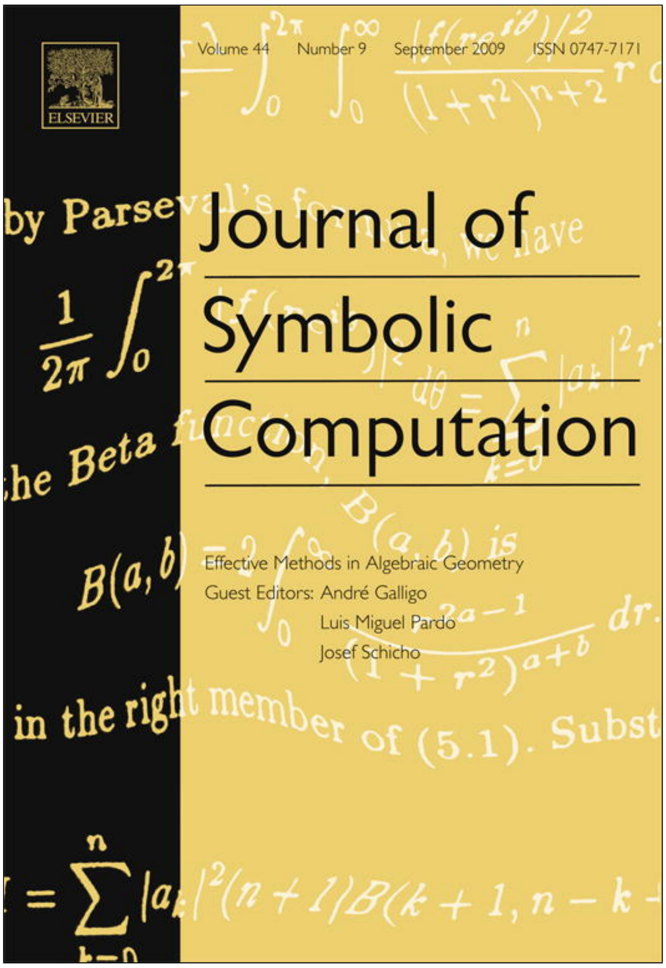
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A quantitative Pólya's Theorem with zeros

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ABSTRACT

Let $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. Pólya's Theorem says that if a form (homogeneous polynomial) $p \in \mathbb{R}[X]$ is positive on the standard n-simplex Δ_n, then for sufficiently large N all the coefficients of $(X_1 + \cdots + X_n)^N p$ are positive. The work in this paper is part of an ongoing project aiming to explain when Pólya's Theorem holds for forms if the condition "positive on Δ_n" is relaxed to "nonnegative on Δ_n", and to give bounds on N. Schweighofer gave a condition which implies the conclusion of Pólya's Theorem for polynomials $f \in \mathbb{R}[X]$. We give a quantitative version of this result and use it to settle the case where a form $p \in \mathbb{R}[X]$ is positive on Δ_n, apart from possibly having zeros at the corners of the simplex.

1. Introduction

Let $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ and let $\mathbb{R}^+[X]$ denote polynomials in $\mathbb{R}[X]$ with nonnegative coefficients. We denote the standard n-simplex $\{(x_1, \ldots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$ by Δ_n.

Pólya's Theorem (1928) says that if p is a homogeneous polynomial in n variables which is positive on the standard n-simplex Δ_n, then for a sufficiently large exponent N, $(X_1 + \cdots + X_n)^N p \in \mathbb{R}^+[X]$. In Powers and Reznick (2001), the second and third authors gave an explicit bound for the exponent N in terms of the degree, the size of the coefficients, and the minimum value of p on the simplex. This result has been used by other authors in applications; for example, in Schweighofer (2002) it is used to give an algorithmic proof of Schmüdgen's Positivstellensatz, and in de Klerk and Pasechnik (2002), it is used to give results on approximating the stability number of a graph.

This paper is part of an ongoing project, begun in Powers and Reznick (2007), aiming to explain exactly when Pólya's Theorem holds if the condition "positive on Δ_n" is relaxed to "nonnegative

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on Δ_n", and to give bounds in this case. Such results should have wide application as the original quantitative Pólya's Theorem did. In this work, we give a quantitative version of a result of Schweighofer which is a "localized" version of Pólya's Theorem. We then use this result to characterize, and give a bound for, forms which are positive on Δ_n apart from having a zero at one vertex and satisfy the conclusion of Pólya's Theorem. This extends the results in Powers and Reznick (2007).

Handelman (1986, 1988) has studied a related question, namely, for which pairs (q, f) of polynomials does there exist $N \in \mathbb{N}$ such that $q^N f$ has nonnegative coefficients? (See also Angelis and Tuncel (2001).) The results in Section 3 without the bound can most likely be deduced from Handelman's work.

2. A localized Pólya's theorem

In Schweighofer (2005, Lemma 7), a condition is given which implies that the conclusion of Pólya's Theorem holds for (not necessarily homogeneous) $f \in \mathbb{R}[X]$. In this section, we give a computational version of this result. The idea is to find a representation of f, which depends on $x \in \Delta_n$, and which implies the conclusion of Pólya's Theorem for coefficients corresponding to X^α, where $\frac{\alpha}{|\alpha|}$ is contained in a neighborhood around x. Our version of this result replaces neighborhoods of x by closed subsets of Δ_n containing x, which allows us to give an explicit bound for the exponent N needed.

We recall the notation of Powers and Reznick (2001). If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Suppose that $p \in \mathbb{R}[X]$ is homogeneous of degree d; we write

$$p(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L(f) := \max_{|\alpha|=d} |b_\alpha|$.

Lemma 1. *Suppose the nonempty set $S \subseteq \Delta_n$ is closed and the polynomial $p \in \mathbb{R}[X]$ is homogeneous of degree d such that $p(x) > 0$ for all $x \in S$. Let λ be the minimum of p on S. Then for*

$$N > \frac{d(d-1)}{2} \frac{L(f)}{\lambda} - d$$

and $\alpha \in \mathbb{N}^n$ such that $\frac{\alpha}{|\alpha|} \in S$, the coefficient of X^α in $(X_1 + \cdots + X_n)^N p$ is nonnegative.

Proof. This follows from the proof of Theorem 1 in Powers and Reznick (2001). □

Proposition 1. *Given $p \in \mathbb{R}[X]$ and a nonempty closed set $S \subseteq \Delta_n$, suppose there exist homogeneous $g_1, \ldots, g_m \in \mathbb{R}[X]$, and $h_1, \ldots, h_m \in \mathbb{R}^+[X]$ with*

(1) $p = g_1 h_1 + \cdots + g_m h_m$,
(2) $g_i(x) > 0$ *for all* $x \in S$.

Suppose further that T is a nonempty closed subset of S and there exists $B \in \mathbb{N}$ with the following property: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ satisfy $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \operatorname{supp}(h_i)$ for some i, and $|\beta| \geq B$, we have $\frac{\beta}{|\beta|} \in S$. Then there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, the coefficient of X^α in $(X_1 + \cdots + X_n)^N p$ is nonnegative.

More precisely, for each i, let $k(i)$ be the bound from Lemma 1 for g_i on S, i.e.,

$$k(i) = \frac{d_i(d_i - 1)}{2} \frac{L(g_i)}{\lambda_i} - d_i,$$

where λ_i is the minimum of g_i on S and $d_i = \deg g_i$. Then we can take any N such that

$$N \geq \max\{k(g_1), \ldots, k(g_m), B\}.$$

Proof. Clearly, it suffices to show that for any $1 \leq j \leq m$, the coefficient of X^α in $(X_1+\cdots+X_n)^N g_j h_j$ is nonnegative. Suppose $\beta, \gamma \in \mathbb{N}^n$ are such that $\beta+\gamma = \alpha$ and the coefficients of X^β in $(X_1+\cdots+X_n)^N g_j$ and X^γ in h_j are nonzero. Since $h_j \in \mathbb{R}^+[X]$, the coefficient of X^γ in h_j is positive. Then since we have $|\beta| \geq N \geq B$ and $\alpha = \beta + \gamma$ for $\gamma \in \operatorname{supp}(h_j)$, $\frac{\beta}{|\beta|} \in S$ by our assumption. Hence, by the choice of $k(j)$, it follows that the coefficient of X^β in $(X_1 + \cdots + X_N)^N g_j$ is nonnegative and we are done. □

We now obtain Schweighofer's result (2005, Lemma 7) as a corollary:

Corollary 1. *Let $f \in \mathbb{R}[X]$. Suppose that for every $x \in \Delta_n$ there are $m \in \mathbb{N}$, homogeneous $g_1, \ldots, g_m \in \mathbb{R}[X]$, and $h_1, \ldots, h_m \in \mathbb{R}^+[X]$ such that*

(1) $f = g_1h_1 + \cdots + g_mh_m$,
(2) $g_i(x) > 0$ *for* $i = 1, \ldots, m$.

Then there exists $N \in \mathbb{N}$ such that the coefficients of $(X_1 + \cdots + X_n)^N f$ are nonnegative.

Proof. Let $B_\epsilon(x)$ denote the open ball of radius ϵ centered at x. For each $x \in \Delta_n$, by continuity of the g_i's there is $\epsilon_x > 0$ such that a representation of f as above exists with $g_i > 0$ on $B_{2\epsilon_x}(x)$. By compactness, we can choose a finite number of $B_{\epsilon_x}(x)$'s covering Δ_n. Then it is enough to show that for each $x \in \Delta_n$ there is an $N_x \in \mathbb{N}$ such that the coefficients of X^α in $(X_1+\cdots+X_n)^{N_x} f$ for $\frac{\alpha}{|\alpha|} \in B_{\epsilon_x}(x)$ are nonnegative. Taking the maximum of the N_x's corresponding to the finite subcover, we are done.

Fix $x \in \Delta_n$, let $M = \max\{\deg(h_i)\}$ and choose $B \geq 2M/\epsilon_x$. Now set $S = \overline{B_{2\epsilon_x}} \cap \Delta_n$ and $T = \overline{B_{\epsilon_x}(x)} \cap \Delta_n$. Then S, T are nonempty and closed and $T \subseteq S$. Moreover, if $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \operatorname{supp}(h_i)$, for some i, and $|\beta| \geq B$, then a series of inequalities, exactly as in the proof of Schweighofer (2005, Lemma 7), shows that $\frac{\beta}{|\beta|} \in S$. □

3. Pólya's Theorem with zeros

If a nonzero form p satisfies the conclusion of Pólya's Theorem, then p cannot have interior zeros. Furthermore, if such a p is zero at some point on the relative interior of a face of Δ_n, then p must vanish on the entire face; see Powers and Reznick (2007, Section 3). In this section, we apply Proposition 1 to give a quantitative version of Pólya's Theorem for forms which have a zero at a corner of Δ_n. This generalizes the main result from Powers and Reznick (2007).

Write $v_1, \ldots, v_n$ for the vertices of Δ_n, i.e., $v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$. For $r \in \mathbb{R}$, $0 < r < 1$ and $j = 1, \ldots, n$, let

$$\Delta_n(j, r) = \left\{(x_1, \ldots, x_n) \in \Delta_n \mid \sum_{i \neq j} x_i \leq r\right\} = \left\{(x_1, \ldots, x_n) \in \Delta_n \mid x_j \geq 1 - r\right\}.$$

In other words, $\Delta_n(j, r)$ is the scaled simplex $r \cdot \Delta_n$ translated by $(1-r)v_j$ and nestled in the v_j corner of Δ_n.

Lemma 2. *Suppose homogeneous $f = cX_1^e + \phi \in \mathbb{R}[X]$ is given, where $c > 0$ and the degree of ϕ in X_1 is less than e. Let U be the sum of the absolute values of the coefficients of f and define*

$$r = \frac{c}{c + 2U}, \quad s = \frac{c}{2}\left(\frac{2U}{c + 2U}\right)^e.$$

Then $f \geq s$ on $\Delta_n(1, r)$.

Proof. Suppose $x = (x_1, \ldots, x_n) \in \Delta_n$ with $x_1 \neq 0$. For $i = 2, \ldots, n$, let $y_i = \frac{x_i}{x_1}$, so that we have $f(x_1, \ldots, x_n) = x_1^e f(1, y_2, \ldots, y_n)$. Let r be as given and suppose $(x_1, \ldots, x_n) \in \Delta_n(1, r)$. Then for each i,

$$y_i = \frac{x_i}{x_1} \leq \frac{r}{1 - r} = \frac{c}{2U}.$$

Since the degree of ϕ in X_1 is less than e, $\phi(1, X_2, \ldots, X_n)$ has no constant term and thus $|\phi(1, y_2, \ldots, y_n)| \leq \frac{c}{2U} U = \frac{c}{2}$. Since $(x_1, \ldots, x_n) \in \Delta_n(1, r)$, we have $x_1 \geq 1 - r$ and thus

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= x_1^e \left(c + \phi(1, y_2, \ldots, y_n)\right) \\ &\geq \left(\frac{2U}{c + 2U}\right)^e \left(c - \frac{c}{2}\right) = s. \quad \square \end{aligned}$$

Let $\mathbb{R}[\tilde{X}]$ denote $\mathbb{R}[X_2, \ldots, X_n]$ and for $\alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{N}^{n-1}$, let $\tilde{X}^\alpha$ denote $X_2^{\alpha_2} \cdot \ldots \cdot X_n^{\alpha_n}$.

Proposition 2. *Suppose $p > 0$ on Δ_n apart from a zero at v_1, and p can be written as*

$$\left(\sum_{k=1}^{m} \tilde{X}^{\beta_k} (c_k X_1^{l_k} + f_k(X))\right) + q(X)$$

where $c_k > 0$, $\deg f_k(X) = l_k$, the degree in X_1 of f_k is strictly less than l_k for all k, and $q(X)$ is a polynomial with only nonnegative coefficients. Let $c = \min_k\{c_k\}$, $d = \deg(p)$, and U be the sum of the absolute values of the coefficients of p. Define

$$r = \frac{c}{c + 2U}, \qquad s = \frac{c}{2}\left(\frac{2U}{c + 2U}\right)^d.$$

Then $(\sum X_i)^N p$ has nonnegative coefficients for

$$N > \max\left\{\frac{d(d-1)}{2}\frac{L(p)}{s}, \frac{d(d-1)}{2}\frac{L(p)}{\lambda}\right\},$$

where λ is the minimum of p on the closure of $\Delta_n \setminus \Delta_n(1, r) = \{x \in \Delta_n \mid x_1 \leq 1 - r\}$.

Proof. For each k, set $g_k := c_k X_1^{l_k} + f_k(X)$ and $h_k := \tilde{X}^{\beta_k}$ and apply Lemma 2 to each g_k. Let r_k, s_k be the constants defined in Lemma 2 for g_k so that $g_k \geq s_k$ on $\Delta_n(1, r_k)$ for each k. Noting that the set of coefficients of each g_k is a subset of the coefficients of p, we have $r_k \leq r$ and $s_k \leq s$. Thus, for each k, $g_k \geq s$ on $\Delta_n(1, r)$.

We now apply Proposition 1 to p with g_k, h_k as above, $S = T = \Delta_n(1, r)$, and $B = 1$. It is straightforward to check that the assumptions of Proposition 1 hold in this case. Applying Proposition 1, noting that $\deg(g_k) \leq \deg(p)$, we can use the bound $\frac{d(d-1)}{2}\frac{L(p)}{s}$ for each g_k in Lemma 1.

Finally, by assumption, $p > 0$ on $\Delta_n \setminus \Delta_n(1, r)$ and hence we can apply Lemma 1 in the case where $\frac{\alpha}{|\alpha|} \in \Delta_n \setminus \Delta_n(1, r)$. Therefore we obtain the bound on N as given. $\square$

Suppose p is positive on Δ_n except for a zero at one v_i; for ease of exposition we may as well assume $i = 1$. Suppose p has degree k as a polynomial in X_1 and write p in the form

$$p = \sum_{i=0}^{k} X_1^i q_i(X_2, \ldots, X_n).$$

Theorem 1. *Given p as above, then the conclusion of Pólya's Theorem holds for p if and only if the following holds: For every term $a\tilde{X}^\alpha$ in some $q_i(\tilde{X})$ with $a < 0$, there exists $j > i$ such that $q_j(\tilde{X})$ contains a term $b\tilde{X}^\beta$ with $b > 0$ and $\tilde{X}^\beta$ dividing $\tilde{X}^\alpha$ (and hence $X_i^j \tilde{X}^\beta = X_1^i \tilde{X}^\alpha$). In particular, if Pólya's Theorem holds we obtain a bound on N as in Proposition 2.*

Proof. If the conclusion of Pólya's Theorem holds, say with exponent N, and there is a term $a\tilde{X}^\alpha$ in some $q_i(\tilde{X})$ with $a < 0$, consider the coefficient of $X_1^{N+i}\tilde{X}^\alpha$ in $(\sum X_i)^N p$. Then it is clear that there exists $j > i$ such that $q_j(\tilde{X})$ contains a term $b\tilde{X}^\beta$ with $b > 0$ and $\tilde{X}^\beta$ dividing $\tilde{X}^\alpha$.

Now suppose that the assumption holds. Given a term in p with a negative coefficient, then it occurs in a summand $X_1^i q_i(\tilde{X})$; say it comes from a term $a\tilde{X}^\alpha$ in $q_i(\tilde{X})$ with $a < 0$. By assumption, there is some $j > i$ such that $q_j(\tilde{X})$ contains $b\tilde{X}^\beta$ with $b > 0$ and $\tilde{X}^\beta$ divides $\tilde{X}^\alpha$. Then in p we have

$$bX_1^j \tilde{X}^\beta + aX_1^i \tilde{X}^\alpha = \tilde{X}^\beta \left(bX_1^j + aX_1^i X_2^{\alpha_2 - \beta_2} + \cdots + X_n^{\alpha_n - \beta_n}\right).$$

By assumption, $j > i$. Since we can do this for each term in p with a negative coefficient, we can write p in the form

$$\left(\sum_{k=1}^{m} \tilde{X}^{\beta_k}(c_k X_1^{l_k} + f_k(X))\right) + q(X)$$

where, for all k, $c_k > 0$, the degree in X_1 of f_k is strictly less than l_k, and q is a polynomial with only nonnegative coefficients. Then by Proposition 2, it is easy to see that Pólya's Theorem holds with the bound on N as stated. □

We obtain as a corollary the main result in Powers and Reznick (2007). We say that a form p of degree d which is nonnegative on Δ_n has a *simple zero* at v_j if the coefficient of X_j^d in p is zero, but the coefficient of $X_j^{d-1}X_i$ is nonzero (and necessarily positive) for each $i \neq j$.

Corollary 2. *Suppose p is positive on Δ_n except for simple zeros at some v_j's. Then Pólya's Theorem holds for p and there is a bound for the exponent N in terms of the degree and coefficients of p, and the minimum of p on Δ_n minus the corner simplices $\Delta_n(j, r_j)$.*

Proof. If p has a simple zero at, say, v_1, then the degree of p in X_1 is $d - 1$, and in the notation of Theorem 1 we have $q_{d-1}(X_2, \ldots, X_n) = b_2X_2 + \cdots + b_nX_n$ where $b_i > 0$ for all i. Given $a\tilde{X}^{\alpha}$ in some $q_i(\tilde{X})$ with $a < 0$, then $i < d - 1$ and we must have $\alpha_j > 0$ for some $2 \le j \le n$. Then we can take $b\tilde{X}^{\beta}$ to be the term b_jX_j from q_{d-1}. □

Example 1. Let

$$q(X_1, X_2, X_3, X_4) := X_1^2(X_2^2 + X_3^2 + X_4^2) - aX_1X_2X_3X_4 + X_2^4 + X_3^4 + X_4^4,$$

where $0 < a < 6$, then q is positive on Δ_n except for a zero at $(1, 0, 0, 0)$; however the term $-aX_1X_2X_3X_4$ is not divisible by any monomial in the leading coefficient of q as a polynomial in X_1. It is not too hard to see that for any $N \in \mathbb{N}$, the coefficient of ${X_1}^{N+1}X_2X_3X_4$ in $(X_1 + \cdots + X_4)^N p$ will always be $-a$ and hence the conclusion of Pólya's Theorem does not hold in this case.

On the other hand, let $p = q + X_1^2X_2X_3$; then all assumptions of Theorem 1 are satisfied. Indeed, we have

$$p = X_2^2(X_1^2 + X_2^2) + X_3^2(X_1^2 + X_3^2) + X_4^2(X_1^2 + X_4^2) + X_2X_3(X_1^2 - aX_1X_4)$$

and we can apply Lemma 2 to $X_1^2 - aX_1X_4$ and hence obtain an explicit N for which $(X_1+X_2+X_3+X_4)^N p$ has nonnegative coefficients.

Remark 1. Theorem 1 surely has a generalization to forms p which are positive on Δ_n except for zeros on a lower dimensional face F of Δ_n, replacing X_1^e by the leading form of p as a polynomial in the variables corresponding to the face F. The statement of such a generalization would be complicated; however for particular examples it should be possible to construct an explicit bound on the exponent N in Pólya's Theorem, using the ideas in Lemma 2 and Theorem 1. This will be the subject of the Ph.D. thesis of the first author under the direction of the second author.

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