\lambda_{20}^3 - \kappa_{10}\kappa_{00}\lambda_{20}^2\lambda_{10} + \kappa_{20}\kappa_{00}\lambda_{20}\lambda_{10}^2 + \kappa_{10}^2\lambda_{20}^2\lambda_{00} - 2\kappa_{20}\kappa_{00}\lambda_{20}^2\lambda_{00} \\ & - \kappa_{20}\kappa_{10}\lambda_{20}\lambda_{10}\lambda_{00} + \kappa_{20}^2\lambda_{20}\lambda_{00}^2 - \kappa_{00}\lambda_{10}^3 + 3\kappa_{00}\lambda_{20}\lambda_{10}\lambda_{00} + \kappa_{10}\lambda_{10}^2\lambda_{00} \\ & - 2\kappa_{10}\lambda_{20}\lambda_{00}^2 - \kappa_{20}\lambda_{10}\lambda_{00}^2 + \lambda_{00}^3. \end{aligned}$$

Then our ninth point, $\rho = (a, b) = \Upsilon(\rho_1, \ldots, \rho_8)$, can be expressed as

$$a = \frac{R_x}{a_1 \cdots a_8}, \qquad b = \frac{R_y}{b_1 \cdots b_8}. \qquad \square$$

While the above formulas involve much division and high degree polynomials, their most significant feature, as has been mentioned before, is that they can be generalized. But this must wait until Section 5.

3. Hilbert

This version of the Cayley–Bacharach theorem was central to Hilbert's 1888 proof that there exist positive semidefinite (psd) ternary sextics which cannot be written as a sum of squares (sos) of real ternary cubics (see [H1888]). Hilbert starts with two real cubic polynomials $F(x, y)$ and $G(x, y)$ which have nine common zeros – $\{\rho_j \mid 1 \le j \le 9\}$, no three on a line, no six on a conic. He shows how to construct a sextic H which is singular at $\{\rho_j \mid 1 \le j \le 8\}$ but so that $H(\rho_9) > 0$. By looking separately at the neighborhoods of the ρ_j's and their complement, Hilbert shows that there exists $\lambda > 0$ so that $f = F^2 + G^2 + \lambda H$ is psd; observe that $f(\rho_j) = 0$ for $1 \le j \le 8$ and $f(\rho_9) > 0$. Suppose now that $f = \sum g_k^2$ for cubics g_k. Then $g_k(\rho_j) = 0$ for $1 \le j \le 8$, and Cayley–Bacharach implies that $g_k(\rho_9) = 0$ for all g_k, a contradiction, which means that f is not sos. The condition on F, G ensures that they (and H) cannot be particularly simple, and no explicit example was constructed in the subsequent eighty years.

The first example of a psd ternary sextic which is not sos was constructed by Motzkin in 1967, and does not follow Hilbert's methodology:

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

In 1969, R. M. Robinson (see [R69]) showed that Hilbert's construction works with a simple pair $\{F, G\}$ which do not satisfy his restriction. He took $F(x, y) = x^3 - x$ and $G(x, y) = y^3 - y$, so that the ρ_j's form the 3×3 grid: $\{-1, 0, 1\}^2$. He then showed (in our notation) that $H(x, y) = (1 - x^2)(1 - y^2)(1 - x^2 - y^2)$ fulfills the conditions of Hilbert's construction and that one may even take $\lambda = 1$. The resulting polynomial homogenizes to an even symmetric ternary sextic form:

$$\begin{aligned} R(x, y, z) &:= (x^3 - xz^2)^2 + (y^3 - yz^2)^2 + (x^2 - z^2)(y^2 - z^2)(z^2 - x^2 - y^2) \\ &= x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2. \end{aligned}$$

Robinson proves that this form is psd, by writing $(x^2 + y^2)R(x, y, z)$ as a sum of squares. Hilbert's argument shows that R itself is not sos. The original eight zeros homogenize to $\{(\pm 1, \pm 1, 1), (\pm 1, 0, 1), (0, \pm 1, 1)\}$ and R itself has two additional zeros "at infinity" $(1, \pm 1, 0)$. The article [R00] contains further historical discussion on psd and sos forms.

To be specific, suppose we take as our "eight points" $\{(1, 0, \pm t), (0, 1, \pm t), (1, \pm 1, \pm 1)\}$, where t is a real parameter. (The Robinson example corresponds to $t = 1$.) The second author [R07] proved that if $t^2 < \frac{1}{2}$, then one can obtain a psd ternary sextic M_t which is not sos with two additional zeros at "at infinity": $(1, 0, 0)$ and $(0, 1, 0)$. Further, as $t \to 0$, the coefficients of M_t converge to those of the Motzkin example.

Choi, Lam and the second author [CLR87] used Robinson's example as a starting point to analyze all psd even symmetric sextics in $n \geq 3$. The second author [R07] generalized Robinson's example and showed that Hilbert's construction applies, as long as no four of the common zeros $\{\rho_j\}$ are on a line, and no seven are on a conic. That paper also contains many worked-out examples.

4. Sparse polynomial systems and sparse resultants

In this section, we collect results on sparse systems of polynomial equations that are necessary for our generalizations of the Cayley–Bacharach theorem.

A *configuration of lattice points*, or a *configuration* is a finite subset A of $\mathbb{Z}^d$. The *dimension* of A, denoted by $\dim(A)$, is the dimension of the smallest affine subspace of $\mathbb{R}^d$ containing all points in A.

If A is a configuration, $\mathrm{conv}(A)$ denotes the convex hull of the elements of A in $\mathbb{R}^d$; $\mathrm{conv}(A)$ is a *convex lattice polytope* (a convex polytope whose vertices have integer coordinates).

A configuration A is called *saturated* if $A = \mathrm{conv}(A) \cap \mathbb{Z}^d$, that is, if A equals the set of all lattice points in its convex hull.

If $A \subset \mathbb{Z}^d$ is a d-dimensional configuration, its *normalized volume*, denoted $\mathrm{vol}(A)$, is the Euclidean volume of $\mathrm{conv}(A)$, renormalized so that the unit simplex in $\mathbb{Z}^d$ has volume one. More explicitly, the $\mathrm{vol}(A)$ is $d!$ times the Euclidean volume of $\mathrm{conv}(A)$.

A Laurent polynomial is *supported on a configuration* A if it is of the form $\sum_{u \in A} \lambda_u x^u \in \mathbb{C}[x_1^{\pm}, \dots, x_d^{\pm}]$.

The following result, due to Kouchnirenko [K76], illustrates the connection between the combinatorics of configurations and systems of polynomial equations.

Theorem 4.1. *Let $A \subset \mathbb{Z}^d$ be a d-dimensional configuration, and let $F_1, \dots, F_d$ be generic Laurent polynomials supported on A. Then the number of common roots of $F_1, \dots, F_d$ (in $(\mathbb{C} \backslash \{0\})^d := (\mathbb{C}^*)^d$) is* $\mathrm{vol}(A)$.

Our next goal is to give the number of solutions to a system of Laurent polynomial equations, when the supports of the equations are not necessarily the same. A note on terminology: when we refer to *sparse systems of equations*, we mean a system of Laurent polynomial equations whose supports have been fixed. We start by introducing a generalization of the notion of volume.

Definition 4.2. If $P, Q \subset \mathbb{R}^d$ are polytopes, their *Minkowski sum* is $P + Q = \{u + v \mid u \in P, v \in Q\}$. Let $A_1, \ldots, A_d$ be configurations, and denote $P_i = \operatorname{conv}(A_i)$. Then the *mixed volume* of $P_1, \ldots, P_d$ is

$$\operatorname{mvol}(P_1, \ldots, P_d) := \frac{1}{d!}\sum_{k=1}^{d}(-1)^{d-k} \sum_{1 \le i_1 < \cdots < i_k \le d} \operatorname{vol}(P_{i_1} + \cdots + P_{i_k}).$$

The following result is known as the Bernstein, Kouchnirenko and Khovanskii (or BKK) theorem. In this form, it first appeared in [B75].

Theorem 4.3. *Let $A_1, \ldots, A_d \subset \mathbb{Z}^d$ be configurations, and denote $P_i = \operatorname{conv}(A_i)$. For $1 \le i \le d$, let $F_i = \sum_{a \in A_i} \lambda_{i,a} x^a$ be a Laurent polynomial with support contained in A_i. If the coefficients $\{\lambda_{i,a} \mid a \in A_i, 1 \le i \le d\}$ are sufficiently generic, the system of polynomial equations $F_1(x) = \cdots = F_d(x) = 0$ has precisely $\operatorname{mvol}(P_1, \ldots, P_d)$ solutions in $(\mathbb{C}^*)^d$.*

We now turn to sparse resultants. While a system of d generic Laurent polynomial equations in d variables has solutions (see Theorem 4.3), a system of $d+1$ Laurent polynomials in d variables in general does not. The coefficients of the polynomials in such a system for which solutions exist are determined by a polynomial called the *resultant*.

As was mentioned in the introduction, resultants can be used to give an expression for the product (of the coordinates) of the roots of a sparse system. The earliest versions of this can be found in [K97, CDS98, R97, PS93]. In this article we use the formulas that appear in [DS15].

Our first task is to introduce resultants in general.

Definition 4.4. Let $A_0, A_1, \ldots, A_d$ be finite subsets of $\mathbb{Z}^d$, and let $\mathscr{A} = (A_0, \ldots, A_d)$. Write $F_i = F_i(\lambda_i, x) = \sum_{u \in A_i} \lambda_{i,u} x^u$ for a Laurent polynomial supported on A_i, where $\lambda_i = (\lambda_{i,u} \mid u \in A_i)$ are variables representing the coefficients of F_i. Set $\lambda = (\lambda_0, \ldots, \lambda_d)$. Let

$$\Omega_{\mathscr{A}} = \{(x, \lambda) \mid F_0(\lambda_0, x) = \cdots = F_d(\lambda_d, x) = 0\} \subset (\mathbb{C}^*)^d \times \prod_{i=0}^{d} \mathbb{P}^{|A_i|-1}.$$

If the closure of the image of $\Omega_{\mathscr{A}}$ under the projection $(\mathbb{C}^*)^d \times \prod_{i=0}^{d} \mathbb{P}^{|A_i|-1} \to \prod_{i=0}^{d} \mathbb{P}^{|A_i|-1}$ has codimension 1, then the *resultant* $\operatorname{Res}_{\mathscr{A}}(F_0, \ldots, F_d)$ is defined to be the unique (up to sign) irreducible polynomial in $\mathbb{Z}[\lambda]$ which vanishes on this hypersurface. If this closure has codimension at least 2, then we define $\operatorname{Res}_{\mathscr{A}}(F_0, \ldots, F_d) = 1$.

Example 4.5. In our definitions, we have used the lattice $\mathbb{Z}^d$ as the ambient lattice without remarking upon it. In general, we may be given configurations that naturally live in lattices other than $\mathbb{Z}^d$, in which case, we need to change the way we compute resultants accordingly.

For instance, consider $A = \{(0,0,0),(0,1,0),(-1,0,2)\} \subset \mathbb{Z}^3$. Then A generates a lattice L which is isomorphic to $\mathbb{Z}^2$. If $F_i = \lambda_{i,1} + \lambda_{i,2}y + \lambda_{i,3}x^{-1}z^2$ are generic Laurent polynomials supported on A for $i = 1,2,3$, the system $F_0(x,y,z) = F_1(x,y,z) = F_2(x,y,z) = 0$ has solutions in $(\mathbb{C}^*)^3$. On the other hand, we can also consider this as a system of 3 equations in the two variables (s,t) induced by the lattice L. This new system is $\lambda_{i,1} + \lambda_{i,2}s + \lambda_{i,3}t = 0$ for $i = 1,2,3$, whose solvability depends on the vanishing of the determinant of the 3×3 matrix $[\lambda_{i,j}]$. Using the convention that the ambient lattice is L, this determinant is the resultant of our system.

In order to use the formulas in [DS15], what is needed is a power $\operatorname{Res}_{\mathscr{A}}(F_0,\ldots,F_d)^{\mu_{\mathscr{A}}}$. The exponent $\mu_{\mathscr{A}}$ can be given a geometric or combinatorial definition; we use the combinatorial definition of $\mu_{\mathscr{A}}$ that can be found in [E10, Section 2.1].

We need a preliminary result. If $J \subset \{0,\ldots,d\}$, set $A_J = \bigcup_{j\in J} A_j$. The following is [S94, Corollary 1.1].

Proposition 4.6. *With the notation of Definition* 4.4, $\operatorname{Res}_{\mathscr{A}}(F_0,\ldots,F_d) \neq 1$ *if and only if there exists a unique subset* $J_0 \subset \{0,\ldots,d\}$ *such that*

(1) $\dim\big(\operatorname{conv}(A_{J_0})\big) - |J_0| = -1$, *and*

(2) *for* $J \subsetneq J_0$, $\dim\big(\operatorname{conv}(A_J)\big) - |J| > -1$.

Definition 4.7. Let $\mathscr{A}$ as in Definition 4.4. Assuming that $\operatorname{Res}_{\mathscr{A}}(F_0,\ldots,F_d) \neq 1$, let $J_0 \subseteq \{0,\ldots,d\}$ be as in Proposition 4.6, and consider $A_{J_0} \times \{1\} \subset \mathbb{Z}^{d+1}$. Let

$$\pi : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d+1}/\operatorname{span}_{\mathbb{R}}(A_{J_0} \times \{1\})$$

be the projection, and let η be a volume form on $\mathbb{R}^{d+1}/\operatorname{span}_{\mathbb{R}}(A_{J_0} \times \{1\})$ such that the volume of $\mathbb{R}^{d+1}/\big(\operatorname{span}_{\mathbb{R}}(A_{J_0} \times \{1\}) + \mathbb{Z}^{d+1}\big)$ with respect to η equals $(d+1-|J_0|)!$. We set

$$\begin{aligned}\mu_{\mathscr{A}} := &\left|\frac{\operatorname{span}_{\mathbb{R}}(A_{J_0} \times \{1\}) \cap \mathbb{Z}^{d+1}}{\operatorname{span}_{\mathbb{Z}}(A_{J_0} \times \{1\})}\right| \\ &\cdot \operatorname{mvol}_\eta\big(\big\{\operatorname{conv}\big(\pi(A_i \times \{1\})\big) \mid i \in \{0,\ldots,d\}\backslash J_0\big\}\big), \end{aligned} \tag{4.1}$$

where mvol_η denotes the mixed volume with respect to the volume form η.

There is one case where $\mu_{\mathscr{A}}$ is easy to compute.

Lemma 4.8. *Use the notation of Definitions* 4.4 *and* 4.7. *If each of the sets* $A_0, \dots, A_d$ *spans* $\mathbb{Z}^d$ *as a lattice, then* $\mu_{\mathscr{A}} = 1$.

Proof. In this case, $J_0 = \{0, \dots, d\}$. We have also $\operatorname{span}_{\mathbb{R}}(A_{J_0} \times \{1\}) \cap \mathbb{Z}^{d+1} = \operatorname{span}_{\mathbb{Z}}(A_{J_0} \times \{1\})$, so that the first factor in (4.1) equals 1, and the second factor does not appear, since $\{0, \dots, d\} \backslash J_0 = \emptyset$. □

Definition 4.9. Using the notation from Definitions 4.4 and 4.7, we set

$$\mathrm{mRes}_{\mathscr{A}}(F_0, \dots, F_d) := \mathrm{Res}_{\mathscr{A}}(F_0, \dots, F_d)^{\mu_{\mathscr{A}}}. \tag{4.2}$$

Remark 4.10. In [E10, DS15], the word *resultant*, and its corresponding notation, is used for the polynomial $\mathrm{mRes}_{\mathscr{A}}(\lambda)$. In this article, we follow the usual convention that resultants are irreducible polynomials.

Our goal is to state a formula from [DS15] that gives the product of the coordinates of the solutions of a sparse system of equations in terms of resultants. To do this, we need to introduce *directional resultants*.

Let $A_1, \dots, A_d$ be finite subsets of $\mathbb{Z}^d$, and let $F_1, \dots, F_d$ be Laurent polynomials such that the support of $F_i = \sum_{u \in A_i} \lambda_{i,u} x^u$ is (contained in) A_i. Let $v \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z}) \cong \mathbb{Z}^d$. The *weight* of $u \in \mathbb{Z}^d$ with respect to v, denoted $v \cdot u$, is the image of u under v. For $1 \le i \le d$, let $A_{i,v}$ be the set of elements of A_i with minimal weight with respect to v, and let $F_{i,v} = \sum_{u \in A_{i,v}} \lambda_{i,u} x^u$ be the restriction of F_i to $A_{i,v}$. Note that $A_{i,v}$ is contained in a translate of the lattice $v^\perp = \{u \in \mathbb{Z}^d \mid v \cdot u = 0\} \cong \mathbb{Z}^{d-1}$. Choose $\beta_{1,v}, \dots, \beta_{d,v} \in \mathbb{Z}^d$ such that $A_{i,v} - \beta_{i,v} := \{u - \beta_{i,v} \mid u \in A_{i,v}\} \subset v^\perp$, and let $G_{i,v} = \sum_{u \in A_i} \lambda_{i,u} x^{u - \beta_{i,v}}$. We denote

$$\mathrm{mRes}_{\mathscr{A},v}(F_{1,v}, \dots, F_{d,v}) := \mathrm{mRes}_{\{A_{1,v}, \dots, A_{d,v}\}}(G_{1,v}, \dots, G_{d,v}). \tag{4.3}$$

In the expression above, the resultant on the right hand side is constructed with respect to the ambient lattice $v^\perp$ as in Example 4.5. We have constructed a polynomial in the coefficients of the $F_{i,v}$, which is called a *directional resultant*. We note that it is independent of the choice of $\beta_{1,v}, \dots, \beta_{d,v}$.

We are now ready to state the main result of this section, which is a special case of [DS15, Corollary 1.3].

Theorem 4.11. *Let* $A_1, \dots, A_d \subset \mathbb{Z}^d$ *be configurations, and let* $F_1, \dots, F_d$ *be Laurent polynomials supported on* $A_1, \dots, A_d$. *Denote by* $\mathscr{Z}(F_1, \dots, F_d)$ *the set of solutions of* $F_1(x) = \cdots = F_d(x) = 0$ *in* $(\mathbb{C}^*)^d$. *If* $\rho \in \mathscr{Z}(F_1, \dots, F_d)$, *let* m_ρ *be its multiplicity as a solution of* $F_1(x) = \cdots = F_d(x) = 0$. *Assume that for all* $v \ne 0$, $v \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$, *we have* $\mathrm{mRes}_{\mathscr{A},v}(F_{1,v}, \dots, F_{d,v}) \ne 0$. *For* $1 \le i \le d$, *we have that*

$$\prod_{\rho \in \mathscr{Z}(F_1,\dots,F_d)} \rho_i^{m_\rho} = \pm \prod \mathrm{mRes}_{\mathscr{A},v}(F_{1,v},\dots,F_{d,v})^{v\cdot e_i}, \tag{4.4}$$

where the product on the right is over the primitive vectors in $\mathrm{Hom}(\mathbb{Z}^d,\mathbb{Z})$*, and* $e_1,\dots,e_d$ *are the standard unit vectors in* $\mathbb{Z}^d$.

We note that by Proposition 4.6, if v is not an inner normal of a codimension 1 face of $\mathrm{conv}(A_1)+\cdots+\mathrm{conv}(A_d)$, then $\mathrm{mRes}_{\mathscr{A},v}(F_{1,v},\dots,F_{d,v})=1$. This implies that the product on the right hand side of (4.4) has finitely many factors different from 1.

The $\pm$ sign in (4.4) is necessary, since resultants are only defined up to sign.

Remark 4.12. The assumption in Theorem 4.11, that the directional resultants do not vanish, is a genericity assumption. It states that we are working with a system of Laurent polynomial equations such that none of the facial subsystems have a common root.

5. Chasles configurations and structures

In this section, we give combinatorial generalizations for the Cayley–Bacharach theorem.

Definition 5.1. Let $A \subset \mathbb{Z}^d$ be a d-dimensional configuration, and write $|A|$ for the cardinality of A. Then A is a *Chasles configuration* if $|A|+1=\mathrm{vol}(A)+d$. A Chasles configuration $A \subset \mathbb{Z}^d$ is *saturated* if $A=\mathrm{conv}(A)\cap\mathbb{Z}^d$.

If A is a Chasles configuration, let $N=\mathrm{vol}(A)-1$, so that $|A|=N+d$. Fix N generic points in $\mathbb{C}^d$. The Laurent polynomials supported on A that vanish on these N points form a d-dimensional vector space. If $F_1,\dots,F_d$ is a basis for this vector space, then by Theorem 4.1, the number of common zeros of $F_1,\dots,F_d$ is $N+1$. This determines a map Υ_A from an open subset of $(\mathbb{C}^d)^N$ to $\mathbb{C}^d$.

Definition 5.2. More generally, a *Chasles structure* consists of: two positive integers N and d, a partition $d=k_1+\cdots+k_\ell$, and configurations $A_1,\dots,A_\ell \subseteq \mathbb{Z}^d$ such that $|A_i|=N+k_i$, and

$$\mathrm{mvol}(\underbrace{A_1,\dots,A_1}_{k_1 \text{ times}},\underbrace{A_2,\dots,A_2}_{k_2 \text{ times}},\dots,\underbrace{A_\ell,\dots,A_\ell}_{k_\ell \text{ times}})=N+1.$$

We denote $\mathscr{A}=\{A_1,\dots,A_\ell\}$. We sometimes abuse notation and call $\mathscr{A}$ a Chasles structure.

Note that a Chasles configuration is a Chasles structure for $N = |A| - \dim(A)$, $d = \dim(A)$, with partition $d = k_1$.

We now come to the main result in this article.

Theorem 5.3. *A Chasles structure $\mathscr{A}$ induces a rational map* $\Upsilon_{\mathscr{A}} : [(\mathbb{C}^*)^d]^N \dashrightarrow (\mathbb{C}^*)^d$.

Proof. A Chasles structure is set up so that, if we fix N general points $\rho_1, \dots, \rho_N \in (\mathbb{C}^*)^d$, then for each $i = 1, \dots, \ell$, those points determine a k_i-dimensional vector space of Laurent polynomials supported on A_i that vanish on them. Picking a basis of each vector space, we obtain $k_1 + \cdots + k_\ell = d$ polynomials $F_1, \dots, F_d$, whose coefficients can be expressed as rational functions on the coordinates of $\rho_1, \dots, \rho_N$.

Since the mixed volume of the corresponding Newton polytopes equals $N + 1$, the Laurent polynomials $F_1, \dots, F_d$ have $N + 1$ common zeros in $(\mathbb{C}^*)^d$ by Theorem 4.3. Let ρ_{N+1} be the point determined in this way. Note that the genericity assumption on $\rho_1, \dots, \rho_N$ implies that $\rho_{N+1} \notin \{\rho_1, \dots, \rho_N\}$.

On the other hand, for fixed $1 \le i \le d$, the product of the ith coordinates of $\rho_1, \dots, \rho_{N+1}$ can be expressed as a rational function on the coefficients of $F_1, \dots, F_d$ via (4.4).

It follows that the ith coordinate of ρ_{N+1} is a rational function of the coordinates of $\rho_1, \dots, \rho_N$. $\square$

We emphasize that, because of the explicit nature of (4.4), the proof of Theorem 5.3 can be used to provide an explicit expression for the map $\Upsilon_{\mathscr{A}}$. This is illustrated in examples in the following section.

6. Examples

6.1. Example: The Cayley–Bacharach theorem. In this case A is the set of lattice points in the triangle in $\mathbb{R}^2$ with vertices $(0,0), (3,0), (0,3)$ (affine or inhomogeneous version) or the set of lattice points in the triangle in $\mathbb{R}^3$ with vertices $(3,0,0), (0,3,0), (0,0,3)$. In either case, A consists of 10 points, $\dim(A) = 2$ and $\mathrm{vol}(A) = 9$, so A is a Chasles configuration.

6.2. Example: Triangle with one interior point. Let $A = \{(0,0), (1,1), (2,1), (1,2)\}$. This is a saturated Chasles configuration, with $\mathrm{vol}(A) = 3$. Given 2 generic points $\rho_1 = (a_1, b_1)$ and $\rho_2 = (a_2, b_2)$, which determine a two-dimensional space of polynomials supported on A that vanish on ρ_1, ρ_2. Denote (a_3, b_3) their third common zero. We pick the following basis of this vector space:

$$F = a_1a_2b_1b_2(a_1b_2 - a_2b_1) + (a_1b_1^2 - a_2b_2^2)x^2y + (a_2^2b_2 - a_1^2b_1)xy^2,$$
$$G = (b_1 - b_2)x^2y + (a_2 - a_1)xy^2 + (a_1b_2 - a_2b_1)xy.$$

In this case, the directional resultants are 2×2 determinants of the coefficients of F and G corresponding to the facets of $\mathrm{conv}(A)$. For example, consider the face $\mathrm{conv}\{(2,1),(1,2)\}$. The normal vector is $v = (1,1)$, and it is valid to translate by $(2,1)$, yielding the configuration $\{(0,0),(-1,1)\}$. The corresponding system is of the form $a + bs^{-1}t = 0 = c + ds^{-1}t$, and this has a common solution $(s,t) \in (\mathbb{C}^*)^2$ if and only if $ac = bd$.

Now the formula (4.4) yields

$$a_1a_2a_3 = \pm\frac{a_1a_2b_1b_2(a_1 - a_2)^2}{(b_1 - b_2)(a_1b_1 - a_2b_2)}, \qquad b_1b_2b_3 = \pm\frac{a_1a_2b_1b_2(b_1 - b_2)^2}{(a_1 - a_2)(a_1b_1 - a_2b_2)}.$$

Checking the signs, we obtain that

$$a_3 = -\frac{b_1b_2(a_1 - a_2)^2}{(b_1 - b_2)(a_1b_1 - a_2b_2)}, \qquad b_3 = -\frac{a_1a_2(b_1 - b_2)^2}{(a_1 - a_2)(a_1b_1 - a_2b_2)}.$$

Note that (a_i, b_i) $i = 1,2,3$ are collinear. To see this note that, since we are working over $(\mathbb{C}^*)^2$, the system $F = G = 0$ is equivalent to the system $F = \frac{1}{xy}G = 0$, and $\frac{1}{xy}G = 0$ is the equation of the line through (a_1, b_1) and (a_2, b_2).

6.3. Saturated planar Chasles configurations. In this section, we show that there are only finitely many isomorphism classes of saturated Chasles configurations of dimension two. We note that there are infinitely two-dimensional Chasles structures involving two different saturated configurations (see Section 6.6).

Recall that a lattice polytope is *reflexive* if its polar polytope is also a lattice polytope. A lattice polygon is reflexive if and only if it contains a unique interior lattice point, but this is not sufficient in higher dimensions. It follows from [S76, H83] that the number of equivalence classes (up to translations and $\mathrm{GL}_n(\mathbb{Z})$) of reflexive polytopes is finite. In the case of dimension 2, the number of equivalence classes is well known to be sixteen; there is an algorithm for computing all such equivalence classes [KS97], which yields 4,319 equivalence classes in dimension three [KS98], and 473,800,776 equivalence classes in dimension four [KS00].

Proposition 6.1. *The saturated Chasles configurations of dimension* 2 *correspond to reflexive polygons. Consequently, there are only sixteen isomorphism classes of saturated Chasles configurations of dimension* 2.

Proof. If $\dim(A) = 2$, let $\mathrm{Int}(A)$ denote the number of lattice points in the interior of $\mathrm{conv}(A)$, and $\mathrm{Bd}(A)$ the number of lattice points on the boundary of $\mathrm{conv}(A)$.

By Pick's formula, the normalized volume of A equals $\mathrm{vol}(A) = 2\,\mathrm{Int}(A) + \mathrm{Bd}(A) - 2$. Combined with the Chasles condition $|A| + 1 = \mathrm{vol}(A) + \dim(A)$, we see that $\mathrm{Int}(A) = 1$. □

Remark 6.2. We point out that Proposition 6.1 does not hold for reflexive polytopes of dimension greater than 2. Consider $A_d = \{\pm e_1, \ldots, \pm e_d\} \subseteq \mathbb{Z}^d$, where $e_1, \ldots, e_d$ are the standard unit vectors. Then $\mathrm{conv}(A)$ is the d-dimensional *cross-polytope*, which is reflexive. Note that $\mathrm{conv}(A_d) \cap \mathbb{Z}^d = A_d \cup \{0\}$. It is also known that $\mathrm{vol}(A_d) = 2^d$. In the two-dimensional case, $|A_2 \cup \{0\}| + 1 = 6 = 2^2 + 2$, so that $A_2 \cup \{0\}$ is a (saturated) Chasles configuration, as Proposition 6.1 states. If $d \geq 3$, we have that $2^d + d > 2d + 2$, so that neither A_d nor $A_d \cup \{0\}$ is a Chasles configuration.

6.4. Example: Cayley octads. For $d = 3$ we consider

$$A = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (2,0,0), (0,2,0), (0,0,2), \\ (1,1,0), (1,0,1), (0,1,1)\},$$

the configuration of all lattice points in twice the standard tetrahedron. Then A is a saturated Chasles configuration, as $\mathrm{vol}(A) + d = 8 + 3 = 11 = |A| + 1$.

From a geometric point of view, a polynomial supported on A gives a quadratic surface, and three of these intersect in eight points generically. Such configurations of eight points are known as *Cayley octads*.

In this case, elegant and compact formulas for the map Υ_A can be found in [PSV11, Proposition 7.1]. We are grateful to Bernd Sturmfels, who directed us to this example.

6.5. Example: A saturated Chasles configuration in dimension $d \geq 3$. For $d \geq 3$ define the configuration $A_d = \{0, e_1, e_2, e_1 + e_2, e_3, 2e_3, e_4, \ldots, e_d\} \subset \mathbb{Z}^d$, where $e_1, \ldots, e_d$ are the coordinate unit vectors. The polytope $\mathrm{conv}(A_3)$ is drawn in Figure 1.

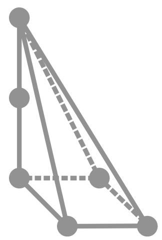

Figure 1. The polytope $\mathrm{conv}(A_3)$

Note that the only lattice points in $\operatorname{conv}(A_d)$ belong to A_d, so that $\operatorname{conv}(A_d)$ contains $d+3$ lattice points. Since $\operatorname{vol}(A_d) = 4$, it follows that A_d is a saturated Chasles configuration.

The formulas for the extra point map are too large to be displayed directly, even for $d = 3$. For instance, one of the directional resultants involved is the resultant of three polynomials supported on the unit square with vertices $\{(0,0), (1,0), (0,1), (1,1)\}$. This is a polynomial of degree 6, with 66 terms, in the 12 coefficients of the corresponding polynomials. And this is without taking into account that those coefficients are themselves rational functions of the coordinates of the given generic points.

6.6. Example: Infinite family of Chasles pairs. Here we produce an infinite family of Chasles structures in the plane.

We let P_n be the quadrangle with vertices $(0,0)$, $(0,n)$, $(1,n+1)$ and $(1,1)$. We let Q_n be the quadrangle with vertices $(1,0)$, $(0,1)$, $(0,n+1)$ and $(1,n)$.

P_n and Q_n are reflections of each other, and contain $2n+2$ lattice points each. Both P_n and Q_n have normalized area $2n$. The Minkowski sum $P_n + Q_n$ is a hexagon with vertices $(1,0)$, $(2,1)$, $(2,2n+1)$, $(1,2n+2)$, $(0,2n+1)$, $(0,1)$, and normalized area $4(2n+1)$. This is illustrated in Figure 2.

For the mixed volume, we see that

$$\begin{aligned}\operatorname{mvol}(P_n + Q_n) &= \frac{1}{2}\big(\operatorname{vol}(P_n + Q_n) - \operatorname{vol}(P_n) - \operatorname{vol}(Q_n)\big)\\ &= \frac{4(2n+1) - 2n - 2n}{2} = 2n+2.\end{aligned}$$

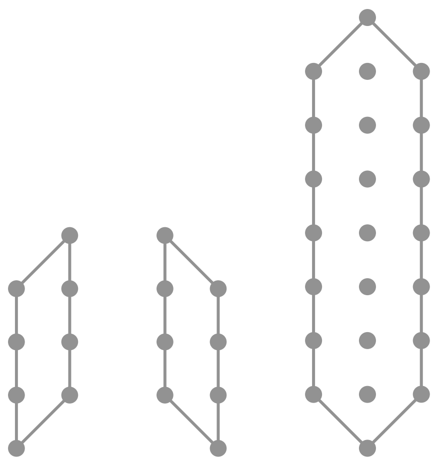

Figure 2. The polygons P_3, Q_3 and $P_3 + Q_3$.

The polygons P_n and Q_n thus correspond to a Chasles structure where $d = 2$, $k_1 = k_2 = 1$ and $N = 2n + 1$. In other words, if we fix $2n + 1$ generic points in $\mathbb{C}^2$, they determine a curve whose defining polynomial has Newton polytope P_n, another curve whose defining polynomial has Newton polytope Q_n, and those two curves meet in $2n + 2$ points.

Since the Minkowski sum $P_n + Q_n$ is a hexagon, there are only six directional resultants appearing in (4.4). Choosing the inner normal vectors $(1, 0)$ or $(-1, 0)$, the corresponding directional resultant is the classical resultant of two univariate polynomials of degree n. The other four inner normal vectors yield directional resultants that are monomials.

Let us give more details in the case $n = 1$. To simplify notation, we consider the points $\rho_1 = (a_1, b_1)$, $\rho_2 = (a_2, b_2)$ and $\rho_3 = (a_3, b_3) = (-1, -1)$. The coefficients of the following polynomials $F(x, y)$, $G(x, y)$ were computed using Cramer's rule to ensure they vanish at ρ_1, ρ_2 and ρ_3:

$$
\begin{aligned}
F(x, y) = {} & b_1 b_2 (a_1 a_2 b_1 - a_1 a_2 b_2 + a_1 b_1 - a_2 b_2 + a_1 - a_2) \\
& - \bigl(a_1 a_2 b_1 b_2 (b_1 - b_2) - a_1 b_1^2 + a_2 b_2^2 - a_1 b_1 + a_2 b_2\bigr) y \\
& + \bigl(b_1 b_2 (a_1 b_1 - a_2 b_2) + a_1 b_1^2 - a_2 b_2^2 - b_1 + b_2\bigr) xy \\
& - \bigl(b_1 b_2 (a_1 - a_2) + a_1 b_1 - a_2 b_2 + b_1 - b_2\bigr) xy^2, \\
G(x, y) = {} & b_1 b_2 (a_2 b_1 - a_1 b_2 - a_1 + a_2 + b_1 - b_2) x \\
& - \bigl(b_1 b_2 (a_2 b_1 - b_1) + b_2^2 (b_1 - a_1) + a_2 b_1 - a_1 b_2\bigr) y \\
& - \bigl(b_1^2 (a_2 - b_2) + b_2^2 (b_1 - a_1) + a_2 b_2 - a_1 b_2\bigr) xy \\
& + \bigl(a_1 a_2 (b_1 - b_2) + b_1 b_2 (a_2 - a_1) + a_2 b_1 - a_1 b_2\bigr) y^2.
\end{aligned}
$$

We know that F and G have a fourth common root $\rho_4 = (a_4, b_4)$, that we wish to find. The next step in this process is to compute the necessary directional resultants. If we choose $v = (1, 0)$, this corresponds to restricting the support of F to the terms 1 and y, and restricting the support of G to the terms y and y^2. The argument in Example 6.2 shows that the directional resultant is the 2×2 determinant of the corresponding coefficients. Similarly, the vector $v = (-1, 0)$ selects the terms xy and xy^2 of F and x and xy of G, with the directional resultant being the corresponding 2×2 determinant. These directional resultants are, respectively

$$
\begin{aligned}
& -a_1 a_2^2 b_1^4 b_2^2 + a_1^2 a_2 b_1^3 b_2^3 + a_1 a_2^2 b_1^3 b_2^3 - a_1^2 a_2 b_1^2 b_2^4 + 2 a_1^2 a_2^2 b_1^3 b_2 - a_1 a_2^2 b_1^4 b_2 \\
& \quad - 4 a_1^2 a_2^2 b_1^2 b_2^2 - a_1^2 a_2 b_1^3 b_2^2 + 2 a_1 a_2^2 b_1^3 b_2^2 + 2 a_1^2 a_2^2 b_1 b_2^3 + 2 a_1^2 a_2 b_1^2 b_2^3 - a_1 a_2^2 b_1^2 b_2^3 \\
& \quad - a_1^2 a_2 b_1 b_2^4 + a_1^2 a_2 b_1^3 b_2 + a_1 a_2^2 b_1^3 b_2 + a_1 a_2 b_1^4 b_2 - 2 a_1^2 a_2 b_1^2 b_2^2 - 2 a_1 a_2^2 b_1^2 b_2^2 \\
& \quad - 2 a_1^2 b_1^3 b_2^2 + a_1 a_2 b_1^3 b_2^2 + a_1^2 a_2 b_1 b_2^3 + a_1 a_2^2 b_1 b_2^3 + a_1 a_2 b_1^2 b_2^3 - 2 a_2^2 b_1^2 b_2^3
\end{aligned}
$$

$$
\begin{aligned}
&+ a_1a_2b_1b_2^4 - a_1^2a_2b_1^3 + a_1a_2b_1^4 + 2a_1^2a_2b_1^2b_2 - a_1a_2^2b_1^2b_2 + 2a_1a_2b_1^3b_2 \\
&- a_1^2a_2b_1b_2^2 + 2a_1a_2^2b_1b_2^2 - 4a_1^2b_1^2b_2^2 + 2a_1a_2b_1^2b_2^2 - 4a_2^2b_1^2b_2^2 - a_1a_2^2b_2^3 \\
&+ 2a_1a_2b_1b_2^3 + a_1a_2b_2^4 - a_1^2a_2b_1^2 + a_1a_2b_1^3 + a_1^2a_2b_1b_2 + a_1a_2^2b_1b_2 \\
&+ a_1a_2b_1^2b_2 - 2a_2^2b_1^2b_2 - a_1a_2^2b_2^2 - 2a_1^2b_1b_2^2 + a_1a_2b_1b_2^2 + a_1a_2b_2^3
\end{aligned}
$$

and

$$
(b_2+1)(b_1+1)(b_2-b_1)(a_1a_2b_1^2 - a_1b_1^2b_2 - a_1a_2b_2^2 - a_2b_1 + a_1b_2).
$$

The other vectors to be considered are $(\pm 1, \pm 1)$. Each of these vectors selects one term of one of the polynomials, and two terms of the other. The corresponding directional resultant is the coefficient of the single term selected. More precisely, using $v = (1,1)$, the directional resultant is the constant coefficient of F; using $v = (1,-1)$ the directional resultant is the coefficient of y^2 in G; using $v = (-1,1)$, the directional resultant is the coefficient of x in G and using $v = (-1,-1)$ the directional resultant is the coefficient of xy^2 in F.

We have listed above all the ingredients necessary to apply the formulas (4.4) and find ρ_4; the resulting expressions, however, are too big to display.

6.7. Example: Non-Chasles configuration and non-rational extra point map. Let A be the set of lattice points in the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,2), (3,1)$, with two interior points $(1,1), (2,1)$, so that A has 5 points, and $\mathrm{vol}(A) = 5$; A is not a Chasles configuration because three zeros, in general, induce two more zeros. Our goal here is to show that the map that assigns the two new zeros to the original three is not rational. It suffices to fix two of the original zeros and make the third variable, and show that the coordinates of the new points involve square roots of the coordinates of the third.

Let us specify that our three points are $(1,1)$, $(2,4)$ and (t, t^2), $t \neq 0, 1, 2$. There is a two-dimensional family of polynomials supported on A that vanish on these three points. The following is a choice of basis for this vector space.

$$
\begin{aligned}
F_1(x,y) &= xy^2 - x^3y = xy(y - x^2); \\
F_2(x,y) &= -8t^3 + (8 + 12t + 14t^2 + 15t^3)xy \\
&\qquad - (12 + 18t + 21t^2 + 7t^3)x^2y + (4 + 6t + 7t^2)x^3y.
\end{aligned}
$$

To find additional common zeros, we note that $F_2(x_0, y_0) \neq 0$ for $x_0y_0 = 0$, and so any zero also lies on $y_0 = x_0^2$. A computation shows that

$$
\begin{aligned}
&F_2(x_0, x_0^2) = (x_0 - 1)(x_0 - 2)(x_0 - t)q(x_0); \\
&\qquad \text{where } q(x_0) = (4 + 6t + 7t^2)x_0^2 + (4t + 6t^2)x_0 + 4t^2.
\end{aligned}
$$

Since the discriminant of q is $-4t^2(12+12t+19t^2)$, which is not a square, the values for x_0 involves square roots of polynomials involving t and hence is not a rational function of $\{1, 1, 2, 4, t, t^2\}$.

Acknowledgments. This project has benefited from visits by the authors to each other's institutions. We also discussed this material during the joint meeting of the AMS and the RSME in Seville in the summer of 2003. We are very grateful to Bernd Sturmfels, for his thoughtful advice, and for directing us to the Cayley octads example in Subsection 6.4. Many thanks to Alicia Dickenstein for a very helpful discussion on resultants. Thanks also to Frank Sottile for productive conversations. We also wish to thank the anonymous referees for their valuable suggestions to improve this work.

LFM was partially supported by NSF Grant DMS-1500832. BR was partially supported by Simons Collaboration Grant 280987.

References

[B75] D. Bernstein, *The number of roots of a system of equations*, Funkcional. Anal. i Priložen., **9** (1975), 1–4.

[CDS98] E. Cattani, A. Dickenstein and B. Sturmfels, *Residues and resultants*, J. Math. Sci. Univ. Tokyo **5** (1998), 119–148.

[CLR87] M. D. Choi, T. Y. Lam and B. Reznick, *Even symmetric sextics*, Math. Z., **195** (1987), 559–580.

[DS15] C. D'Andrea and M. Sombra, *A Poisson formula for the sparse resultant*, Proc. London Math. Soc. **110** (2015), no. 4, 932–964.

[E10] A. Esterov, *Newton polyhedra of discriminants of projections*, Discrete Comput. Geom. **44** (2010), 96–148.

[EGH96] D. Eisenbud, M. Green and J. Harris, *Cayley-Bacharach theorems and conjectures*, Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 3, 295–324.

[M2] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, http://www.math.uiuc.edu/Macaulay2/.

[H83] D. Hensley, *Lattice vertex polytopes with interior lattice points*, Pacific J. Math., **105** (1983), no. 1, 183–191.

[H1888] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann. **32** (1888), 342–350; see Ges. Abh. 2, 154–161, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.

[K97] A. Khovanskii, *Newton polyhedra, a new formula for mixed volume, product of roots of a system of equations*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., 1999, pp. 325–364.

[K76] A. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.

[KS97] M. Kreuzer and H. Skarke, *On the classification of reflexive polyhedra*, Comm. Math. Phys. **185** (1997), no. 2, 495–508.

[KS98] M. Kreuzer and H. Skarke, *Classification of reflexive polyhedra in three dimensions*, Adv. Theor. Math. Phys. **2** (1998), no. 4, 853–871.

[KS00] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four dimensions*, Adv. Theor. Math. Phys. **4** (2000), no. 6, 1209–1230.

[PS93] P. Pedersen and B. Sturmfels, *Product formulas for resultants and Chow forms*, Math. Z. **214** (1993), 377–396.

[PSV11] D. Plaumann, B. Sturmfels and C. Vinzant, *Quartic curves and their bitangents*, J. Symbolic Comput. **46** (2011), no. 6, 712–733.

[RRS15] Q. Ren, J. Richter-Gebert and B. Sturmfels, *Cayley-Bacharach formulas*, Amer. Math. Monthly **122** (2015), no. 9, 845–854.

[R00] B. Reznick, *Some concrete aspects of Hilbert's 17th Problem*, Contemp. Math., **253** (2000), 251–272.

[R07] B. Reznick, *On Hilbert's construction of positive polynomials*, arXiv:0707.2156.

[R69] R. M. Robinson, *Some definite polynomials which are not sums of squares of real polynomials*, in: Selected questions of algebra and logic, Izdat. "Nauka" Sibirsk. Otdel. Novosibirsk, 1973, pp. 264–282. Abstract in Notices AMS, **16** (1969), p. 554.

[R97] J. M. Rojas, *Toric laminations, sparse generalized characteristic polynomials, and a refinement of Hilbert's tenth problem*, Foundations of computational mathematics (Rio de Janeiro, 1997), 369–381, Springer, Berlin, 1997.

[S76] P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc., **15** (1976), 395–399.

[S94] B. Sturmfels, *On the Newton Polytope of the Resultant*, J. Algebraic Combin. **3** (1994), 207–236.

Received February 28, 2019; revision received July 14, 2020

L. F. Matusevich, Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

E-mail: laura@math.tamu.edu

B. Reznick, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

E-mail: reznick@math.uiuc.edu