ON HILBERT'S CONSTRUCTION OF POSITIVE POLYNOMIALS

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Abstract. In 1888, Hilbert described how to find real polynomials which take only non-negative values but are not a sum of squares of polynomials. His construction was so restrictive that no explicit examples appeared until the late 1960s. We revisit and generalize Hilbert's construction and present many such polynomials.

1. History and Overview

A real polynomial $f(x_1,\dots,x_n)$ is *psd* or *positive* if $f(a)\ge 0$ for all $a\in\mathbb{R}^n$; it is *sos* or a *sum of squares* if there exist real polynomials h_j so that $f=\sum h_j^2$. For forms, we follow the notation of [4] and use $P_{n,m}$ to denote the cone of real psd forms of even degree m in n variables, $\Sigma_{n,m}$ to denote its subcone of sos forms and let $\Delta_{n,m}=P_{n,m}\smallsetminus\Sigma_{n,m}$. The Fundamental Theorem of Algebra implies that $\Delta_{2,m}=\emptyset$; $\Delta_{n,2}=\emptyset$ follows from the diagonalization of psd quadratic forms.

The first suggestion that a psd form might not be sos was made by Minkowski in the oral defense of his 1885 doctoral dissertation: Minkowski proposed the thesis that not every psd form is sos. Hilbert was one of his official "opponents" and remarked that Minkowski's arguments had convinced him that this thesis should be true for ternary forms. (See [14], [15] and [24].) Three years later, in a single remarkable paper, Hilbert [11] resolved the question. He first showed that $F\in P_{3,4}$ is a sum of three squares of quadratic forms; see [23] and [26] for recent expositions and [17, 18] for another approach. Hilbert then described a construction of forms in $\Delta_{3,6}$ and $\Delta_{4,4}$; after multiplying these by powers of linear forms if necessary, it follows that $\Delta_{n,m}\ne\emptyset$ if $n\ge 3$ and $m\ge 6$ or $n\ge 4$ and $m\ge 4$.

The goal of this paper is to isolate the underlying mechanism of Hilbert's construction, show that it applies to situations more general than those in [11], and use it to produce many new examples.

In [11], Hilbert first worked with polynomials in two variables, which homogenize to ternary forms. Suppose $f_1(x,y)$ and $f_2(x,y)$ are two relatively prime real cubic polynomials with nine distinct real common zeros – $\{\pi_i\}$, indexed arbitrarily – so that

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no three of the π_i's lie on a line and no six lie on a quadratic. By counting coefficients, one sees that there exists a non-zero quadratic $\phi(x,y)$ with zeros at $\{\pi_1,\dots,\pi_5\}$ and a non-zero quartic $\psi(x,y)$ with the same zeros, and which is singular at $\{\pi_6,\pi_7,\pi_8\}$: the sextic $\phi\psi$ is thus singular at $\{\pi_1,\dots,\pi_8\}$. Hilbert showed that $(\phi\psi)(\pi_9)\neq 0$ and that there exists $c\neq 0$ so that the perturbed polynomial $p = f_1^2 + f_2^2 + c\phi\psi$ is positive. If $p = \sum h_j^2$, then each h_j would be a cubic which vanishes on $\{\pi_1,\dots,\pi_8\}$. But Cayley-Bacharach implies that $h_j(\pi_9) = 0$ for each j, hence $p(\pi_9) = 0$, a contradiction. Thus, p homogenizes to a form $P \in \Delta_{3,6}$.

Hilbert also considered in [11] three relatively prime real quadratic polynomials, $f_i(x,y,z)$, $1 \le i \le 3$, with eight distinct real common zeros – $\{\pi_i\}$, indexed arbitrarily – so that no four of the zeros lie on a plane. There exists a non-zero linear $\phi(x,y,z)$ with zeros at $\{\pi_1,\pi_2,\pi_3\}$ and a non-zero cubic $\psi(x,y,z)$ with the same zeros, and which is singular at $\{\pi_4,\pi_5,\pi_6,\pi_7\}$. Similarly, $(\phi\psi)(\pi_8)\neq 0$ and there exists $c\neq 0$ so that $f_1^2+f_2^2+f_3^2+c\phi\psi$ is positive and not sos. This homogenizes to a form in $\Delta_{4,4}$.

In 1893, Hilbert [12] showed that if $F \in P_{3,m}$ with $m \ge 4$, then there exists a form $G \in P_{3,m-4}$ and forms H_k, $1 \le k \le 3$, so that $GF = H_1^2 + H_2^2 + H_3^2$. (Hilbert's construction does not readily identify G or the H_k's.) In particular, if $F \in P_{3,6}$, then there exists $Q \in P_{3,2}$ so that $QF \in \Sigma_{3,8}$; since $Q \cdot QF \in \Sigma_{3,10}$, F is a sum of squares of rational functions with common denominator Q. An iteration of this argument shows that if $F \in P_{3,m}$, then there exists G so that G^2F is sos. Hilbert's 17th Problem [13] asked whether this representation as a sum of squares of rational functions exists for forms in $P_{n,m}$ when $n \ge 4$. For much more on the history of this subject up to 1999, see [21]. Recently, Blekherman [3] has shown that for fixed degree m, the "probability" that a psd form is sos goes to 0 as n increases. This result highlights the importance of understanding psd forms which are not sos.

Hilbert's restriction on the common zeros meant that no very simple or symmetric example could be constructed, and the first explicit example of any $P \in \Delta_{n,m}$ did not appear for many decades. The only two detailed references to Hilbert's construction before the late 1960s (known to the author) are by Terpstra [27] (on biquadratic forms, related to $\Delta_{4,6}$, thanks to Roland Hildebrand for the reference), and an exposition [10, pp.232-235] by Gel'fand and Vilenkin of the sextic case only.

At a 1965 conference on inequalities, Motzkin [16] presented a specific sextic polynomial $m(x,y)$ which is positive by the arithmetic-geometric inequality and not sos by the arrangement of monomials in its Newton polytope. (Hilbert's last assistant, Olga Taussky-Todd, who had a lifelong interest in sums of squares, heard Motzkin speak, and informed him that $m(x,y)$ was the first specific polynomial known to be positive but not sos.) After homogenization, Motzkin's example is

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \in \Delta_{3,6}. \tag{1.1}$$

Around the same time and independently, R. M. Robinson [22, p.264] wrote that he saw "an unpublished example of a ternary sextic worked out recently by W. J. Ellison using Hilbert's Method. It is, as would be expected, very complicated.

After seeing this, I discovered that an astonishing simplification would be possible by dropping some unnecessary assumptions made by Hilbert." Robinson observed that the cubics $f_1(x,y) = x^3 - x$ and $f_2(x,y) = y^3 - y$ have nine common zeros: the 3×3 square $\{-1,0,1\}^2$. There are eight lines which each contain three of the zeros. Still, the sextic $(x^2-1)(y^2-1)(1-x^2-y^2)$ is positive at (0,0) and singular at the other eight points. By taking the maximum value for c in Hilbert's construction and homogenizing, Robinson showed that

$$R(x,y,z) = x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2 \tag{1.2}$$

is in $\Delta_{3,6}$. Similarly, by taking the three quadratics $x^2 - x$, $y^2 - y$ and $z^2 - z$, whose common zeros are $\{0,1\}^3$, choosing $(1,1,1)$ as the eighth point, and then homogenizing, Robinson showed that

$$\tilde{R}(x,y,z,w) = x^2(x-w)^2 + y^2(y-w)^2 + z^2(z-w)^2 + 2xyz(x+y+z-2w) \tag{1.3}$$

is in $\Delta_{4,4}$. (The only other published implementation of Hilbert's Method known to the author is a 1979 sextic studied by Schmüdgen [25] using $\{-2,0,2\}^2$, with ninth point $(2,0)$.)

The papers of Motzkin and Robinson renewed interest in these polynomials, and two more examples in the style of M were presented by Choi and Lam [4, 5]:

$$S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \in \Delta_{3,6}, \tag{1.4}$$

$$Q(x,y,z,w) = x^2y^2 + x^2z^2 + y^2z^2 + w^4 - 4wxyz \in \Delta_{4,4}. \tag{1.5}$$

Here is an overview of the rest of the paper.

In section two, we present some preliminary material, mainly from curve theory; it is important to consider reducible (as well as irreducible) polynomials.

In section three, we present our version of Hilbert's Method (see Theorem 3.4), based on more general perturbations and contradictions. There is a class of perturbations of a given positive polynomial with fixed zeros by a polynomial which is singular at these zeros, in which positivity is preserved. By counting dimensions, under certain circumstances, there are polynomials of degree $2d$ which are singular on a set A, but are not in the vector space generated by products of pairs of polynomials of degree d which vanish on A. If such a polynomial is positive, it cannot be sos. In Robinson's work, the set of cubics vanishing at the eight points is spanned by $\{f_1, f_2\}$, but the vector space of sextics which are singular at the eight points has dimension four and so cannot be spanned by $\{f_1^2, f_1f_2, f_2^2\}$. It is not necessary to construct ϕ and ψ to find this new sextic, although its behavior at the ninth point must be analyzed to show that a successful perturbation is possible.

We show in Theorem 4.1 that Hilbert's Method works when f and g are ternary cubics with exactly nine real intersections, whether or not three are on a line or six on a quadratic. (In other words, Robinson's "astonishing simplification" always works.) We also show that Hilbert's Method applies to the set of cubics which vanish on a set of seven zeros, no four on a line, not all on a quadratic; see Theorem 4.3.

Example 1.1. Let

$$\mathcal{A} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1)\},$$
$$F_1(x,y,z) = x(y^2-z^2), F_2(x,y,z) = y(z^2-x^2), F_3(x,y,z) = z(x^2-y^2), \tag{1.6}$$
$$G(x,y,z) = (x^2-y^2)(x^2-z^2)(y^2-z^2).$$

It is easy to show that the F_k's span the set of ternary cubics which vanish on $\mathcal{A}$ and that G is singular on $\mathcal{A}$ and not in the span of the F_jF_k's. It follows from Theorem 4.3 that for some $c > 0$, $P_c = F_1^2 + F_2^2 + F_3^2 + cG$ is psd and not sos. In fact, $P_1 = 2S$, providing a new construction of (1.4).

In section five, we look at the sections of the cones $P_{3,6}$ and $\Sigma_{3,6}$ consisting of ternary sextics with the eight zeros of Theorem 4.1. In addition to some general results, we give a one-parameter family $\{R_t : t > 0\}$ of forms in $\Delta_{3,6}$ with ten zeros and such that $R_1 = R$:

$$R_t(x,y,z) :=$$
$$\left(\frac{t^4+2t^2-3}{3}\right)(x^3-xz^2)^2 + \left(\frac{1+2t^2-3t^4}{3t^4}\right)(y^3-yz^2)^2 + R(x,y,z). \tag{1.7}$$

We give necessary and sufficient conditions for a sextic polynomial $p(x,y)$ with zeros at $\{-1,0,1\}^2 \setminus (0,0)$ to be psd and to be sos.

In section six, we present more examples in $\Delta_{3,6}$. This paper would not be complete without an explicit illustration of Hilbert's Method under his original restrictions. Theorems 4.1 and 4.3 and other techniques are then applied to produce new forms in $\Delta_{3,6}$, including one-parameter families which include R, S and M. For $t^2 < \frac{1}{2}$, let

$$\begin{aligned} M_t(x,y,z) &= (1-2t^2)(x^4y^2+x^2y^4) + t^4(x^4z^2+y^4z^2) \\ &-(3-8t^2+2t^4)x^2y^2z^2 - 2t^2(x^2+y^2)z^4 + z^6; \end{aligned} \tag{1.8}$$

$M_t \in \Delta_{3,6}$ has ten zeros and $M_0 = M$. Let

$$\begin{aligned} S_t(x,y,z) &= t^4(x^6+y^6+z^6) + (1-2t^6)(x^4y^2+y^4z^2+z^4x^2) \\ &+(t^8-2t^2)(x^2y^4+y^2z^4+z^2x^4) - 3(1-2t^2+t^4-2t^6+t^8)x^2y^2z^2; \end{aligned} \tag{1.9}$$

$S_t \in \Delta_{3,6}$ has ten zeros if $t > 0$. Note that $S_0 = S$ and $S_1 = R$, so S_t provides a "homotopy" between S and R in $\Delta_{3,6}$ in the set of forms with ten zeros. We also show that

$$U_c(x,y,z) = x^2y^2(x-y)^2 + y^2z^2(y-z)^2 + z^2x^2(z-x)^2 + cxyz(x-y)(y-z)(z-x) \tag{1.10}$$

is psd if and only if $|c| \le 4\sqrt{\sqrt{2}-1}$ and sos only if $c = 0$. We conclude the section by returning to a subject brought up by Robinson: $(ax^2+by^2+cz^2)R(x,y,z)$ is sos if and only if $a,b,c \ge 0$ and $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the sides of a (possibly degenerate) triangle.

In section seven, we discuss the zeros of extremal ternary forms, using the perturbation argument from Hilbert's Method and show that if $p \in \Delta_{3,6}$ has exactly ten zeros, then it is extremal in the cone $P_{3,6}$. We present supporting evidence for the conjecture that, at least in a limiting sense, all extremal forms in $\Delta_{3,6}$ have ten zeros.

Finally, in section eight, we apply Hilbert's Method to provide a family of positive polynomials in two variables in even degree ≥ 6 which are not sos. We also speculate on the general applicability of Hilbert's Method in higher degree.

Bezout's Theorem becomes more complicated in more variables, and for that reason, we have confined our discussions to ternary forms. However, we wish to record a somewhat unexpected connection between $\tilde{R}$ and Q (c.f. (1.3), (1.5)):

$$\tilde{R}(x-w, y-w, z-w, x+y+z-w) = 2Q(x,y,z,w). \tag{1.11}$$

Robinson's example, after homogenization and this change in variables, gives a new derivation of the Choi-Lam example. The set of quaternary quadratics which vanish on

$$\begin{gathered}\mathcal{A} = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),\\ (1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1)\}\end{gathered} \tag{1.12}$$

is spanned by $\{xy - zw, xz - yw, xw - yz\}$, and any such quadratic also vanishes at $(0,0,0,1)$. The form Q is evidently psd by the arithmetic-geometric inequality, singular on $\mathcal{A}$ and positive at $(0,0,0,1)$, and so is not sos.

Parts of this paper have been presented at many conferences over the last several years. The author thanks the organizers for their many invitations to speak, and his friends and colleagues for their encouragement and suggestions.

2. Preliminaries

Throughout this paper, we toggle between forms F in k variables and polynomials f in $k-1$ variables, with the ordinary convention that

$$\begin{gathered}f(x_1,\dots,x_{k-1}) := F(x_1,\dots,x_{k-1},1),\\ F(x_1,\dots,x_k) := x_k^d f(\tfrac{x_1}{x_k},\dots,\tfrac{x_{k-1}}{x_k}),\end{gathered} \tag{2.1}$$

where $d = \deg f$. For even d, it is easy to see that F and f are simultaneously psd or sos. It is usually more convenient to use forms, since $F \in P_{k,m}$ if and only if $F(u) \geq 0$ for u in the compact set S^{k-1}, simplifying perturbation. On the other hand, the zeros of f can be isolated, whereas those of F are not.

Following [6], we define the *zero-set* of any k-ary m-ic form F by

$$\mathcal{Z}(F) := \{(a_1,\dots,a_k) \in \mathbb{R}^k \ : \ F(a_1,\dots,a_k) = 0\}. \tag{2.2}$$

We have $0 \notin \mathcal{Z}(F)$ by convention, $|\mathcal{Z}(F)|$ will be interpreted as the number of lines in $\mathcal{Z}(F)$ and only only one representative of each line need be given. If $a \in \mathcal{Z}(F)$ and $a_k \neq 0$, then a corresponds to a unique zero of f; if $a_k = 0$, then a corresponds to a *zero of f at infinity.* We also define

$$\mathcal{Z}(f) := \{(a_1,\dots,a_{k-1}) \in \mathbb{R}^{k-1} \ : \ f(a_1,\dots,a_{k-1}) = 0\}, \tag{2.3}$$

for non-homogeneous f. It is possible for a strictly positive f to be have zeros at infinity. Consider $f(x,y) = x^2 + (xy-1)^2$ (and $F(x,y,z) = x^2z^2 + (xy-z^2)^2$): clearly, $f(a,b) > 0$ for $(a,b) \in \mathbb{R}^2$ and $\mathcal{Z}(F) = \{(1,0,0),(0,1,0)\}$.

If f is positive and $a \in \mathcal{Z}(f)$, then of course $\frac{\partial f}{\partial x_i}(a) = 0$ for all i. We shall say that f is *round at a* if f_a, the second-order component of the Taylor series to f at a, is a positive definite quadratic form. This is a "singular non troppo" zero for a positive polynomial. The corresponding second-order component of Taylor series for F is psd but not positive definite, since F vanishes on lines through the origin.

If $F \in P_{n,m}$ (resp. $\Sigma_{n,m}$), and G is derived from F by an invertible linear change of variables, then $G \in P_{n,m}$ (resp. $\Sigma_{n,m}$). Thus, it is harmless to assume when convenient that $\mathcal{Z}(F)$ avoids the hyperplane $a_n = 0$; that is, f has no zeros at infinity.

Let $\mathbb{R}_{n,d} \subset \mathbb{R}[x_1,\dots,x_n]$ denote the $\binom{n+d}{n}$-dimensional vector space of real polynomials $f(x_1,\dots,x_n)$ with $\deg f \le d$. Suppose $A = \{\pi_1,\dots,\pi_r\} \subset \mathbb{R}^n$ is given. Let $I_{s,d}(A)$ denote the vector space of those $p \in \mathbb{R}_{n,d}$ which have an s-th order zero at each π_j. In particular,

$$I_{1,d}(A) = \{p \in \mathbb{R}_{n,d} \ : \ p(\pi_j) = 0, \quad 1 \le j \le r\};$$
$$I_{2,2d}(A) = \left\{p \in \mathbb{R}_{n,2d} \ : \ p(\pi_j) = \tfrac{\partial p}{\partial x_i}(\pi_j) = 0, \quad 1 \le i \le n, \quad 1 \le j \le r\right\}. \tag{2.4}$$

Since an s-th order zero in n variables imposes $\binom{n+s-1}{n}$ linear conditions,

$$\dim I_{s,d}(A) \ge \binom{n+d}{n} - r\binom{n+s-1}{n}. \tag{2.5}$$

In Hilbert's sextic construction, $A = \{\pi_1,\dots,\pi_9\}$ is the set of common zeros of $f_1(x,y)$ and $f_2(x,y)$, and $\dim(I_{1,3}(A)) = 2 > \binom{5}{2} - 9\binom{2}{2}$.

Let

$$I_{1,d}^2(A) := \left\{\sum f_i g_i \ : \ f_i, g_i \in I_{1,d}(A)\right\}. \tag{2.6}$$

Clearly, $I_{1,d}^2(A) \subseteq I_{2,2d}(A)$. It is essential to Hilbert's Method that this inclusion may be strict; for example, $\phi\psi(\pi_9) > 0$ so $\phi\psi \in I_{2,6}(A) \smallsetminus I_{1,3}^2(A)$.

We also need to consider the "forced" zeros, familiar from the Cayley-Bacharach Theorem; see [9]. Suppose $A \subset \mathbb{R}^n$ and $I_{1,d}(A)$ are given as above. Let

$$\tilde{A} := \bigcap_{j=1}^{r} \mathcal{Z}(f_j) \smallsetminus A = \mathcal{Z}\left(\sum_{j=1}^{r} f_j^2\right) \smallsetminus A. \tag{2.7}$$

Unfortunately, this notation fails to capture forced zeros at infinity. Accordingly, for $A \subset \mathbb{R}^n$, define the associated projective set $\mathcal{A} \subset \mathbb{R}^{n+1}$ by

$$(a_1,\dots,a_n) \in A \iff (a_1,\dots,a_n,1) \in \mathcal{A}. \tag{2.8}$$

As before, we define $I_{s,d}(\mathcal{A})$ to be the set of d-ic forms $F(x_1, \dots, x_{n+1})$ which have s-th order zeros on $\mathcal{A}$. Then $f \in I_{s,d}(A)$ if and only if $F \in I_{s,d}(\mathcal{A})$. We define

$$\tilde{\mathcal{A}} := \bigcap_{j=1}^{r} \mathcal{Z}(F_j) \smallsetminus \mathcal{A} = \mathcal{Z}\left(\sum_{j=1}^{r} F_j^2\right) \smallsetminus \mathcal{A}. \tag{2.9}$$

Given $A \subset \mathbb{R}^n$, $\tilde{\mathcal{A}} = \emptyset$ when there are no forced zeros, even at infinity.

We say that $I_{1,d}(A)$ is *full* if, for any $\pi \in A$ and $v \in \mathbb{R}^n$, there exists $f \in I_{1,d}(A)$ such that $\vec{\nabla} f(\pi) = v$. Equivalently, if $\{f_1, \dots, f_s\}$ is a basis for $I_{1,d}(A)$ and $f = \sum_j f_j^2$, then $I_{1,d}(A)$ is full if and only if f is round at each $\pi \in A$.

Bezout's Theorem in a relatively simple form is essential to our proofs. Suppose $f_1(x, y)$ and $f_2(x, y)$ are relatively prime polynomials of degrees d_1 and d_2. Let $\mathcal{Z} \subset \mathbb{C}^2$ denote the set of common (complex) zeros of f_1 and f_2. Then

$$d_1 d_2 = \sum_{\pi \in \mathcal{Z}} \mathcal{I}_\pi(f_1, f_2), \tag{2.10}$$

where $\mathcal{I}_\pi(f_1, f_2)$ measures the singularity of the intersection of the curves $f_1 = 0$ and $f_2 = 0$ at π. In particular, $\mathcal{I}_\pi(f_1, f_2) = 1$ if and only if the curves $f_1 = 0$ and $f_2 = 0$ are nonsingular at π and have different tangents. Thus, $\mathcal{I}_\pi(f_1, f_2) = 1$ if and only if $f_1^2 + f_2^2$ is round at π, and $\mathcal{I}_\pi(f_1, f_2) \ge 2$ otherwise. If f_1 and f_2 are both singular at π, then $\mathcal{I}_\pi(f_1, f_2) \ge 4$.

Lemma 2.1. *Suppose $f_1(x, y), f_2(x, y) \in \mathbb{R}_{2,d}$ and $|\mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)| = d^2$. If $A \subseteq \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$ is such that $I_{1,d}(A)$ has basis $\{f_1, f_2\}$, then A is full.*

Proof. It follows from (2.10) that any common zero of f_1 and f_2 must be real, and that $\mathcal{I}_\pi(f_1, f_2) = 1$ for each common zero π. It follows that A is full. □

The next proposition collects some useful information from curve theory. As is customary, if $f(\pi) = 0$, we say that π *lies on* f or f *contains* π.

Proposition 2.2. *All polynomials herein are assumed to be in $\mathbb{R}[x, y]$, and all enumerated sets of points are assumed to be distinct. These results apply to ternary forms with the obvious modifications.*

1. *If a quadratic q is singular at π and $q(\pi') = 0$ for some $\pi' \ne \pi$, then $q = \ell_1 \ell_2$ is a product of two linear forms ℓ_j containing π.*
2. *If a set of eight points $A = \{\pi_1, \dots, \pi_8\}$ is given, no four on a line and no seven on a quadratic, then* $\dim I_{1,3}(A) = 2$.
3. *In the last situation, if $A_j = A \smallsetminus \{\pi_j\}$, then there exists a cubic f so that $f|_{A_j} = 0$, but $f(\pi_j) \ne 0$; in particular,* $\dim I_{1,3}(A_j) = 3$.
4. *Suppose $f(x, y)$ and $g(x, y)$ are cubics, $A = \mathcal{Z}(f) \cap \mathcal{Z}(g) = \{\pi_1, \dots, \pi_9\}$ and $A_j = A \smallsetminus \{\pi_j\}$. For each j, $I_{1,3}(A_j) = I_{1,3}(A)$. In other words, if eight of the points lie on a cubic h, then so will the ninth.*

(5) *Under the same conditions as (4), no four of the π_i's lie on a line and no seven lie on a quadratic. Three of the π_i's lie on a line if and only if the other six lie on a quadratic if and only if $I_{1,3}(A)$ contains a reducible cubic.*

Proof. For (1), write $q(x,y) = a+bx+cy+dx^2+exy+fy^2$ and assume by translation that $\pi = (0,0)$. Then $a = b = c = 0$ and $q(x,y) = dx^2 + exy + fy^2$. If $\pi' = (r,s) \neq (0,0)$, then $sx - ry$ is a factor of q. The next two assertions are classical and proofs can be found, for example, in [2, Ch.15]; (4) is well-known and is often attributed to Cayley-Bacharach, but it was discovered by Chasles; see [9].

For (5), if four π_i's lie on a line ℓ, then ℓ divides both f and g by Bezout, so that $|\mathcal{Z}(f) \cap \mathcal{Z}(g)| = \infty$. If seven π_i's lie on a reducible quadratic $q = \ell_1\ell_2$, then at least four lie on one ℓ_i, and we are in the earlier case. If they lie on an irreducible q, then it must be indefinite, and again, q divides both f and g by Bezout, so that $|\mathcal{Z}(f) \cap \mathcal{Z}(g)| = \infty$.

Suppose now that three points of A, say $\{\pi_1, \pi_2, \pi_3\}$, lie on the line ℓ and let q be the quadratic containing $\{\pi_4, \dots, \pi_8\}$. Then $\ell q = 0$ on A_8, so by (4), $(\ell q)(\pi_9) = 0$. Since $\ell(\pi_9) \neq 0$, we must have $q(\pi_9) = 0$; thus six zeros lie on q and $\ell q \in I_{1,3}(A_j)$. (A similar proof follows if we start with six points lying on the quadratic q.) Finally, if $\ell q \in I_{1,3}(A)$, then at most three of the π_i's can lie on ℓ, and at most six can lie on q, hence these numbers are exact. □

Lemma 2.3. *Suppose A is a set of eight distinct points, no four on a line and no seven on a quadratic, and let $\{f_1, f_2\}$ be a basis for $I_{1,3}(A)$. Then f_1 and f_2 are relatively prime.*

Proof. If f_1 and f_2 have a common quadratic factor q, then $f_i = \ell_i q$ and at most six points of A lie on q, so ℓ_1 and ℓ_2 share two points and so are proportional, a contradiction. If f_1 and f_2 have only a common linear factor ℓ, then $f_i = \ell q_i$, and at most three points of A lie on ℓ, so q_1 and q_2 share five points and so are proportional, again a contradiction. □

In the situation of Lemma 2.3, Bezout's Theorem has one of three possible implications: (a) there is a ninth point $\pi \in \tilde{A}$ so that $f_1(\pi) = f_2(\pi) = 0$; (b) $\tilde{A} = \emptyset$, but $(a,b,0) \in \tilde{\mathcal{A}}$ is a common zero of f_1 and f_2 at infinity; (c) $\mathcal{I}_\pi(f_1,f_2) = 2$ for some $\pi \in A$. The first two cases are essentially the same: if (b) occurs, we homogenize and change variables so that the zero is no longer at infinity after dehomogenization. Any necessary construction can then be performed, and the variables changed back. The third case is singular, but seems to be difficult to identify in advance, and is equivalent to the existence of a cubic in $I_{1,3}(A)$ which is singular at some $\pi \in A$.

We shall say that a set of eight points A for which (a) or (b) occurs is *copacetic.* Since f_1 and f_2 are real, $f_1(\pi) = f_2(\pi) = 0 \implies f_1(\bar{\pi}) = f_2(\bar{\pi}) = 0$. Bezout implies that $\pi = \bar{\pi}$; that is, the ninth point π must be real. We have the following corollary to Lemma 2.1.

Lemma 2.4. *If A is copacetic, then it is full.*

The following lemma was probably known a hundred years ago.

Lemma 2.5. *Suppose seven points $A = \{\pi_1, \dots, \pi_7\}$ in the plane are given, not all on a quadratic and no four on a line. Then, up to multiple, there is a unique cubic $f(x, y)$ which is singular at π_1 and contains $\{\pi_2, \dots, \pi_7\}$.*

Proof. Since $1 \cdot 3 + 6 \cdot 1 < 10$ linear conditions are given, at least one such nonzero f exists. Suppose f_1 and f_2 satisfy these properties and are not proportional. Then $\sum_j \mathcal{I}_{\pi_j}(f_1, f_2) \geq 2^2 + 6 \cdot 1 > 3 \cdot 3$, hence f_1 and f_2 have a common factor. The common factor could be an irreducible quadratic, a reducible quadratic, or linear.

In the first case, $f_1 = \ell_1 q$ and $f_2 = \ell_2 q$, where $q(\pi_1) = \ell_i(\pi_1) = 0$ by Prop. 2.2(1). At least one point, say π_7, does not lie on q, hence $\ell_i(\pi_7) = 0$ as well. Thus the two ℓ_i's share two zeros and are proportional, a contradiction.

In the second case, we have $f_1 = \ell_1\ell_2\ell_3$ and $f_2 = \ell_1\ell_2\ell_4$, and π_1 lies on at least two of $\{\ell_1, \ell_2, \ell_3\}$ and two of $\{\ell_1, \ell_2, \ell_4\}$. If $\ell_1(\pi_1) = \ell_2(\pi_1) = 0$, then ℓ_1 and ℓ_2 together can contain at most four of the six points $\{\pi_2, \dots, \pi_7\}$, hence ℓ_3 and ℓ_4 must each contain at least two points in common, and so are proportional, again a contradiction. Otherwise, without loss of generality, $\ell_1(\pi_1) = 0$ and $\ell_2(\pi_1) \neq 0$, hence $\ell_3(\pi_1) = \ell_4(\pi_1) = 0$. In this case, ℓ_1 and ℓ_2 can together contain at most five of the six points $\{\pi_2, \dots, \pi_7\}$, so that ℓ_3 and ℓ_4 must contain also some π_j other than π_1. This is again a contradiction.

Finally, suppose $f_1 = \ell q_1$ and $f_2 = \ell q_2$, where q_1 and q_2 are relatively prime quadratics, so they share at most four points. If $\ell(\pi_1) = 0$, then $q_j(\pi_1) = 0$ as well and since at least four of $\{\pi_2, \dots, \pi_7\}$ do not lie on ℓ, they must lie on both q_1 and q_2. Thus q_1 and q_2 share five points, a contradiction. If $\ell(\pi_1) \neq 0$, then $h_1 = \ell\ell_1\ell_2$ and $h_2 = \ell\ell_3\ell_4$, where the ℓ_i's are distinct lines containing π_1. But if $\ell(\pi_j) \neq 0$ (which is true for at least four π_j's) then π_j must also lie on one of $\{\ell_1, \ell_2\}$ and one of $\{\ell_3, \ell_4\}$. That is, the line through π_1 and π_j divides both $\ell_1\ell_2$ and $\ell_3\ell_4$, a final contradiction. □

The last lemma in this section is used in the proof of Theorem 4.3.

Lemma 2.6. *If d=3 and A is a set of seven points in $\mathbb{R}^2$, no four on a line and not all on a quadratic, then A is full and $\tilde{\mathcal{A}} = \emptyset$.*

Proof. Choose π_8 to avoid any line between two points of A and any quadratic determined by five points of A. Then $A \cup \{\pi_8\}$ has no four points in a line and no seven on a quadratic, and so $\dim I_{1,3}(A) = 3$ by Prop. 2.2(3). Suppose $\{f_1, f_2, f_3\}$ is a basis for $I_{1,3}(A)$ and for each j, consider the map

$$T_j : (c_1, c_2, c_3) \mapsto \sum_{k=1}^{3} c_k \vec{\nabla} f_k(\pi_j). \tag{2.11}$$

By Lemma 2.5, $\dim(\ker(T_j)) = 1$, hence each T_j is surjective, and so A is full.

Suppose $\pi \in \tilde{\mathcal{A}}$; after an invertible linear change, we may assume without loss of generality that $\pi \in \tilde{A}$. By the contrapositive to Prop. 2.2(3), either $A \cup \{\pi\}$ has four points in a line or has seven points on a quadratic. Again, choose π_8 so that $A_1 = A \cup \{\pi_8\}$ has no four points in a line and no seven on a quadratic. By Prop. 2.2(2), we may assume without loss of generality that $I_{1,3}(A_1)$ has basis $\{f_1, f_2\}$, so $\pi \in \tilde{A}_1$. Let $A_2 = A_1 \cup \{\pi\}$. Thus f_1 and f_2 are two cubics which vanish on a set A_2 with four points on a line ℓ or seven points on a quadratic q, and so f_1 and f_2 have a common factor by Bezout, a contradiction. □

3. Hilbert's Method

We begin this section with a general perturbation result.

Lemma 3.1 (The Perturbation Lemma)**.** *Suppose $f, g \in \mathbb{R}_{n,2d}$ satisfy the following conditions:*

1. *The polynomial f is positive with no zeros at infinity, and $2d = \deg f \ge \deg g$;*
2. *There is a finite set V_1 so that if $v \in V_1$, then f is round at v and g vanishes to second-order at v;*
3. *The set $V_2 := \mathcal{Z}(f) \smallsetminus V_1$ is finite and if $w \in V_2$, then $g(w) > 0$.*

Then there exists $c = c(f,g) > 0$ so that $f + cg$ is a positive polynomial.

Proof. For $v \in V_1$, let g_v denote the second-order (lowest degree) term of the Taylor series for g at v. Since f_v is positive definite, there exists $\alpha(v) > 0$ so that $f_v + \alpha g_v$ is positive definite for $0 \le \alpha \le \alpha(v)$. If $\alpha_0 = \min_v \alpha(v)$, then there exist neighborhoods $\mathcal{N}_v$ of each v so that $f + \alpha_0 g$ is positive on each $\mathcal{N}_v \smallsetminus \{v\}$. Further, for $w \in V_2$, $(f + \alpha_0 g)(w) = \alpha_0 g(w) > 0$, hence there is a neighborhood $\mathcal{N}_w$ of w on which $f + \alpha_0 g$ is positive. It follows that $f + \alpha_0 g$ is non-negative on the open set $\mathcal{N} = \cup \mathcal{N}_v \cup \mathcal{N}_w$.

Homogenize f, g to forms F, G of degree $2d$ in $n+1$. For $x \in \mathbb{R}^n$, let $||x|| = (1 + \sum_i x_i^2)^{1/2}$ and let $\widetilde{\mathcal{N}}$ be the image of $\mathcal{N}$ under the map

$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{||x||}, \dots, \frac{x_n}{||x||}, \frac{1}{||x||} \right) \in S^n. \tag{3.1}$$

Then $\widetilde{\mathcal{N}}$ is open and $(F + \alpha G)(x) \ge 0$ for $x \in \widetilde{\mathcal{N}}$. By hypothesis, $\mathcal{Z}(F) \subset \widetilde{\mathcal{N}}$, hence F is positive on the complement $\widetilde{\mathcal{N}}^c$, so it achieves a positive minimum on the compact set $\widetilde{\mathcal{N}}^c$. Since G is bounded on S^n, there exists $\beta > 0$ so that $(F + \beta G)(x) \ge 0$ for $x \in \widetilde{\mathcal{N}}^c$. It follows that $F + cG$ is psd, where $c = \min\{\alpha_0, \beta\}$. The desired result follows upon dehomogenizing. □

The following two theorems generalize the contradiction of Hilbert's construction.

Theorem 3.2. *If $p \in I_{2,2d}(A)$ is sos, then $p \in I_{1,d}^2(A)$.*

Proof. If $p = \sum_k h_k^2$, then $p(a) = 0$ for $a \in A$, hence $h_k(a) = 0$, and so $h_k \in I_{1,d}(A)$, implying $p \in I_{1,d}^2(A)$. □

Let $I_{1,d}(A)$ have basis $\{f_1, \dots, f_r\}$, and suppose the $\binom{r+1}{2}$ polynomials $f_i f_j, 1 \le i \le j \le r$ are linearly independent; in other words, for each $p \in I_{1,d}^2(A)$ there is a unique quadratic form Q so that $p = Q(f_1, \dots, f_r)$. We call this the *independent case.* (We have been unable to find $I_{1,d}(A)$ for which this does not hold.) Let

$$R_f := \{(f_1(x), \dots, f_r(x)) \ : \ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^r. \tag{3.2}$$

denote the range of the basis as an r-tuple.

Theorem 3.3. *Suppose $p = Q(f_1, \dots, f_r) \in I_{1,d}^2(A)$ in the independent case:*

(1) *p is sos if and only if Q is an sos quadratic form;*
(2) *p is psd if and only if $Q(u) \geq 0$ for $u \in R_f$;*
(3) *if $n = 2$, $r = 2$, and f_1 and f_2 are relatively prime polynomials with odd degree d, then $R_f = \mathbb{R}^2$, hence $p \in I_{1,d}^2(A)$ is psd if and only if it is sos.*

Proof. If $p = \sum_k h_k^2$ is sos, then as in the last proof, $h_k \in I_{1,d}(A)$. To be specific, if $h_k = \sum_\ell c_{k\ell} f_\ell$, then by the uniqueness of Q, $Q(u_1, \dots, u_r) = \sum_\ell (\sum_\ell c_{k\ell} u_\ell)^2$. Conversely, if $Q = \sum_\ell T_\ell^2$ for linear forms T_ℓ, then $p = \sum_\ell T_\ell(f_1, \dots, f_r)^2$.

The assertion in (2) is immediate.

For (3), we first note that, since $(f_1(\lambda x), f_2(\lambda x)) = \lambda^d (f_1(x), f_2(x))$, it suffices to show that every line through the origin intersects R_f. By hypothesis, $\mathcal{Z}(f_1)$ and $\mathcal{Z}(f_2)$ are infinite sets, but $|\mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)| \leq d^2$. It follows that there exist π and π' so that $(f_1(\pi), f_2(\pi)) = (1, 0)$ and $(f_1(\pi'), f_2(\pi')) = (0, 1)$. Now take a curve $\gamma(t) \in \mathbb{R}^2$ and so that $\gamma(0) = \pi$, $\gamma(1) = \pi'$ and $\gamma(2) = -\pi$ and $\gamma(t) \notin \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$, and let $h(t) = (f_1(\gamma(t)), f_2(\gamma(t)))$. We have $h(0) = (1, 0)$, $h(1) = (0, 1)$, $h(2) = (-1, 0)$ and $h(t) \neq (0, 0)$, so by continuity, each line through the origin contains some $h(t)$, $0 \leq t \leq 2$. □

The hypotheses of Theorem 3.3(3) applies in Hilbert's original construction, with $d = 3$. We show in Example 3.1 below that $R_f \neq \mathbb{R}^r$ in general, and combine this with Theorem 3.3(2) to give one instance of a positive form in a $I_{1,d}^2(A)$ which is not sos.

Theorem 3.4 (Hilbert's Method)**.** *Suppose a finite set $A \subset \mathbb{R}^n$ is such that $I_{1,d}(A)$ has basis $\{f_1, \dots, f_s\}$, where $\tilde{A}$ is finite, A is full and $f = \sum_j f_j^2$ has no zeros at infinity. Further, suppose there exists $g \in I_{2,2d}(A) \setminus I_{1,d}^2(A)$ so that $g(w) > 0$ for each $w \in \tilde{A}$. Then there exists $c > 0$ so that*

$$p_c = \sum_{j=1}^{s} f_j^2 + cg \tag{3.3}$$

is positive and not sos.

Proof. In the notation of Lemma 3.1, let $V_1 = A$ and $V_2 = \tilde{A}$. Since f has no zeros at infinity, $\deg f = 2d$, and A is full, the hypotheses of Lemma 3.1 are satisfied. Thus there exists $c > 0$ so that p_c is positive, and since $p_c \notin I_{1,d}^2(A)$, it is not sos by Theorem 3.2. □

Remarks.

(1) If $\tilde{A} = \emptyset$, then the Perturbation Lemma can be applied to $(f, \pm g)$ for both signs, so that $f \pm cg$ is positive for some $c > 0$ and both choices of sign.
(2) In any particular case, the condition that f is round at $v \in V_1$ can be relaxed in the Perturbation Lemma, so long as a stronger condition is imposed on g to insure that $f + \alpha g$ is positive in some punctured neighborhood $\mathcal{N}_v$ of v.
(3) Since Hilbert's Method applies to any basis of $I_{1,d}(A)$, we may replace $\sum_j f_j^2$ by any positive definite quadratic form in the f_j's.
(4) Hilbert's original sextic contradiction follows from $(\phi\psi)(\pi_9) \neq 0$, which implies that $\phi\psi \in I_{2,6}(A) \smallsetminus I_{1,3}^2(A)$.
(5) Theorem 4.3 covers a situation in which $\tilde{A} = \emptyset$, but that $I_{2,2d}(A) \smallsetminus I_{1,d}^2(A)$ is non-empty, so Hilbert's Method still applies.

Example 3.1. We revisit Example 1.1, keeping the notation of (1.6). It is easy to check that

$$\{F_1^2, F_2^2, F_3^2, F_1F_2, F_1F_3, F_2F_3\} \tag{3.4}$$

is linearly independent, so that Theorem 3.3 applies. Let

$$Q(u_1, u_2, u_3) = 5u_1^2 + 5u_2^2 + 5u_3^2 - 6u_1u_2 - 6u_1u_3 - 6u_2u_3; \tag{3.5}$$

evidently, Q is not a psd quadratic form. We show now (in two ways) that

$$T := Q(F_1, F_2, F_3) \tag{3.6}$$

is psd; note that T is not sos by Theorem 3.3(1).

Let

$$\begin{aligned} P(v_1, v_2, v_3) &:= v_1^4 + v_2^4 + v_3^4 - 2v_1^2v_2^2 - 2v_1^2v_3^2 - 2v_2^2v_3^2 \\ &= (v_1 + v_2 + v_3)(v_1 + v_2 - v_3)(v_1 - v_2 + v_3)(v_1 - v_2 - v_3). \end{aligned} \tag{3.7}$$

A computation shows that

$$P(F_1, F_2, F_3) = (x^2 - y^2)^2(x^2 - z^2)^2(y^2 - z^2)^2 \tag{3.8}$$

is psd, hence $R_F \subseteq \{(x, y, z) : P(x, y, z) \geq 0\}$. We claim that that $Q \geq 0$ on R_F and so T is psd by Theorem 3.3(2). Since

$$5u_1^2 + 5u_2^2 + 5u_3^2 - 6u_1u_2 - 6u_1u_3 \tag{3.9}$$

is psd, if $\bar{u}_2\bar{u}_3 < 0$, say, then $Q(\bar{u}_1, \bar{u}_2, \bar{u}_3) \geq 0$. By symmetry, it follows that $Q(v_1, v_2, v_3) \geq 0$ unless the v_i's have the same sign, and it suffices to suppose $v_1 \geq v_2 \geq v_3 \geq 0$. The first three linear factors of P in (3.7) are always positive, so $P(v_1, v_2, v_3) \geq 0$ if and only if $v_1 = v_2 + v_3 + t$ with $t \geq 0$. Since

$$Q(v_2 + v_3 + t, v_2, v_3) = 4(v_2 - v_3)^2 + t(4v_2 + 4v_3 + 5t), \tag{3.10}$$

the claim is verified.

The second proof is direct. We note that T is symmetric:

$$T(x, y, z) = 5\sum^{6} x^4y^2 + 6\sum^{3} x^4yz + 6\sum^{3} x^3y^3 - 6\sum^{6} x^3y^2z - 30x^2y^2z^2. \tag{3.11}$$

A calculation shows that

$$\begin{aligned} 2(x^2+y^2+z^2-xy-xz-yz)T(x,y,z) &= (x-y)^4(xy+3xz+3yz+z^2)^2 \\ &+(x-z)^4(xz+3xy+3yz+y^2)^2+(y-z)^4(yz+3xy+3xz+x^2)^2, \end{aligned} \tag{3.12}$$

so T is psd. Although $|\mathcal{Z}(T)| = 7$, the zeros at $(1,1,-1),(1,-1,1),(-1,1,1)$ are not round. In fact, $T(1+t,1-t,-1) = 48t^4+4t^6$, etc. These singularities are useful in constructing the representation (3.12).

4. Two applications of Hilbert's Method to ternary sextics

In this section we show that Robinson's simplification of Hilbert's Method works in general. By Theorem 2.2(5), the assumption that no three of the nine points are on a line and no six are on a quadratic is equivalent to saying that no $\alpha f_1 + \beta f_2$ is reducible. Theorem 4.1 removes this restriction. In Theorem 4.3, we show that Hilbert's Method also applies to the set of ternary sextics which share seven zeros, no four in a line, no seven on a quadratic.

Theorem 4.1. *Suppose $f_1(x,y)$ and $f_2(x,y)$ are two relatively prime real cubics with exactly nine distinct real common zeros. Then Hilbert's Method applies to any subset A of eight of the common zeros.*

Proof. Lemma 2.4 shows that if $A = \{\pi_1,\dots,\pi_8\}$ is copacetic, as is assumed here, then $\tilde{A} = \{\pi_9\}$ and A is full. It follows from (2.5) that $\dim I_{2,6}(A) \ge \binom{8}{2} - 3\cdot 8 = 4$. Since $I^2_{1,3}(A)$ is spanned by $\{f_1^2, f_1f_2, f_2^2\}$, there exists $0 \ne g \in I_{2,6}(A) \smallsetminus I^2_{1,3}(A)$. If we can show that $g(\pi_9) \ne 0$, then $\pm g(\pi_9) > 0$ for some choice of sign, and Theorem 3.4 applies.

Suppose to the contrary that $g(\pi_9) = 0$. Either g is singular at π_9, or there exists $(\alpha_1,\alpha_2) \ne (0,0)$ so that the tangents of g and $\alpha_1 f_1 + \alpha_2 f_2$ are parallel at π_9. Since the choice of basis for $I_{1,3}(A)$ was arbitrary, we may assume without loss of generality that $(\alpha_1,\alpha_2) = (1,0)$ from the beginning. In either case, $\mathcal{I}_{\pi_9}(f_1,g) \ge 2$, so

$$\sum_{j=1}^{9} \mathcal{I}_{\pi_j}(f_1,g) \ge 2\cdot 9 = \deg(f_1)\cdot\deg(g). \tag{4.1}$$

Since f_1 is a real cubic, there exists $\pi_0 \notin A \cup \tilde{A}$ so that $f_1(\pi_0) = 0$ and, necessarily, $f_2(g_0) \ne 0$. Now let

$$\tilde{g} = g - \frac{g(\pi_0)}{f_2^2(\pi_0)} f_2^2, \tag{4.2}$$

so that $\tilde{g}(\pi_0) = 0$. Observe that $\tilde{g} \in I_{2,6}(A) \smallsetminus I^2_{1,3}(A)$, and g and $\tilde{g}$ agree to second-order at π_9. In particular, they are either both singular or have the same tangents. Thus, we may replace g by $\tilde{g}$ for purposes of the argument, and assume that $g(\pi_0) = 0$. Combining $\mathcal{I}_{\pi_0}(f_1,g) \ge 1$ with (4.1), we see that f_1 and g have a common factor by Bezout. Let $d = \deg(\gcd(f_1,g))$.

If $d = 3$, then $g = f_1 k$ for some cubic k. Since g is singular on A and f_1 is singular at no point of A, we must have $k \in I_{1,3}(A)$, so that $g \in I_{1,3}^2(A)$, a contradiction. (Under Hilbert's restrictions, f_1 is irreducible, so this is the only case.)

Suppose $d = 2$ and write $f_1 = \ell q$ and $g = pq$, where ℓ is linear, q is quadratic and p is quartic and ℓ and p are relatively prime. Then $\ell = 0$ on exactly three of the π_i's. After reindexing, there are two cases: either $\ell = 0$ on $\{\pi_1, \pi_2, \pi_3\}$ or $\ell = 0$ on $\{\pi_1, \pi_2, \pi_9\}$, with $q = 0$ on the complementary sets. In the first case, $q(\pi_i) \neq 0$ for $i = 1, 2, 3$, so p is singular at these three points and $\mathcal{I}_{\pi_1}(\ell, p) + \mathcal{I}_{\pi_2}(\ell, p) + \mathcal{I}_{\pi_3}(\ell, p) \geq 6 > 1 \cdot 4$. Since ℓ and p are relatively prime, this is a contradiction by Bezout. In the second case, p is still singular at π_1, π_2 and $q(\pi_9) \neq 0$, so $p(\pi_9) = 0$ and $\mathcal{I}_{\pi_1}(\ell, p) + \mathcal{I}_{\pi_2}(\ell, p) + \mathcal{I}_{\pi_9}(\ell, p) \geq 5 > 2 + 2 + 1$, another contradiction.

Finally, suppose $d = 1$ and write $f_1 = \ell q$ and $g = \ell p$, where ℓ is linear, q is quadratic and p is quintic and q and p are relatively prime. With either case for ℓ as above, $\ell \neq 0$ and p is singular at $\pi_4, \dots, \pi_8$ and $\mathcal{I}_{\pi_4}(q, p) + \cdots + \mathcal{I}_{\pi_8}(q, p) \geq 10 = 2 \cdot 5$. In the first case, $\ell(\pi_9) \neq 0$, so $\mathcal{I}_{\pi_9}(q, p) \geq 1$; in the second case, $\ell(\pi_3) \neq 0$, so $\mathcal{I}_{\pi_3}(q, p) \geq 2$. In either case Bezout implies that q and p are not relatively prime, and this contradiction completes the proof. □

It is possible for g and the f_i's to have a common factor, provided it does not contain π_9. This happens in Robinson's example: $f_1 = x(x^2 - 1)$, $f_2 = y(y^2 - 1)$ and $g = (x^2 - 1)(y^2 - 1)(1 - x^2 - y^2)$.

Corollary 4.2. *If A is copacetic, then there exists a positive sextic polynomial $p(x, y)$ so that $A \subseteq \mathcal{Z}(p)$ and p is not sos.*

Theorem 4.3. *Suppose $A = \{\pi_1, \dots, \pi_7\} \subset \mathbb{R}^2$, with no four π_i's in a line and not all seven on one quadratic. Then Hilbert's Method applies to A.*

Proof. It follows from Lemma 2.6 that A is full and $\tilde{\mathcal{A}} = \emptyset$. We have $\dim I_{1,3}(A) = 3$, so $\dim I_{1,3}^2(A) \leq 6$, but by (2.5), $\dim I_{2,6}(A) \geq \binom{8}{2} - 7 \cdot \binom{3}{2} = 7$. Thus there exists $g \in \dim I_{2,6}(A) \smallsetminus I_{1,3}^2(A)$ and since $\tilde{\mathcal{A}} = \emptyset$, Hilbert's Method can be applied. □

Theorem 4.3 is implemented in Examples 1.1 and 6.3.

Corollary 4.4. *If A is a set of seven points in $\mathbb{R}^2$, no four on a line and not all on a quadratic, then there exists a positive sextic polynomial $p(x, y)$ so that $A \subseteq \mathcal{Z}(p)$ and p is not sos.*

5. Psd and sos sections

We now consider $I_{2,6}(\mathcal{A}) \cap P_{3,6}$ and $I_{2,6}(\mathcal{A}) \cap \Sigma_{3,6}$ in detail. Our motivation is that $P_{3,6}$ and $\Sigma_{3,6}$ lie in $\mathbb{R}^{28}$ and are difficult to visualize. These two sections, in general, lie in $\mathbb{R}^4$, and thus are more comprehensible. We work in the homogeneous case.

Theorem 5.1. *In the notation of Theorem 4.1, suppose*

$$P = c_1 F_1^2 + 2c_2 F_1 F_2 + c_3 F_2^2 + c_4 G. \tag{5.1}$$

If P is sos, then $c_4 = 0$. If $c_4 = 0$, then P is sos if and only if P is psd if and only if $c_1 \geq 0$, $c_3 \geq 0$ and $c_1 c_3 \geq c_2^2$.

Proof. These are Theorems 3.2 and 3.3(1),(2) in the homogeneous case. □

Because G is only defined modulo $I_{1,d}^2(\mathcal{A})$, it is difficult to make any general statements about the circumstances under which P is psd. However, one can identify the possible zeros of P.

Theorem 5.2. *Suppose $P = c_1F_1^2 + 2c_2F_1F_2 + c_3F_2^2 + c_4G$ is psd, where $c_4 \neq 0$ and let J be the Jacobian of F_1, F_2 and G. Then*

$$\mathcal{Z}(P) \subseteq \mathcal{Z}(F_1) \cup \mathcal{Z}(F_2) \cup \mathcal{Z}(J). \tag{5.2}$$

Proof. If $P(a) = 0$ and $(F_1(a), F_2(a)) \neq (0, 0)$, then P and $(F_1(a)F_2 - F_2(a)F_1)^2$ are linearly independent sextics which are both singular at a. Thus the Jacobian of $(F_1^2, F_1F_2, F_2^2, G)$, when evaluated at a, has rank ≤ 2. In particular, the 3×3 minor omitting F_1F_2 vanishes; this minor reduces to $4F_1F_2J$. □

A maximal perturbation might not lead to a new zero, but rather to a greater singularity at a pre-existing zero; see Example 3.1.

In the special case of Robinson's example, we are able to give a much more precise description of these sections. Let $A = \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. A routine calculation shows that $f_1(x, y) = x^3 - x$ and $f_2(x, y) = y^3 - y$ span $I_{1,3}(A)$ and f_1^2, f_1f_2, f_2^2 and $g(x, y) = (x^2 - 1)(y^2 - 1)(1 - x^2 - y^2)$ span $I_{2,6}(A)$. It is convenient to replace g with $f_1^2 + f_2^2 + g$, which homogenizes to R.

Consider now

$$\begin{aligned} \Phi[c_1, c_2, c_3, c_4](x, y, z) &:= c_1F_1^2 + 2c_2F_1F_2 + c_3F_2^2 + c_4R \\ &= c_1(x^3 - xz^2)^2 + 2c_2(x^3 - xz^2)(y^3 - yz^2) + c_3(y^3 - yz^2)^2 \\ &+c_4(x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2). \end{aligned} \tag{5.3}$$

This is the general form of $\Phi \in I_{2,6}(\mathcal{A})$, where

$$\mathcal{A} = \{(\pm 1, 0, 1), (0, \pm 1, 1), (\pm 1, \pm 1, 1)\}. \tag{5.4}$$

Theorem 5.1 implies that $\Phi(c_1, c_2, c_3, 0)$ is psd if and only if it is sos if and only if $c_1, c_3, c_1c_3 - c_2^2 \geq 0$, so we may henceforth assume that $c_4 \neq 0$.

We begin our discussion of positivity with a collection of short observations.

Lemma 5.3. *Suppose $\Phi[c_1, c_2, c_3, c_4]$ is psd. Then the following are true:*

(1) $c_4 \geq 0$;
(2) $\Phi[c_1, -c_2, c_3, c_4]$ *and* $\Phi[c_3, c_2, c_1, c_4]$ *are psd;*
(3) $\Gamma(x, y) := (c_1 + c_4)x^6 - c_4x^4y^2 + 2c_2x^3y^3 - c_4x^2y^4 + (c_3 + c_4)y^6$ *is psd;*
(4) $\Phi[c_1, 0, c_3, c_4]$ *is psd.*

Proof. The first observation follows from evaluation at $(0, 0, 1)$, the second from taking $(x, y, z) \mapsto (x, -y, z), (y, x, z)$, the third from setting $z = 0$, and the fourth from averaging the psd forms $\Phi[c_1, \pm c_2, c_3, c_4]$. □

In view of Lemma 5.3(1), it suffices now to assume $c_4 = 1$. For $t > 0$, let

$$\alpha(t) = \frac{2t^2 + t^4}{3}, \qquad \beta(t) = \frac{1 + 2t^2}{3t^4}, \qquad \gamma(t) = \beta(\alpha^{-1}(t)). \tag{5.5}$$

Then $\beta(t) = \alpha(t^{-1})$, and as t increases from 0 to ∞, so does $\alpha(t)$, monotonically.

Lemma 5.4. *For $t > 0$, the sextic $\Phi_t(x, y) := \alpha(t)x^6 - x^4y^2 - x^2y^4 + \beta(t)y^6$ is positive with zeros at $(1, \pm t)$.*

Proof. A computation shows that

$$\Phi_t(x, y) = \frac{(t^2x^2 - y^2)^2((t^4 + 2t^2)x^2 + (2t^2 + 1)y^2)}{3t^4}. \tag{5.6}$$

□

Let $K = \{(x, y) : x > 0,\ y \ge \gamma(x))\}$ denote the region lying above the curve $C = \{(\alpha(t), \beta(t)) : t > 0\}$, which partially parametrizes the quartic curve $27x^2y^2 - 18xy - 4x - 4y - 1 = 0$. For this reason,

$$\gamma(x) = \frac{2 + 9x + 2(1 + 3x)^{3/2}}{27x^2}. \tag{5.7}$$

Lemma 5.5. *The binary sextic $\Psi(x, y) = rx^6 - x^4y^2 - x^2y^4 + sy^6$ is psd if and only if $(r, s) \in K$.*

Proof. A necessary condition for the positivity of Ψ is $r > 0$. Let $t_0 = \alpha^{-1}(r) > 0$, so

$$\Psi(x, y) = \Phi_{t_0}(x, y) + (s - \gamma(t_0))y^6. \tag{5.8}$$

If $(r, s) \in K$; that is, if $s \ge \gamma(t_0)$, then Lemma 5.4 and (5.8) show that Ψ is positive. Conversely, $\Psi(1, t_0) = (s - \gamma(t_0))t_0^6$, so if Ψ is positive, then $s \ge \gamma(t_0)$. □

Theorem 5.6. *The sextic $\Phi[c_1, 0, c_3, 1]$ is psd if and only if $(1 + c_1, 1 + c_3) \in K$.*

Proof. One direction is clear by Lemmas 5.3(3) and 5.5. For the converse, note that $(1 + c_1, 1 + c_3) \in K$ if and only if $1 + c_1 = \alpha(t_0)$ implies $1 + c_3 \ge \beta(t_0)$. In other words, we need to show that, with $\lambda = 1 + c_3 - \beta(t_0)$,

$$\Phi[\alpha(t_0) - 1, 0, \beta(t_0) + \lambda - 1, 1] = \Phi[\alpha(t_0) - 1, 0, \beta(t_0) - 1, 1] + \lambda F_2^2 \tag{5.9}$$

is psd whenever $\lambda \ge 0$. To this end, for $t > 0$, define

$$R_t(x, y, z) := \Phi[\alpha(t) - 1, 0, \beta(t) - 1, 1](x, y, z) =$$
$$\left(\frac{t^4 + 2t^2 - 3}{3}\right)^2 F_1^2(x, y, z) + \left(\frac{1 + 2t^2 - 3t^4}{3t^4}\right) F_2^2(x, y, z) + R(x, y, z). \tag{5.10}$$

Note that $R_1 = R$, $R_{1/t}(x, y, z) = R_t(y, x, z)$ and that for $t \ne 1$, the coefficients of F_1^2 and F_2^2 have opposite sign. The following algebraic identity gives Q_tR_t as a sum of four squares for a psd quadratic form $Q_t(x,y)$, which implies that R_t is psd, and completes the proof.

$$\begin{aligned}(5.11)\qquad &((2t^4+t^2)x^2+(t^2+2)y^2)3t^4R_t(x,y,z)\\ &=3t^6(1+2t^2)x^2z^2(x^2-z^2)^2+3t^4(2+t^2)y^2z^2(y^2-z^2)^2\\ &\quad+t^2(t^2-1)^2x^2y^2(t^2x^2-y^2+(1-t^2)z^2)^2\\ &\quad+(2+t^2)(1+2t^2)(t^4x^4-y^4-t^4x^2z^2+y^2z^2)^2.\end{aligned}$$

□

For $t=1$, (5.11) essentially appears in [22, p.273]. In view of the foregoing, $\mathcal{Z}(R_t)$ contains, at least, $\mathcal{A}\cup\{(1,\pm t,0)\}$. If $R_t(a,b,c)=0$, then each of the squares in (5.9) vanishes. In particular, $cF_1(a,b,c)=cF_2(a,b,c)=0$, so either $c=0$ or $(a,b,c)\in\mathcal{A}\cup\{(0,0,1)\}$. These cases have already been discussed and we may conclude that $\mathcal{Z}(R_t)=\mathcal{A}\cup\{(1,\pm t,0)\}$ and $|\mathcal{Z}(R_t)|=10$.

We now complete our discussion of the psd case.

Theorem 5.7. *The sextic $\Phi[c_1,c_2,c_3,1]$ is psd if and only if $(c_1,c_3)\in K$ and $|c_2|\le\sigma(c_1,c_3)$ for a function $\sigma(c_1,c_3)\ge 0$ defined on K (see (5.15)). If $c_2=\pm\sigma(c_1,c_3)$, then $\Phi[c_1,c_2,c_3,1]=R_t+\alpha(t^3F_1\pm F_2)^2$ (for suitable t,α and choice of sign).*

Proof. First, suppose $\Phi[c_1,c_2,c_3,1]$ is psd. Then $(c_1,c_3)\in K$ by Lemma 5.3(4) and Theorem 5.6. Setting $z=0$, we obtain the psd binary sextic

$$(5.12)\qquad \Gamma(x,y)=(1+c_1)x^6-x^4y^2+2c_2x^3y^3-x^2y^4+(1+c_3)y^6.$$

Define t_0 so that $1+c_1=\alpha(t_0)$. If $1+c_3=\beta(t_0)$, then $\Gamma(1,\pm t_0)=\pm c_2t_0^3$ implies that $c_2=0$; otherwise, $(1+c_1,1+c_3)$ lies strictly above C. Suppose now that $c_2<0$ without loss of generality (taking $y\mapsto -y$ if necessary), so that for $u>0$,

$$(5.13)\qquad \Gamma(1,-u)>\Gamma(1,u)=(1+c_1)-u^2-2|c_2|u^3-u^4+(1+c_3)u^6\ge 0.$$

Let $\Psi(u)=(1+c_1)u^{-3}-u^{-1}-u+(1+c_3)u^3$, so that

$$(5.14)\qquad 0\le u^{-3}\Gamma(1,u)=u^3(\Psi(u)-2|c_2|).$$

Now define

$$(5.15)\qquad \sigma(c_1,c_3):=\min_{u>0}\tfrac{1}{2}\Psi(u)=\tfrac{1}{2}\Psi(v);$$

since $\Psi(u)\to\infty$ as $t\to 0$ or $t\to\infty$, the minimum exists. It follows that $|c_2|\le\sigma(c_1,c_3)$. (Although $\sigma(c_1,c_3)$ is computable explicitly, it is quite complicated. For example, $2\sigma(1,0)$ is the unique real positive root of the sextic $729x^6-22518x^4+182774x^2-111392$, approximately .81392.)

We must now show that every $\Phi[c_1,\pm\sigma(c_1,c_3),c_3,1]$ is psd. Since $\Psi'(v)=0$, we have the system

$$\begin{aligned}(5.16)\qquad &\sigma(c_1,c_3)=\frac{1}{2}\left((1+c_3)v^3-v-v^{-1}+(1+c_1)v^{-3}\right);\\ &3(1+c_3)v^2-1+v^{-2}-3(1+c_1)v^{-4}=0.\end{aligned}$$

A calculation shows that (5.17) implies

$$\Phi[c_1, -\sigma(c_1,c_3), c_3, 1] = R_v + \mu(v^3F_1 - F_2)^2, \tag{5.17}$$

where R_v is defined in (5.10) and

$$\mu = \frac{3(1+c_3)v^4 - (2v^2+1)}{3v^4}. \tag{5.18}$$

We are done if we can show that $\mu \ge 0$. By hypothesis, both sides of (5.17) vanish at $(1, v, 0)$. But if we evaluate (5.17) at $(1, -v, 0)$, we have already seen that the left-hand side is positive, and the right-hand side is $0 + 4v^6\mu$, hence $\mu > 0$. □

If $\Phi[c_1, c_2, c_3, 1](a, b, c) = 0$, then Theorem 5.2 implies that $(a, b, c) \in \mathcal{A}$ or

$$abc(a^2-c^2)(b^2-c^2)(a^2-ab+b^2-c^2)(a^2+ab+b^2-c^2) = 0. \tag{5.19}$$

This includes the new zeros of R_t on $c = 0$ but also the extraneous points (a, b, c) for which $a^2 + b^2 - c^2 = \pm ab$, which never appear non-trivially as zeros for any R_t.

To sum up, we have described sections of the two cones

$$\begin{aligned} P &= \{(c_1,c_2,c_3,c_4) : c_1F_1^2 + 2c_2F_1F_2 + c_3F_2^2 + c_4R \in P_{3,6}\} \subseteq \mathbb{R}^4, \\ \Sigma &= \{(c_1,c_2,c_3,c_4) : c_1F_1^2 + 2c_2F_1F_2 + c_3F_2^2 + c_4R \in \Sigma_{3,6}\} \subseteq \mathbb{R}^4; \end{aligned} \tag{5.20}$$

at $c_4 = 0$ and at $c_4 = 1$. In the first case, the sections coincide and are literally a right regular cone. In the second case Σ disappears, and if we think of (c_1, c_3) as lying in a plane and c_2 as the vertical dimension, then P is a kind of clam-shell, with a convex boundary curve C lying in the plane and rays emanating at varying angles from the points on the boundary.

6. More ternary sextic examples

Example 6.1. Let $A = \{\pi_i\} = \{(a_i, b_i)\}$ be given by $\pi_1 = (-1, 0), \pi_2 = (-1,-1), \pi_3 = (0, 1), \pi_4 = (0,-1), \pi_5 = (1, 0), \pi_6 = (2, 2), \pi_7 = (2,-2), \pi_8 = (1,-3)$. By looking at the 3×3 minors of the matrix with rows $(1, a_i, b_i)$ and the 6×6 minors of the matrix with rows $(1, a_i, b_i, a_i^2, a_ib_i, b_i^2)$, one can check that no three of the π_i's lie in a line, and no six on a quadratic. According to Mathematica, $I_{1,3}(A)$ is spanned by

$$\begin{aligned} f_1(x,y) &= -42 + 49x + 42x^2 - 49x^3 - 20y - 38xy + 4x^2y + 42y^2 + 20y^3, \\ f_2(x,y) &= -22 + 31x + 22x^2 - 31x^3 - 12y - 18xy + 22y^2 + 4xy^2 + 12y^3, \end{aligned} \tag{6.1}$$

and $\tilde{A} = \{(\frac{2516}{1297}, \frac{4991}{2594})\}$, so A is copacetic. In Hilbert's notation, $\phi(x, y) = x^2 - xy + y^2 - 1$ and

$$\begin{aligned} \psi(x,y) = &-6136 + 2924x + 5784x^2 - 2924x^3 + 352x^4 \\ &-2804y - 7000xy + 6299x^2y - 1049x^3y + 5818y^2 \\ &-7803xy^2 + 1811x^2y^2 + 2804y^3 - 1402xy^3 + 318y^4. \end{aligned} \tag{6.2}$$

It follows that there exists $c > 0$ so that $f_1^2 + f_2^2 + c\phi\psi$ is psd and not sos. We do not offer an estimate for c.

In the examples in the rest of this section, the symmetries are more clearly seen when the polynomials are homogenized.

Example 6.2. We present one of several ways to generalize Robinson's original set of eight points. For $t > 0$, let

$$A_t = \{(\pm 1, \pm 1), (\pm t, 0), (0, \pm t)\}. \tag{6.3}$$

It is not hard to see that A_t is copacetic (with ninth point $(0,0)$) unless $t = \sqrt{2}$, in which case A_t lies on $x^2 + y^2 = 2$. Since $A_t \mapsto A_{2/t}$ under the invertible map $(x, y) \mapsto ((x + y)/t, (x - y)/t)$, we may assume $0 < t < \sqrt{2}$. After homogenizing to $\mathcal{A}_t$, we note that a basis of $I_{1,3}(\mathcal{A}_t)$ is given by

$$\{F_{1,t}, F_{2,t}\} = \{x(x^2 + (t^2 - 1)y^2 - t^2z^2), y((t^2 - 1)x^2 + y^2 - t^2z^2)\} \tag{6.4}$$

and that $\tilde{\mathcal{A}}_t = (0, 0, 1)$. It is not hard to see that

$$G_t(x, y, z) = (x^2 + (t^2 - 1)y^2 - t^2z^2)((t^2 - 1)x^2 + y^2 - t^2z^2)(-x^2 - y^2 + t^2z^2) \tag{6.5}$$

is singular on $\mathcal{A}_t$ and is positive on $(0, 0, 1)$. (Robinson's example is recovered by setting $t = 1$.)

Consider now

$$\begin{aligned} P_t := F_{1,t}^2 + F_{2,t}^2 + 1 \cdot G_t^2 = (2 - t^2)(x^6 - x^4y^2 - x^2y^4 + y^6) + \\ (2t^4 - 3t^2)(x^4 + y^4)z^2 + (6t^2 - 4t^4 + t^6)x^2y^2z^2 - t^6(x^2z^4 + y^2z^4 - z^6). \end{aligned} \tag{6.6}$$

The proof that P_t is psd follows from the identity

$$\begin{aligned} (x^2 + y^2)P_t = (2 - t^2)(x^2 - y^2)^2(x^2 + y^2 - t^2z^2)^2 + \\ t^2x^2z^2(x^2 + (t^2 - 1)y^2 - t^2z^2)^2 + t^2y^2z^2((t^2 - 1)x^2 + y^2 - t^2z^2)^2. \end{aligned} \tag{6.7}$$

For $t = 1$, this formula is in [22]. For $t = 0, \sqrt{2}$, P_t is sos. It is not hard to show that if $0 < t < \sqrt{2}$, then $\mathcal{Z}(P_t) = \mathcal{A}_t \cup \{(1, \pm 1, 0)\}$ has 10 points and P_t is not sos.

Example 6.3. Let

$$\mathcal{A} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \tag{6.8}$$

It is again simple to show that $I_{1,3}(\mathcal{A})$ is spanned by

$$F_1(x, y, z) = xy(x - y), \quad F_2(x, y, z) = yz(y - z), \quad F_3(x, y, z) = zx(z - x), \tag{6.9}$$

and that

$$G(x, y, z) = xyz(x - y)(y - z)(z - x) \tag{6.10}$$

is in $I_{2,6}(\mathcal{A}) \smallsetminus I_{1,3}^2(\mathcal{A})$. Accordingly, by Theorem 4.3, there exists $c > 0$ so that

$$U_c(x, y, z) = x^2y^2(x-y)^2 + y^2z^2(y-z)^2 + z^2x^2(z-x)^2 + cxyz(x-y)(y-z)(z-x) \tag{6.11}$$

is psd and not sos. Since $U_c(x,y,z) \ge 0$ whenever $xyz = 0$, we define

$$Q_c(x,y,z) := \frac{U_c(x,y,z)}{x^2y^2z^2} \tag{6.12}$$
$$= \frac{(x-y)^2}{z^2} + \frac{(y-z)^2}{x^2} + \frac{(z-x)^2}{y^2} + c\left(\frac{x-y}{z}\right)\left(\frac{y-z}{x}\right)\left(\frac{z-y}{x}\right).$$

It is now sensible to make a substitution: let

$$u := \frac{x-y}{z}; \quad v := \frac{y-z}{x}; \quad w := \frac{z-x}{y}. \tag{6.13}$$

Then $Q_c = u^2 + v^2 + w^2 + cuvw$; somewhat surprisingly, $\{u, v, w\}$ is not algebraically independent: in fact,

$$u + v + w + uvw = 0. \tag{6.14}$$

An application of Lagrange multipliers to minimize Q_c, subject to (6.14), shows that two of $\{u, v, w\}$ are equal; by symmetry, we may take $u = v$, so that $w = -\frac{2u}{u^2+1}$, and

$$Q_c\left(u, u, -\tfrac{2u}{u^2+1}\right) = \frac{2u^2(u^4 + 2u^2 + 3 - cu(1+u^2))}{(1+u^2)^2}. \tag{6.15}$$

Let $\sigma = \sqrt{\sqrt{2}+1}$. A little calculus shows that the numerator is psd provided $|c| \le c_0 := 4/\sigma$, with $Q_{c_0} = 0$ when $u = \pm\sigma$. Solving back for (x,y,z) yields, up to multiple, that $(1+\sigma, 1+\sigma^2, 1-\sigma)$ and its cyclic images are in $\mathcal{Z}(U_{c_0})$, together with (6.8). Here, $|\mathcal{Z}(U_{c_0})| = 10$.

Example 6.4. The Motzkin form M cannot be derived directly from Theorems 4.1 or 4.3 because $|\mathcal{Z}(M)| = 6$; however, M has zeros at $(1,0,0)$ and $(0,1,0)$ which vanish to the sixth order in the z-direction. It is possible to construct psd ternary sextics M_t with $|\mathcal{Z}(M_t)| = 10$ for $t > 0$ and such that $M_t \to M$ as $t \to 0$. We do this with an Ansatz by supposing that there is a non-zero even ternary sextic which is symmetric in (x,y) and lies in $I_{2,6}(\mathcal{A}_t)$ for

$$\mathcal{A}_t = \{(1,0,0), (0,1,0), (1,0,\pm t), (0,1,\pm t), (1,\pm 1,\pm 1)\}. \tag{6.16}$$

Although these impose 30 equations on the 28 coefficients of a ternary sextic, there is some redundancy, and it can be verified that

$$M_t(x,y,z) = (1-2t^2)(x^4y^2 + x^2y^4) + t^4(x^4z^2 + y^4z^2) \tag{6.17}$$
$$-(3 - 8t^2 + 2t^4)x^2y^2z^2 - 2t^2(x^2+y^2)z^4 + z^6$$

satisfies this criterion. It is not clear that M_t is psd; in fact, it is not psd when $t^2 > 1/2$. We note that $M_0 = M$ and M_t is a square when $t^2 = 1/2$. The proof that M_t is psd for $t^2 < 1/2$ is given by an sos representation of Q_tM_t:

$$(x^2+y^2)M_t(x,y,z) = (1-2t^2)x^2y^2(x^2+y^2-2z^2)^2 + \tag{6.18}$$
$$y^2z^2(t^2(x^2-y^2) - (x^2-z^2))^2 + x^2z^2(t^2(y^2-x^2) - (y^2-z^2))^2.$$

This equation also shows that, at least when $t^2 < 1/2$, $\mathcal{Z}(M_t) = \mathcal{A}_t$. We may also derive M_t using Theorem 4.1, by first choosing any eight points in $\mathcal{A}_t$.

Example 6.5. Similarly, one can approach $S(x, y, z)$ by Ansatz and look for a cyclically symmetric even sextic S_t which is singular at

$$\mathcal{A}_t = \{(\pm t, 1, 0), (0, \pm t, 1), (1, 0, \pm t), (1, \pm 1, \pm 1)\}. \tag{6.19}$$

Again, although there is no reason to expect a non-zero solution, there is one:

$$\begin{aligned} S_t(x, y, z) &= t^4(x^6 + y^6 + z^6) + (1 - 2t^6)(x^4y^2 + y^4z^2 + z^4x^2) \\ &+(t^8 - 2t^2)(x^2y^4 + y^2z^4 + z^2x^4) - 3(1 - 2t^2 + t^4 - 2t^6 + t^8)x^2y^2z^2. \end{aligned} \tag{6.20}$$

We find that $t^8 S_{1/t}(x, y, z) = S_t(x, z, y)$, $S_0(x, y, z) = S(x, y, z)$ and $S_1(x, y, z) = R(x, y, z)$. The proof that S_t is psd follows from yet another algebraic identity:

$$\begin{aligned} (x^2 + y^2)S_t(x, y, z) &= (t^2x^4 + x^2y^2 - t^4x^2y^2 - t^2y^4 - x^2z^2 + t^4y^2z^2)^2 \\ &+y^2z^2(y^2 - x^2 + t^2(x^2 - z^2))^2 + t^4x^2z^2(y^2 - z^2 + t^2(x^2 - y^2))^2 \\ &+(t^2 - 1)^2x^2y^2((z^2 - x^2) + t^2(y^2 - z^2))^2. \end{aligned} \tag{6.21}$$

When $t = 1$, (5.11) and (6.21) coincide. This example was announced, without proof, in [21, p.261].

Robinson [22, p.273] observed that $(ax^2 + by^2 + cz^2)R(x, y, z)$ is sos, "at least if $0 \le a \le b + c$, $0 \le b \le a + c$, $0 \le c \le a + b$." We revisit this situation and simultaneously illustrate the method used to discover (5.11), (6.7), (6.18) and (6.21).

Theorem 6.1. *If $r, s, t \ge 0$, then $(r^2x^2 + s^2y^2 + t^2z^2)R(x, y, z)$ is sos if and only if $r \le s + t$, $s \le r + t$ and $t \le r + s$.*

Proof. It was shown in [7, p.569] (by a polarization argument) that an even sos polynomial F has an sos representation $F = \sum H_j^2$ in which each H_j^2 is even. Suppose

$$(r^2x^2 + s^2y^2 + t^2z^2)R(x, y, z) = \sum_{j=1}^{r} H_j^2(x, y, z) \tag{6.22}$$

is such an "even" representation. Then $\mathcal{Z}(R) \subseteq \mathcal{Z}(H_j)$ for the quartic H_j's (c.f. (5.4)). It follows that

$$\begin{aligned} H_j(x, y, z) &= c_{1j}xy(x^2 - y^2) + c_{2j}xz(x^2 - z^2) + c_{3j}yz(y^2 - z^2) \\ &+(c_{4j}(x^2 - z^2)(x^2 - y^2 + z^2) + c_{5j}(y^2 - z^2)(-x^2 + y^2 + z^2)). \end{aligned} \tag{6.23}$$

Each H_j^2 is even, so the only cross-terms which can appear in any H_j^2 are $c_{4j}c_{5j}$ and

$$\begin{aligned} (r^2x^2 + s^2y^2 + t^2z^2)R(x, y, z) &= \lambda_1x^2y^2(x^2 - y^2)^2 + \lambda_2x^2z^2(x^2 - z^2)^2 \\ &+\lambda_3y^2z^2(y^2 - z^2)^2 + \lambda_4(x^2 - z^2)^2(x^2 - y^2 + z^2)^2+ \\ &2\lambda_5(x^2 - z^2)(x^2 - y^2 + z^2)(y^2 - z^2)(-x^2 + y^2 + z^2) \\ &+\lambda_6(y^2 - z^2)^2(-x^2 + y^2 + z^2)^2, \end{aligned} \tag{6.24}$$

for λ_j's, defined by

$$\begin{aligned} \lambda_1 = \sum_j c_{1j}^2, \quad \lambda_2 = \sum_j c_{2j}^2, \quad \lambda_3 = \sum_j c_{3j}^2, \\ \lambda_4 = \sum_j c_{4j}^2, \quad \lambda_5 = \sum_j c_{4j}c_{5j}, \quad \lambda_6 = \sum_j c_{5j}^2. \end{aligned} \tag{6.25}$$

We solve for the λ_j in (6.24):

$$\lambda_1 = t^2, \quad \lambda_2 = s^2, \quad \lambda_3 = r^2, \quad \lambda_4 = r^2, \quad \lambda_6 = s^2, \quad \lambda_5 = (t^2 - r^2 - s^2)/2. \tag{6.26}$$

There exist c_{ij} to satisfy (6.25) and (6.26) if and only if

$$0 \le \lambda_4\lambda_6 - \lambda_5^2 = \frac{1}{4}(r+s-t)(r+t-s)(s+t-r)(r+s+t) \tag{6.27}$$

If, say, $r \ge s \ge t \ge 0$, then $r + s \ge t$ and $r + t \ge s$ automatically, and so (6.27) holds if and only if $s + t \ge r$. By symmetry, we see that (6.27) is true if and only if all three inequalities hold. □

7. Extremal psd ternary forms

In 1980, Choi, Lam and the author [6] studied $|\mathcal{Z}(F)|$ for $F \in P_{3,m}$. Let

$$\alpha(m) := \max\left(\frac{m^2}{4}, \frac{(m-1)(m-2)}{2}\right). \tag{7.1}$$

By Theorem 3.5 in [6], if $F \in P_{3,m}$, then $|\mathcal{Z}(F)| > \alpha(m)$ implies $|\mathcal{Z}(F)| = \infty$, and this occurs if and only if F is divisible by the square of an indefinite form. Let

$$B_{3,m} = \{\sup |\mathcal{Z}(F)| \ : \ F \in P_{3,m}, \ |\mathcal{Z}(F)| < \infty\}. \tag{7.2}$$

Then by Theorem 4.3 in [6],

$$\begin{aligned} \frac{m^2}{4} \le B_{3,m} \le \frac{(m-1)(m-2)}{2}; \\ B_{3,6k} \ge 10k^2, \quad B_{3,6k+2} \ge 10k^2 + 1, \quad B_{3,6k+4} \ge 10k^2 + 4. \end{aligned} \tag{7.3}$$

In particular, $B_{3,6} = 10$. Further, if $F \in P_{3,6}$, and $|\mathcal{Z}(F)| > 10$, then $|\mathcal{Z}(F)| = \infty$ and $F \in \Sigma_{3,6}$ is a sum of three squares (Theorem 3.7). If G is a ternary sextic and $|\mathcal{Z}(G)| = 10$, then one of $\pm G$ is psd and not sos (Corollary 4.8). We wrote (p.12): "it would be of interest to determine, if possible, *all* forms $p \in P_{3,6}$ with exactly 10 zeros. From a combinatorial point of view, it would already be of interest to determine (or classify) all configurations of 10-point sets $S \subset \mathbb{P}^2$ for which there exist $p \in P_{3,6}$ such that $S = \mathcal{Z}(p)$ ... The only known psd ternary sextic with 10 zeros is R." Sections five and six of this paper are inspired by this remark.

Lemma 7.1. *If $F \in P_{3,6}$ is reducible, then $F \in \Sigma_{3,6}$.*

Proof. If F has an indefinite factor H, then $F = H^2G$, where $G \in P_{3,2d} = \Sigma_{3,2d}$ for $2d \le 4$. If $F = F_1F_2$ for definite F_i, then $\deg F_i \le 4$ again implies $F \in \Sigma_{3,6}$. □

A form F in the closed convex cone $P_{n,m}$ is *extremal* if $F = G_1 + G_2$ for $G_j \in P_{n,m}$ implies that $G_j = \lambda_j F$ for $0 \le \lambda_j \in \mathbb{R}$. Equivalently, F is extremal if $F \ge G \ge 0$ implies $G = \lambda F$. The set of extremal forms in $P_{n,m}$ is denoted by $E(P_{n,m})$.

Theorem 7.2. *Suppose $F \in P_{3,6}$ and $|\mathcal{Z}(F)| = 10$. Then $F \in E(P_{3,6})$.*

Proof. Since $F \in \Delta_{3,6}$ by [6], Lemma 7.1 implies that F is irreducible. Suppose $F \ge G \ge 0$. Then F and G are both singular at the ten zeros of F, and since $10 \cdot 2^2 > 6 \cdot 6$, Bezout implies that F and G have a common factor. Thus $G = \lambda F$ and F is extremal. □

Theorems 5.1 and 5.7 imply that if $F \in E(P_{3,6})$ has Robinson's 8 zeros, then either $F = P_t \in \Delta_{3,6}$ for some $t > 0$ has ten zeros, or $F = (\alpha F_1 + \beta F_2)^2 \in E(\Sigma_{3,6})$.

We can use the Perturbation Lemma to put a strong restriction on those extremal forms which only have round zeros.

Theorem 7.3. *If $P \in E(P_{3,2d}) \cap \Delta_{3,2d}$ and all zeros of P are round, then $|\mathcal{Z}(P)| \ge \frac{(d+1)(d+2)}{2}$.*

Proof. Suppose P is psd, all its zeros are round, and $|\mathcal{Z}(P)| < \frac{(d+1)(d+2)}{2}$. Then there exists a non-zero $H \in I_{1,d}(\mathcal{Z}(P))$ and the Perturbation Lemma applies to $(P, \pm H^2)$. It follows that $P \pm cH^2$ is psd for some $c > 0$ and P is not extremal because

$$P = \tfrac{1}{2}(P - cH^2) + \tfrac{1}{2}(P + cH^2); \tag{7.4}$$

$P \ne \lambda H^2$ since P is not sos. □

Corollary 7.4. *If $p \in E(P_{3,6}) \cap \Delta_{3,6}$ and all zeros of P are round, then $|\mathcal{Z}(p)| = 10$.*

Lemma 7.5. *If $P \in P_{3,6}$, and $\mathcal{Z}(P)$ contains four points in a line or seven points on a quadratic, then $P \in \Sigma_{3,6}$.*

Proof. If $\mathcal{Z}(P)$ contains four points π_i on the line L, then since P is singular at its zeros, Bezout implies that L divides P and $P \in \Sigma_{3,6}$ by Lemma 7.1. Similarly, if $\mathcal{Z}(P)$ contains seven points π_i on the quadratic Q, then Bezout again implies that P is reducible. □

Theorem 7.6. *If $P \in E(P_{3,6}) \cap \Delta_{3,6}$ and all zeros of P are round, then P can be derived by Hilbert's Method using Theorem 4.3.*

Proof. Let A denote any subset of seven of the ten zeros of P. By Lemma 7.5, A meets the hypothesis of Theorem 4.3. □

Given positive $f \in \mathbb{R}_{n,2d}$ and $\pi \in \mathbb{R}^n$, let $E(f,\pi)$ denote the set of $g \in \mathbb{R}_{n,d}$ such that there exists a neighborhood $\mathcal{N}_g$ of π and $c > 0$ so that $f - cg^2$ is non-negative on $\mathcal{N}_g$.

Lemma 7.7. *$E(f,\pi)$ is a subspace of $\mathbb{R}_{n,d}$.*

Proof. Clearly, $g \in E(f,\pi)$ implies $\lambda g \in E(f,\pi)$ for $\lambda \in \mathbb{R}$. Suppose $g_1, g_2 \in E(f,\pi)$; specifically, $f - c_1 g_1^2 \ge 0$ on $\mathcal{N}_1$ and $f - c_2 g_2^2 \ge 0$ on $\mathcal{N}_2$, and let $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$ and $c = \min(c_1, c_2)$. The identity

$$f - \tfrac{c}{4}(g_1+g_2)^2 = \tfrac{1}{2}(f - cg_1^2) + \tfrac{1}{2}(f - cg_2^2) + \tfrac{c}{4}(g_1 - g_2)^2 \tag{7.5}$$

shows that $g_1 + g_2 \in E(f,\pi)$. □

If $f(\pi) > 0$, then $E(f,\pi) = \mathbb{R}_{n,d}$. Let

$$\delta(f,\pi) := \binom{n+d}{d} - \dim E(f,\pi) \tag{7.6}$$

measure the singularity of the zero of f at π; the argument of the Perturbation Lemma shows that $\delta(f,\pi) = 1$ if and only if f has a round zero at π. These definitions also apply in the obvious way to the homogeneous case.

Theorem 7.8. *If $P \in E(P_{3,2d}) \cap \Delta_{3,2d}$, then*

$$\delta(P) := \sum_{\pi \in \mathcal{Z}(P)} \delta(P,\pi) \ge \frac{(d+1)(d+2)}{2}. \tag{7.7}$$

Proof. If $f(\pi) > 0$, then $E(f,\pi) = \mathbb{R}_{n,d}$. Let

$$\mathcal{E} := \bigcap_{\pi \in \mathcal{Z}(P)} E(f,\pi). \tag{7.8}$$

Since

$$\dim \mathcal{E} \ge \frac{(d+1)(d+2)}{2} - \delta(P), \tag{7.9}$$

if (7.7) fails, then there exists $0 \ne H \in \mathcal{E}$. The argument of the Perturbation Lemma applies to $(P, \pm H^2)$, so that (7.4) holds for some $c > 0$, and P is not extremal. □

It can be checked that M has round zeros at $(1,\pm 1,\pm 1)$. Let $\pi = (1,0,0)$. If $M - cF^2$ is non-negative near $(1,0,0)$ for a ternary cubic F, then by the method of cages (see [8, §3]), x^3, x^2z, xz^2 cannot appear in F, whereas every other monomial is in $E(M,\pi)$, and so $\delta(M,\pi) = 3$. By symmetry, $\delta(M,(0,1,0)) = 3$, so that $\delta(M) = 4\cdot 1 + 2\cdot 3 = 10$. A similar calculation for S shows that it has round zeros at $(1,\pm 1,\pm 1)$ and that $\delta(S,e_i) = 2$ at the unit vectors e_i so $\delta(S) = 4\cdot 1 + 3\cdot 2 = 10$ as well. Examples 6.4 and 6.5 were constructed under a heuristic in which "coalescing" zeros explain higher-order singularities. These lead to a perhaps overly-optimistic conjecture:

Conjecture 7.9. *If $P \in E(P_{3,6}) \cap \Delta_{3,6}$, then $\delta(P) = 10$, and either P has ten round zeros, or is the limit of psd extremal ternary sextics with ten round zeros.*

These results are likely more complicated in higher degree. The ternary octic

$$T(x,y,z) = x^4y^4 + x^2z^6 + y^2z^6 - 3x^2y^2z^4 = x^4y^4z^6 M(1/x,1/y,1/z) \tag{7.10}$$

is in $E(P_{3,8})\cap\Delta_{3,8}$; see [19, p.372]. It has five round zeros at $(0,0,1)$ and $(1,\pm1,\pm1)$, and more singular zeros at $(1,0,0)$ and $(0,1,0)$ at which $\delta=5$, so that $\delta(T)=15$. On the other hand, for

$$U(x,y,z)=x^2(x-z)^2(x-2z)^2(x-3z)^2+y^2(y-z)^2(y-2z)^2(y-3z)^2\in\Sigma_{3,8}, \tag{7.11}$$

$\mathcal{Z}(U)=\{(i,j,1):0\le i,j\le 3\}$, so $\delta(U)=16$. Thus, there is no threshold value for δ separating $\Sigma_{3,8}$ and $\Delta_{3,8}$, as there is for sextics.

8. Ternary forms in higher degree

For $d\ge 3$, let

$$T_d=\{(i,j)\ :\ 0\le i,j,\ i+j\le d\}\subset\mathbb{Z}^2 \tag{8.1}$$

denote a right triangle of $\frac{(d+1)(d+2)}{2}$ lattice points. Define the falling product by

$$(t)_m=\prod_{j=0}^{m-1}(t-j). \tag{8.2}$$

The following construction is due to Biermann [1], see [20, pp.31-32]. For $(r,s)\in T_d$, let

$$\phi_{r,s,d}(x,y):=\frac{(x)_r(y)_s(d-x-y)_{d-r-s}}{r!s!(d-r-s)!}. \tag{8.3}$$

Lemma 8.1. *If $(i,j)\in T_d$, then $\phi_{r,s,d}(i,j)=0$ if $(i,j)\ne(r,s)$ and $\phi_{r,s,d}(r,s)=1$.*

Proof. Observe that $(n)_m=0$ if $n\in\{0,\dots,m-1\}$ and $(m)_m=m!$. If $(i,j)\in T_d$, then $0\le i$, $0\le j$ and $0\le d-i-j$. Thus $\phi_{r,s,d}(i,j)=0$ unless $i\ge r$, $j\ge s$ and $d-i-j\ge d-r-s$, or $i+j\le r+s$; that is, unless $(i,j)=(r,s)$. The second assertion is immediate. □

Theorem 8.2. *Suppose $B\subseteq T_d$ and $A=T_d\smallsetminus B$. Then a basis for $I_{1,d}(A)$ is given by $\{\phi_{r,s,d}:(r,s)\in B\}$.*

Proof. The set $\{\phi_{r,s,d}:(r,s)\in T_d\}$ consists of the correct number of linearly independent polynomials and so is a basis for $\mathbb{R}_{2,d}$. If $p\in\mathbb{R}_{2,d}$, then upon evaluation at $(r,s)\in T_d$, we immediately obtain

$$p(x,y)=\sum_{(r,s)\in T_d}p(r,s)\phi_{r,s,d}(x,y). \tag{8.4}$$

If $p\in I_{1,d}(A)$, then $\phi_{r,s,d}$ has non-zero coefficient in (8.4) only if $(r,s)\in B$. □

We use this construction in the following example, which was inspired by looking at the regular pattern of pine trees below the Sulphur Mountain tram, during a break in the October 2006 BIRS program on "Positive Polynomials and Optimization".

Example 8.1 (The Banff Gondola Polynomials). Suppose $d \ge 3$ and let

$$A_d = T_d \setminus \{(d,0),(0,d)\} = \{(i,j) : 0 \le i,j \le d-1, i+j \le d\}. \tag{8.5}$$

By Theorem 8.2, $I_{1,d}(A_d)$ is spanned by $f_1(x,y) = \phi_{d,0,d}(x,y) = (x)_d$ and $f_2(x,y) = \phi_{0,d,d}(x,y) = (y)_d$, and it is easy to see that $\mathcal{Z}(f_1) \cap \mathcal{Z}(f_2) = \{0,\dots,d-1\}^2$, so that

$$\tilde{A}_d = \{(i,j) : 0 \le i,j \le d-1, i+j \ge d+1\}. \tag{8.6}$$

Note that $(i,j) \in \tilde{A}_d$ implies that $i,j \ge 2$. Let

$$\begin{aligned} g_d(x,y) &= (x)_2(y)_2(x+y-2)_{d-1}(x+y-4)_{d-3} \\ &= x(x-1)y(y-1)(x+y-2)(x+y-3)\prod_{k=0}^{d-4}(x+y-4-k)^2, \end{aligned} \tag{8.7}$$

We claim that g_d is singular at $\pi \in A_d$ and positive at $\pi \in \tilde{A}_d$. First, it is easy to check that each point in A_3 lies on at least two of the lines, and $g_3(2,2) = 8$. Now suppose $d \ge 4$ and $(r,s) \in A_d$. If $4 \le r+s \le d$, then (r,s) lies on a squared factor; if $2 \le r+s \le 3$, then (r,s) lies on $x+y-2=0$ or $x+y-3=0$, but also, at least one of $\{r,s\}$ is 0 or 1. Finally, if $0 \le r+s \le 1$, then $\{r,s\} \subseteq \{0,1\}$. If $(r,s) \in \tilde{A}_d$ for any d, then $r,s \ge 2$ and $r+s \ge d+1$, so each factor in g_d is positive at (r,s). It follows from Theorem 3.4 that there exists $c_d > 0$ so that

$$(x)_d^2 + (y)_d^2 + c_d(x)_2(y)_2(x+y-2)_{d-1}(x+y-4)_{d-3} \tag{8.8}$$

is positive and not a sum of squares. Note that this polynomial has at least $|A_d|$ zeros, so $B_{3,2d} \ge \frac{d^2+3d-2}{2}$. This improves the lower bound in (7.3) for $2d = 8, 10$. It can be shown that $c(3) = 4/3$ (exactly) and that $c(d) \le 12d^{-2}$, so $c(d) \to 0$.

We conclude with some speculations about Hilbert's Method in degree $d \ge 4$. Suppose A is a set of $\binom{d+2}{2} - 2$ points in general position, so that $I_{1,d}(A)$ has basis $\{f_1, f_2\}$. By Bezout, we can only say that $|\tilde{A}| \le d^2 - |A| = \binom{d-1}{2}$ as the common zeros do not have to be real or distinct. We have $\dim I_{1,d}^2(A) = 3$ and, from (2.5),

$$\dim I_{2,2d}(A) \ge \binom{2d+2}{2} - 3\left(\binom{d+2}{2} - 2\right) = \binom{d-1}{2} + 3. \tag{8.9}$$

There exist $\binom{d-1}{2}$ linearly independent polynomials in $I_{2,2d}(A) \setminus I_{1,d}^2(A)$, and it is plausible that one is positive on $\tilde{A}$. If so, then Hilbert's Method could be applied.

If $r \ge 3$, and A is a set of $\binom{d+2}{2} - r$ points in general position, so that $\dim I_{1,d}(A) = r$, then it is plausible to expect $\tilde{\mathcal{A}} = \emptyset$. We have

$$\begin{aligned} \dim I_{2,2d}(A) &\ge \binom{2d+2}{2} - 3\left(\binom{d+2}{2} - r\right) = \binom{d-1}{2} + 3r - 3 \\ &= \frac{r(r+1)}{2} + \frac{(d+1-r)(d+r-4)}{2} \ge \dim I_{1,d}^2 + \frac{(d+1-r)(d+r-4)}{2}, \end{aligned} \tag{8.10}$$

so if $r \leq d$, $I_{2,2d}(A) \smallsetminus I_{1,d}^2(A)$ would be non-empty, and again Hilbert's Method could be applied. We hope to return to these questions elsewhere.

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