CLEAN LATTICE TETRAHEDRA

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Abstract. A clean lattice tetrahedron is a non-degenerate tetrahedron with the property that the only lattice points on its boundary are its vertices. We present some new proofs of old results and some new results on clean lattice tetrahedra, with an emphasis on counting the number of its interior lattice points and on computing its lattice width.

1. Introduction and Overview

Let $T = T(v_1, \dots, v_n) = conv(v_1, \dots, v_n)$ be a non-degenerate simplex with vertices $v_j \in \mathbb{Z}^n$. We say that T is *clean* if there are no non-vertex lattice points on the boundary of T. Let $i(T) = \#\{int(T) \cap \mathbb{Z}^n\}$ denote the number of lattice points in the interior of a clean lattice simplex T. If $i(T) = k$, then T is called a *k-point lattice simplex*. If $i(T) = 0$, then T is called *empty*. This paper is mainly concerned with clean tetrahedra.

Pick's Theorem says that the area of a clean lattice triangle T is equal to $i(T)+1/2$. Reeve [13] showed in 1957 that there are empty lattice tetrahedra having arbitrarily large volume. By constrast, if T is a (not necessarily clean) lattice tetrahedron with $k \ge 1$ interior points, then Hensley [5] showed in 1983 that there is an upper bound on the volume of T depending on k. Any lattice tetrahedron determines an (affine) lattice Λ, and if $|\mathbb{Z}^3/\Lambda| = m$; that is, if there are m lattice points in the fundamental parallelepiped, then the volume of that parallelepiped equals m. The volume of the corresponding tetrahedron is then equal to $m/6$, but there seems to be no easy way to determine the number of lattice points it contains.

In this paper, we give a unified discussion of clean lattice tetrahedra. We begin with preliminaries in section two. Two tetrahedra T and T' are *equivalent* if there is an affine unimodular map which takes the vertices of T into the vertices of T' in some order. For $(a, b, n) \in \mathbb{Z}^3$, we define the tetrahedron $T_{a,b,n}$, which has vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and (a, b, n), and we give necessary and sufficient conditions under which $T_{a,b,n}$ and $T_{a',b',n'}$ are equivalent. A crucial "hidden" parameter is $c = 1 - a - b$.

In section three, we show that every clean lattice tetrahedron is equivalent to some $T_{a,b,n}$, where $\gcd(a, n) = \gcd(b, n) = \gcd(c, n) = 1$ and $0 \le a, b \le n - 1$. (Reeve had

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originally discussed these conditions in the context of empty lattice tetrahedra.) We then review our 1986 result [15] that

$$i(T_{a,b,n}) = \#\left\{t : 1 \le t \le n-1 \quad \text{and} \quad \{\tfrac{t(a+b-1)}{n}\} + \{\tfrac{-ta}{n}\} + \{\tfrac{-tb}{n}\} + \{\tfrac{t}{n}\} = 1\right\}.$$

Using this, we give a new and shorter proof of White's 1964 theorem [17] that a lattice tetrahedron is empty if and only if it is equivalent to $T_{0,0,0}$ or some $T_{1,b,n}$, where $n \ge 2$, $1 \le b \le n-1$ and $\gcd(b,n) = 1$. We also discuss bounds on $i(T_{a,b,n})$. It is not hard to show that $i(T_{3k,3k,3k+1}) = k$. Han Duong has proved that $i(T_{2k+1,4k+3,12k+8}) = k$, and used the formula above to conjecture that for all clean tetrahedra,

$$\frac{n-1}{3} \ge i(T_{a,b,n}) \ge \frac{n-8}{12},$$

with the extreme examples given by the two aforementioned families.

Section four is devoted to 1-point lattice tetrahedra. Suppose T is such a tetrahedron, with interior point w. We give a new proof of our earlier result that there are only seven possible sets of barycentric coordinates for w with respect to the vertices of T. If two 1-point lattice tetrahedra are equivalent, then their interior points have the same barycentric coordinates, but the converse is false: $T_{3,3,4}$ and $T_{3,7,20}$ have different volumes and so are not equivalent, but each has a single interior lattice point at the centroid. Mazur has [9] recently showed that, up to equivalence, these are the only two such 1-point lattice tetrahedra. (This was the fruit of an undergraduate research project.) We show that in the other six cases of barycentric coordinates, there is exactly one equivalence class of 1-point tetrahedra. After an early version of this paper was distributed, Julian Pfeifle pointed out that this result had been proved recently by A. Kasprzyk [6] . The proof here seems sufficiently different to merit publication. Kasprzyk's paper is motivated by a question in toric varieties on the classification of toric Fano 3-folds with terminal singularities. In his discussion, the 1-point tetrahedra are arranged so that the interior point is the origin.

Finally, in section five, we discuss the lattice width of clean tetrahedra. White's Theorem showed that an empty tetrahedron lies in two consecutive planes of lattice points; that is, an empty tetrahedron has lattice width one. We show that each 1-point tetrahedron lies in three consecutive planes of lattice points, and so has lattice width two. Since $T_{3k,3k,3k+1}$ also has lattice width two, there is no deterministic connection between $i(T)$ and its lattice width; however, we conjecture that the lattice width of a clean tetrahedron T is bounded above by $i(T)+1$. We also show that the lattice width of T is $\mathcal{O}(n^{1/3})$ and that T_{n,n^2,n^3-1} has lattice width n, so this bound is asymptotically best possible, up to multiplicative constant.

Most of the literature is not fastidious about the existence of lattice points on the boundary of a simplex, unless there are no interior points. A principal result of [15] was that, if $S \in \mathbb{R}^n$ is a clean lattice simplex with exactly k interior points, then there is an upper bound, depending on k and n, for the denominators of the barycentric coordinates of these points. This result was subsumed by the stronger and essentially simultaneous work of D. Hensley [5], who proved that if $S \in \mathbb{R}^n$ is a lattice simplex with $k \geq 1$ interior points, then there are bounds on the volume of S. These bounds were subsequently improved by J. Lagarias and G. Ziegler [8] and by O. Pikhurko [12]. A special case of Pikhurko's bound shows that the volume of a lattice tetrahedron with one interior point is $\leq \frac{31^3}{3!352} < \frac{85}{6}$. This result could be combined with Theorem 4(i) below and some computer searching to determine all 1-point lattice tetrahedra up to equivalence. Nevertheless, we believe it is worthwhile to give a proof in which all computations are explicitly presented. Note that the tetrahedron with vertices at the origin and $\{4e_j\}$, $1 \leq j \leq 3$, has volume $\frac{64}{6}$ and a single interior point $(1,1,1)$, along with many boundary points. It is plausible to believe that this volume is maximal among such tetrahedra.

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2. Preliminaries

Let $T = T(v_1, v_2, v_3, v_4) = conv(v_1, v_2, v_3, v_4)$ be a non-degenerate tetrahedron in $\mathbb{R}^3$. Every point $w \in \mathbb{R}^3$ has a unique set of *barycentric coordinates* with respect to T; namely $\lambda_j := \lambda_{j,T}(w) \in \mathbb{R}$, $1 \leq j \leq 4$, so that

$$w = \sum_{j=1}^{4} \lambda_j v_j, \qquad \sum_{j=1}^{4} \lambda_j = 1. \tag{1}$$

If (1) holds, we write $BC_T(w) := (\lambda_1(w), \lambda_2(w), \lambda_3(w), \lambda_4(w))$. If the vertices of T are permuted, T as a geometric object is unchanged, but the coordinates of $BC_T(w)$ are permuted.

Observe that $w \in T$ if and only if $\lambda_j(w) \geq 0$ for all j and $w \in int(T)$ if and only if $\lambda_j(w) > 0$ for all j. If T is clean, $w \in T$ and $\lambda_j(w) = 0$ for some j, then $w = v_k$ for some k and $BC_T(w)$ is a unit vector.

Recall that $x \in \mathbb{R}$ can be written $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor \in \mathbb{Z}$ and $\{x\} \in [0,1)$. If $\sum x_j \in \mathbb{Z}$, then so is $\sum \{x_j\}$. In particular, if $x, y \in \mathbb{R}$ and $x + y \in \mathbb{Z}$, but $x, y \notin \mathbb{Z}$, then $\{x\} + \{y\} = 1$; thus, if $x \notin \mathbb{Z}$, $\lfloor -x \rfloor + \lfloor x \rfloor = -1$. Further, if m is an integer and $x + m \in [0,1)$, then $m = -\lfloor x \rfloor$ and $x + m = \{x\}$. If a and n are integers, then $a \equiv n\{\frac{a}{n}\} \pmod{n}$.

Let $\mathcal{L}$ denote the set of affine unimodular maps $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by $f(v) = Mv + u$, where $M \in M_3(\mathbb{Z})$, $\det(M) = \pm 1$ and $v \in \mathbb{Z}^3$. Then $f^{-1} \in \mathcal{L}$ as well and f is an bijection of $\mathbb{Z}^3$ to itself. Since

$$\sum_{j=1}^{4} \lambda_j f(v_j) = \sum_{j=1}^{4} \lambda_j (Mv_j + u) = M\left(\sum_{j=1}^{4} \lambda_j v_j\right) + \left(\sum_{j=1}^{4} \lambda_j\right) u = f\left(\sum_{j=1}^{4} \lambda_j v_j\right),$$

f preserves barycentric coordinates; thus, $f(T) \cap \mathbb{Z}^3 = f(T \cap \mathbb{Z}^3)$, with boundary and interior points mapped to boundary and interior points. For this reason, it makes sense to classify lattice tetrahedra up to the action of $\mathcal{L}$. Following [15], given lattice tetrahedra $T = T(v_j)$ and $T' = T(v'_j)$, we say that T and T' are *equivalent* ($T \approx T'$) if there exists $f \in \mathcal{L}$ so that $\{v'_j\} = \{f(v_j)\}$. It is not necessary that f preserve the order of the vertices.

The class $\mathcal{L}$ contains translations and reflections, of course. It also contains shears; of particular interest is the map $(x, y, z) \mapsto (x - mz, y - nz, z)$ for $m, n \in \mathbb{Z}$, where m and n are chosen by the Euclidean algorithm so that $0 \le x - mz, y - nz < |z|$. We call this a *Euclidean shear*. If $r, s \in \mathbb{Z}$ and $g = \gcd(r, s)$, then there exist $r', s', m, n \in \mathbb{Z}$ so that $r = gr', s = gs'$, and $mr' + ns' = 1$. The map sending (x_j, x_k) to $(mx_j + nx_k, -s'x_j + r'x_k)$, and fixing the other coordinate, has determinant $mr' + ns' = 1$ and sends (r, s) to $(g, 0)$. We call this a *tweak*.

For $(a, b, n) \in \mathbb{Z}^3$, $n \ne 0$, we define a standard family of tetrahedra:

$$T_{a,b,n} := T((0,0,0), (1,0,0), (0,1,0), (a,b,n)).$$

The face containing $(0,0,0), (1,0,0), (0,1,0)$ is the *base* of $T_{a,b,n}$ and (a, b, n) is the *top*. The reflection $(x, y, z) \mapsto (x, y, -z)$ shows that $T_{a,b,n} \approx T_{a,b,-n}$. A Euclidean shear fixes the vertices of the base and shows that every $T_{a,b,n}$ is equivalent to some $T_{a',b',|n|}$, with $0 \le a', b' \le |n| - 1$. Given (a, b, n), we define

$$c = 1 - a - b.$$

Note that $c \equiv 1 \pmod{n}$ if and only if $n \mid a + b$. As we shall see, c is an "equal partner" of a and b in $T_{a,b,n}$.

If $T \approx T'$, then $vol(T) = vol(T')$. However, equal volumes do not imply equivalence, even for clean tetrahedra.

Lemma 1. *(*[15, Thm. 5.6]*,* [4, pp. 144-145]*,* [7, Thm. 5.1]*) We have $T_{a,b,n} \approx T_{d,e,n'}$ if and only if $|n| = |n'|$ and d and e are congruent* $\pmod{|n|}$ *to two of the elements in one of the following triples:*

$$(2) \quad (a, b, c), \quad (a^{-1}, -ba^{-1}, -ca^{-1}), \quad (b^{-1}, -ab^{-1}, -cb^{-1}), \quad (c^{-1}, -ac^{-1}, -bc^{-1}).$$

(If any of $\{a, b, c\}$ is not invertible mod $|n|$, then the corresponding triple does not appear in (2)*.)*

Proof. Since $f \in \mathcal{L}$ preserves volume, $|n| = |n'|$ is necessary. Suppose $|n| = |n'|$, and after a possible reflection, suppose $n' = n > 0$. An affine map in $\mathbb{R}^3$ is determined by its values on the vertices non-degenerate tetrahedron; since any affine map taking

$T_{a,b,n}$ to $T_{d,e,n}$ will be volume-preserving, the issue is whether its coefficients are integral. There are $4! = 24$ cases. If the base of $T_{a,b,n}$ is mapped to the base of $T_{d,e,n}$ by a map in $\mathcal{L}$, then f permutes $(0,0,0),(1,0,0),(0,1,0)$, and so f fixes z and sends (x,y) to two of $\{x,y,1-x-y\}$. In this case, (d,e) is congruent, mod n, to two of the elements (a,b,c), in some order.

We do one case to stand for the remaining 18: If $f(0,0,0) = (0,0,0)$, $f(1,0,0) = (1,0,0)$, $f(0,1,0) = (d,e,n)$ and $f(a,b,n) = (0,1,0)$, then

$$f(x,y,z) = \left(x + dy - \left(\frac{a+bd}{n}\right) z,\ ey + \left(\frac{1-eb}{n}\right) z,\ ny - bz\right).$$

Observe that $f \in \mathcal{L}$ if and only if $\gcd(b,n) = 1$, $d \equiv -ab^{-1} \bmod n$ and $e \equiv b^{-1} \bmod n$. The other cases are numbingly similar. □

If we suspend $\mathbb{R}^3$ in a hyperplane of $\mathbb{R}^4$ via $(x,y,z) \mapsto (1-x-y-z,x,y,z)$, then $T_{a,b,n}$ is sent to the tetrahedron with vertices $(1,0,0,0),(0,1,0,0),(0,0,1,0)$ and $(1-a-b-n,a,b,n)$. The six maps which permute the base correspond to permutations of the first three coordinates in $\mathbb{R}^4$, and we can see directly that $T_{a,b,n} \approx T_{a,c,n} \approx T_{b,a,n} \approx T_{b,c,n} \approx T_{c,a,n} \approx T_{c,b,n}$.

3. Clean and Empty Lattice Tetrahedra

The following criterion was first studied by Reeve [13] in 1957.

Lemma 2. *Suppose T is a non-degenerate lattice tetrahedron and suppose the only lattice points on the face $v_1v_2v_3$ are its vertices. Then there exists $f \in \mathcal{L}$ so that $f(v_1) = (0,0,0)$, $f(v_2) = (1,0,0)$ and $f(v_3) = (0,1,0)$. Thus, $T \approx T_{a,b,n}$ for some (a,b,n) with $n \geq 1$.*

Proof. We construct the equivalence explicitly. First translate by the first vertex, so that $v_1 = (0,0,0)$; say that v_2 is now (r,s,t). There are no lattice points on the open edge v_1v_2, hence $1 = \gcd(r,s,t) = \gcd(r,\gcd(s,t))$. Let $g = \gcd(s,t)$. Tweak the last two coordinates, sending v_2 to $(r,g,0)$, and then tweak the first two coordinates, so $(r,g,0) \mapsto (1,0,0)$. These tweaks fix v_1, and at this point, we have $v_1 = (0,0,0)$ and $v_2 = (1,0,0)$. Suppose now that $v_3 = (i,j,k)$; again $\gcd(i,j,k) = 1$; again tweak the last two coordinates so that $(i,j,k) \mapsto (i,q,0)$, fixing v_1 and v_2. A Euclidean shear fixes v_1, v_2 and sends $v_3 = (i,q,0)$ to $(p,q,0)$, where $p \equiv i \pmod q$ and $0 \leq p \leq q-1$. Since $1 = \gcd(p,q)$, if $q > 1$ then $p \geq 1$. We claim that $q = 1$, so $p = 0$. Suppose otherwise that $q \geq 2$. Note that the non-vertex lattice point

$$(1,1,0) = \left(\frac{p-1}{q}\right) \cdot (0,0,0) + \left(\frac{q-p}{q}\right) \cdot (1,0,0) + \left(\frac{1}{q}\right) \cdot (p,q,0)$$

is on the face $v_1v_2v_3$, violating the cleanliness of T. Therefore, $q = 1$. (This last step can be replaced by an appeal to Pick's Theorem; see [15, p.233].) □

Theorem 3. *([13, pp.389-390]) The lattice tetrahedron T is clean if and only $T \approx T_{0,0,1}$ or $T \approx T_{a,b,n}$, where*

$$n \geq 2, \quad 0 \leq a, b \leq n-1, \quad \gcd(a,n) = \gcd(b,n) = \gcd(c,n) = 1. \tag{3}$$

This equivalence can be effected with $f \in \mathcal{L}$ sending v_1, v_2, v_3, in that order, to $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$.

Proof. First, observe that $T_{0,0,1}$ is clean. Suppose $T = T_{a,b,n}$, where (3) holds. Then the face $v_1v_3v_4$ is contained in the plane $nx = az$, and if $w = (r,s,t)$ is a lattice point on this face, then $nr = at$. Since $\gcd(a,n) = 1$, it follows that $n \mid t$. But $v_1v_3v_4$ lies between the planes $z = 0$ and $z = n$, so $0 \leq t \leq n$ and hence $t = 0$ (so $r = 0$ and $w = v_1$ or v_3) or $t = n$ (so $r = a$ and $w = v_4$.) It follows that no non-vertex lattice points lie on the face $v_1v_3v_4$. Since $v_1v_2v_4$ is contained in the plane $ny = bz$ and $v_2v_3v_4$ lies in the plane $n(x+y-1) = (a+b-1)z$, similar arguments, applied to $\gcd(b,n) = 1$ and $\gcd(c,n) = \gcd(1-a-b,n) = 1$ show that T is clean.

Conversely, suppose T is clean and $vol(T) = n/6$. Apply Lemma 2 so that $T \approx T_{a,b,n}$, with $0 \leq a,b \leq n-1$. If $n = 1$, then $T \approx T_{0,0,1}$. Otherwise, $n \geq 2$. We need to show that $\gcd(a,n) = \gcd(b,n) = \gcd(c,n) = 1$. Suppose $g = \gcd(a,n) > 1$, and write $(a,n) = (ga', gn')$. If $b = gb'$ for $b' \in \mathbb{Z}$, then $v_4 = (ga', gb', gn')$ and the lattice point (a', b', n') is on the open edge v_1v_4, which is impossible. Accordingly, $b/g \notin \mathbb{Z}$; let $m = \lfloor b/g \rfloor$ and $\ell = b - gm$, noting that $1 \leq \ell \leq g-1$. Observe that the non-vertex lattice point

$$(a', m+1, n') = \left(\frac{\ell - 1}{g}\right) \cdot (0,0,0) + \left(\frac{g-\ell}{g}\right) \cdot (0,1,0) + \left(\frac{1}{g}\right) \cdot (ga', b, gn')$$

is on the face $v_1v_3v_4$, a contradiction. Similar arguments apply to b and c. □

The number of interior lattice points in $T_{a,b,n}$ for $n \geq 2$ requires a non-trivial computation, in contrast to the n points in the corresponding fundamental parallelepiped. The following result can be pieced together from Theorems 4.5, 4.7 and 5.2 in [15]; we prove it here directly.

Theorem 4. *(i) Suppose $T = T_{a,b,n}$ is clean (so (3) holds.) Let*

$$A_t := \left\{\frac{t(n-c)}{n}\right\} + \left\{\frac{t(n-a)}{n}\right\} + \left\{\frac{t(n-b)}{n}\right\} + \left\{\frac{t}{n}\right\}. \tag{4}$$

Then $i(T) = \#\{t : 1 \leq t \leq n-1 \text{ and } A_t = 1\}$.

(ii) Suppose T is a clean tetrahedron and $w \in int(T) \cap \mathbb{Z}^3$. Then $BC(w) = (d_1/N, d_2/N, d_3/N, d_4/N)$ for positive integers d_j, $\sum d_j = N$, so that $\gcd(d_j, N) = 1$.

(iii) Given positive integers d_j, $\sum d_j = N$, so that $\gcd(d_j, N) = 1$, let $\lambda = (d_1/N, d_2/N, d_3/N, d_4/N)$. Then there exists a clean tetrahedron T with at least one interior lattice point w for which $BC_T(w) = \lambda$.

Proof. We first remark that if $w = (r,s,t) \in \mathbb{Z}^3$ and $T = T_{a,b,n}$, then a routine computation shows that

$$BC_T(w) = \left(1 - r - s + \frac{t(a+b-1)}{n},\ r - \frac{ta}{n},\ s - \frac{tb}{n},\ \frac{t}{n}\right). \tag{5}$$

Observe for later use that $\{\frac{-ta}{n}\} = \{\frac{t(n-a)}{n}\}$, $\{\frac{-tb}{n}\} = \{\frac{t(n-b)}{n}\}$ and $\{\frac{t(a+b-1)}{n}\} = \{\frac{t(n-c)}{n}\}$, because in each case, the two fractions differ by an integer.

(i) If $w \in int(T)$, (5) implies that $1 \le t \le n-1$, and since $\lambda_2(w), \lambda_3(w)$ lie in $(0,1)$, we must have $r := r(t) = -\lfloor\frac{-ta}{n}\rfloor$ and $s := s(t) = -\lfloor\frac{-tb}{n}\rfloor$, so that $\lambda_2(w) = \{\frac{t(n-a)}{n}\}$ and $\lambda_3(w) = \{\frac{t(n-b)}{n}\}$. Thus the only possible interior points have the shape $w_t := (r(t), s(t), t)$, and $w_t \in int(T)$ if and only if $\lambda_1(w_t) \in (0,1)$. Since $\lambda_1(w_t) = 1 - \sum_{j=2}^4 \lambda_j(w_t)$ differs from $\frac{t(n-c)}{n}$ by an integer, we see that $\lambda_1(w_t) \in (0,1)$ if and only if it actually equals $\{\frac{t(n-c)}{n}\}$; that is, $w_t \in T$ if and only if $A_t = 1$.

(ii) The unimodular map taking T to $T_{a,b,n}$ maps w to w_t for some t with $1 \le t \le n-1$; let $g = \gcd(t,n)$. By taking reduced fractions in

$$BC(w_t) = \left(\left\{\frac{t(n-c)}{n}\right\}, \left\{\frac{t(n-a)}{n}\right\}, \left\{\frac{t(n-b)}{n}\right\}, \left\{\frac{t}{n}\right\}\right), \tag{6}$$

and recalling (3), we see that (6) gives the desired shape with $N = n/g$.

(iii) Since $\gcd(d_4, N) = 1$, we can choose $m \in \mathbb{Z}$ so that $md_4 = 1 + sN$. Now, let $T = T_{N-md_2, N-md_3, N}$ and $w = (d_4 - sd_2, d_4 - sd_3, d_4)$. It is easy to check that $BC_T(w) = \lambda$. □

There is an important clean tetrahedron in which the only interior point has $g > 1$. Let $T = T_{3,7,20}$; since 3, 7 and $c(3,7) = -9$ are relatively prime to 20, $T_{3,7,20}$ is clean. A routine computation shows that $\{\frac{9t}{20}\} + \{\frac{17t}{20}\} + \{\frac{13t}{20}\} + \{\frac{t}{20}\} = 1$ for $1 \le t \le 19$ if and only if $t = 5$, so T is a 1-point tetrahedron with $BC(w) = (\{\frac{45}{20}\}, \{\frac{85}{20}\}, \{\frac{65}{20}\}, \{\frac{5}{20}\}) = (\frac14, \frac14, \frac14, \frac14)$. We show in Theorem 14 that, up to equivalence, T is the only 1-point tetrahedron for which $g > 1$.

Let $\Lambda = \left\{\sum_{j=1}^4 r_j v_j : r_j \in \mathbb{Z},\ \sum_{j=1}^4 r_j = 1\right\}$ denote the affine lattice determined by the vertices of T. When $T = T_{a,b,n}$, the fundamental region of $\mathbb{Z}^3/\Lambda$ can be taken to be a parallelepiped with edges $v_j - v_i$, $v_k - v_i$, $v_\ell - v_i$, for any permutation of the vertices. It follows from (5) that $\{w_t : 0 \le t \le n-1\}$ is a set of representatives of $\mathbb{Z}^3/\Lambda$. Let $M(a,b,n)$ denote the $(n-1) \times 4$ matrix whose t-th row is $BC(w_t)$; namely, $\left(\left\{\frac{t(n-c)}{n}\right\}, \left\{\frac{t(n-a)}{n}\right\}, \left\{\frac{t(n-b)}{n}\right\}, \frac{t}{n}\right)$. Permutation of the first three columns yields $M(a,c,n), M(c,b,n)$, etc. In order to involve the fourth column, we must permute the rows as well to insure that the last entry in the first row is $1/n$. If, for example, $u \equiv -b^{-1} \pmod{n}$, then the u-th row of $M(a,b,n)$ is

$$\left(\frac{n-(-b^{-1}c)}{n}, \frac{n-(-b^{-1}a)}{n}, \frac{1}{n}, \frac{n-b^{-1}}{n}\right),$$

and after permuting the last two columns, we obtain the first row of $M(-b^{-1}a, b^{-1}, n)$. If $gcd(u,n) = 1$, then $\{ut \pmod n : 1 \le t \le n-1\}$ is simply a permutation of $\{t \pmod n : 1 \le t \le n-1\}$, so replacing t by ut in the definition of $M(a,b,n)$ permutes its rows. It follows in this way from Lemma 1 that $T_{a,b,n} \equiv T_{a',b',n}$ if and only if $M(a',b',n)$ can be derived from $M(a,b,n)$ after a permutation of rows and columns.

As an application of Theorem 4(i), consider $T_{n-1,n-1,n}$; since $c = 3-2n$, we assume $\gcd(n,3) = 1$. In this case, $BC(w_t) = (\{\frac{-3t}{n}\}, \frac{t}{n}, \frac{t}{n}, \frac{t}{n})$, and $A_t = 1$ precisely for $1 \le t \le \lfloor \frac{n}{3} \rfloor$, so $i(T) = \lfloor \frac{n}{3} \rfloor$. Han Duong has computed $i(T)$ for all clean $T_{a,b,n}$ with $1 \le n \le 100$ and found that, for $k = i(T)$, the inequality $3k+1 \le n \le 12k+8$ holds. In every case that $n = 3k+1$, the tetrahedra are equivalent to $T_{3k,3k,3k+1}$; in every case that $n = 12k+8$, the tetrahedra are equivalent to $T_{2k+1,4k+3,12k+8}$. Additionally, he has shown that $i(T_{2k+1,4k+3,12k+8}) = k$ for all $k \ge 1$. Duong and the author conjecture that these bounds are, in fact, sharp for all $k \ge 1$. Since $A_t + A_{n-t} = 4$, in the proof of Theorem 4, it follows that $n \ge 2k+1$ in any case. Heuristically, one would "expect" $k \approx \frac{n}{6}$ if the lattice points were evenly spaced in the fundamental parallelepiped; the conjecture suggests bounds of roughly $\frac{n}{3}$ and $\frac{n}{12}$. This conjecture will be discussed at greater length in [2].

Duong has an elegant proof of the bound $n = 3k+1$, though without uniqueness. Suppose T is a lattice tetrahedron (not necessarily clean) with k interior points. We use these points to subdivide T. Every point used in this subdivision is either interior to one of the tetrahedra, or on an interior face or on an interior edge. In these cases, the subdivision creates 3, 4 and 3 new tetrahedra, respectively, and so in the end, T is a union of at least $3k+1$ lattice tetrahedra, each of which has volume $\ge 1/6$.

The characterization of empty tetrahedra was first made by White [17, pp.390-394], using a longish combinatorial proof. P. Noordzij [11] gave a generalization with a longer, elementary proof. There have since been a short, but sophisticated proof using L-functions by D. Morrison and G. Stevens [10, pp. 16-17], and combinatorial proofs by H. E. Scarf [16, pp. 411-413] (based in part on work of R. Howe) and Handelman [4, pp. 145-148]. We present yet another elementary proof.

Theorem 5. *The clean lattice tetrahedron T is empty if and only if $T \approx T_{0,0,1}$ or $T \approx T_{1,d,n}$, where $\gcd(d,n) = 1$ and $1 \le d \le n-1$. This equivalence can be effected by a unimodular map sending v_4 to $(1,d,n)$.*

Proof. If T is empty and $vol(T) = \frac{1}{6}$, then $T \approx T_{0,0,1}$, which is clearly empty.

We may now assume $vol(T) = \frac{n}{6} > \frac{1}{6}$, so $T \approx T_{a,b,n}$, $n \ge 2$. Observe that $T_{1,d,n}$ is contained in the slab $0 \le x \le 1$. Thus, if $w = (r,s,t) \in T_{1,d,n} \cap \mathbb{Z}^3$, then $r = 0$ (and w is on the edge v_1v_3) or $r = 1$ (and w is on the edge from v_2v_4.) Since $\gcd(d,n) = 1$, this implies that w is a vertex, and so $T_{1,d,n}$ is empty. (Put another way, $T_{1,d,n}$ has lattice width 1, because it lies in the consecutive planes $x = 0$ and $x = 1$; White expressed his result by saying that the vertices of T lie in consecutive lattice planes. We return to this topic in section five.)

Conversely, suppose $T_{a,b,n}$ is empty, where $n \geq 2$. Referring to (4), it follows from Theorem 4(i) that $A_t > 1$ for $1 \leq t \leq n-1$. Further,

$$A_t + A_{n-t} = \left(\left\{\frac{t(n-c)}{n}\right\} + \left\{\frac{(n-t)(n-c)}{n}\right\}\right) + \cdots$$

can be arranged into a sum of four pairs of terms each of which sums to 1, hence $A_t + A_{n-t} = 4$, and so $A_t = 2$ for $1 \leq t \leq n-1$. It is now convenient to let $n-1 \geq a_3 \geq a_2 \geq a_1$ be the ordered rearrangement of $\{n-a, n-b, n-c\}$. In view of Lemma 1, it suffices to show that $a_3 = n-1$. Writing $a_0 = 1$, we have $A_t = \sum_{j=0}^{3}\{ta_j/n\} = 2$ for $1 \leq t \leq n-1$, and taking $t = 1$, we see that $a_0 + a_1 + a_2 + a_3 = 2n$. Define $B_t := \sum_{j=0}^{3}\lfloor ta_j/n\rfloor$. Then

$$A_t + B_t = \sum_{j=0}^{3}\left(\left\{\frac{ta_j}{n}\right\} + \left\lfloor\frac{ta_j}{n}\right\rfloor\right) = \sum_{j=0}^{3}\frac{ta_j}{n} = t(A_1 + B_1),$$

and since $A_1 = 2$ and $B_1 = 0$, we conclude that $B_t = 2(t-1)$ for $1 \leq t \leq n-1$. In particular, $B_2 = 2$, and since $\lfloor 2a_j/n\rfloor \in (0,2)$, we must have $a_3 \geq a_2 \geq n/2 > a_1$. Write $a_3 = n - b_2$ and $a_2 = n - b_1$, so that $b_2 \leq b_1 < n/2$ and note that $(n-b_1) + (n-b_2) + a_1 + 1 = 2n$, hence $b_1 + b_2 = a_1 + 1$.

We need to show that $b_2 = n - a_3 = 1$. Suppose not; then $2 \leq b_2 \leq b_1 \leq a_1 - 1 < n/2$. We have

$$2(t-1) = \left\lfloor\frac{t(n-b_1)}{n}\right\rfloor + \left\lfloor\frac{t(n-b_2)}{n}\right\rfloor + \left\lfloor\frac{ta_1}{n}\right\rfloor + \left\lfloor\frac{t}{n}\right\rfloor. \tag{7}$$

But $t/n \in (0,1)$ and $t(n-b_j)/n \notin \mathbb{Z}$ implies that $\lfloor t(n-b_j)/n\rfloor = t + \lfloor t(-b_j)/n\rfloor = t - \lfloor tb_j/n\rfloor - 1$. Thus, it follows from (7) that

$$2(t-1) = (t-1) - \left\lfloor\frac{tb_1}{n}\right\rfloor + (t-1) - \left\lfloor\frac{tb_2}{n}\right\rfloor + \left\lfloor\frac{ta_1}{n}\right\rfloor,$$

and so

$$C_t := \left\lfloor\frac{ta_1}{n}\right\rfloor = \left\lfloor\frac{tb_1}{n}\right\rfloor + \left\lfloor\frac{tb_2}{n}\right\rfloor. \tag{8}$$

For $r \in \mathbb{R}\setminus\mathbb{Z}$ and $k \in \mathbb{Z}$, let $\Delta_k(r) := \lfloor (k+1)r\rfloor - \lfloor kr\rfloor$. Since $\Delta_k(r) = r + \{kr\} - \{(k+1)r\}$, we have $|\Delta_k(r) - r| < 1$, hence $\Delta_k(r) = \lfloor r\rfloor$ or $\lfloor r\rfloor + 1$. In particular, if $r \in (0,1)$, then $\Delta_k(r) \in \{0,1\}$. Further, $\Delta_k(r) = 1$ if and only if there is an integer h so that $kr < h \leq (k+1)r$; that is, $k < h/r \leq k+1$, or $k+1 = \lceil h/r\rceil$. If we also know that $h/r \notin \mathbb{Z}$, then $k = \lfloor h/r\rfloor$, and so for fixed $r \in (0,1)$, $\Delta_k(r) = 1$ on a sequence of k's with jumps of size $\lfloor 1/r\rfloor$ or $\lfloor 1/r\rfloor + 1$.

For $1 \leq t \leq n-2$, let $D_t = C_{t+1} - C_t$. It follows from (8) that

$$D_t = \Delta_t\left(\frac{a_1}{n}\right) = \Delta_t\left(\frac{b_1}{n}\right) + \Delta_t\left(\frac{b_2}{n}\right), \tag{9}$$

so in particular, $D_t = 0$ or 1. Let $S = \{t : D_t = 1\}$. It follows from (9) that

$$S = \left\{ \left\lfloor \frac{n}{a_1} \right\rfloor, \ \dots, \ \left\lfloor \frac{(a_1-1)n}{a_1} \right\rfloor \right\}.$$

(Observe that $kn/a_1 \notin \mathbb{Z}$ for $1 \le k \le a_1 - 1$.) Letting $g = \lfloor n/a_1 \rfloor \ge 2$, we see that S contains $a_1 - 1$ integers, with jumps of g or $g+1$. Let

$$S_j = \left\{ \left\lfloor \frac{n}{b_j} \right\rfloor, \ \dots, \ \left\lfloor \frac{(b_j-1)n}{b_j} \right\rfloor \right\}.$$

for $j = 1, 2$; it also follows from (9) that $S = S_1 \cup S_2$, where $|S_j| = b_j - 1$; note that $a_1 - 1 = (b_1 - 1) + (b_2 - 1)$ and $kn/b_j \notin \mathbb{Z}$ for $1 \le k \le b_j - 1$. Since $b_1 \ge b_2$, $\lfloor n/b_1 \rfloor \le \lfloor n/b_2 \rfloor$ and so $g = \lfloor n/b_1 \rfloor$. Thus S_1 is also a sequence with jumps of size g or $g+1$. Since $2g > g+1$, any elements in S that are not in S_1 must come from the ends of S, not the middle. But S and S_1 have the same first element g and the last element,

$$\left\lfloor \frac{(a_1-1)n}{a_1} \right\rfloor = \left\lfloor n - \frac{n}{a_1} \right\rfloor = n - (g+1) = \left\lfloor \frac{(b_1-1)n}{b_1} \right\rfloor,$$

which gives the contradiction. We conclude that $b_2 = 1$, completing the proof. □

Reeve [13] observed that $T_{1,d,n}$ is always empty, but that $a = 1$ or $b = 1$ is not a necessary condition, as $T_{2,5,7}$ is empty. In our notation, $c(2,5) \equiv 1-2-5 \pmod 7 = 1$. M. Khan [7] has given a formula for the number of equivalence classes of empty tetrahedra of volume $n/6$.

We conclude this section with an observation that will be essential in the next section.

Corollary 6. *Suppose T is a clean tetrahedron. Then T is empty if and only if, for every $w = (r,s,t) \in \mathbb{Z}^3$, the $\lambda_{j,T}(w)$'s sum pairwise to integers.*

Proof. If T is empty, then $T \approx T_{0,0,1}$ or $T \approx T_{1,d,n}$ by Theorem 5. (Permutation of the vertices is irrelevant to the condition given.) For $T = T_{1,d,n}$, and $w = (r,s,t)$, (5) shows that $\lambda_{1,T}(w) + \lambda_{3,T}(w) = 1 - r$ and $\lambda_{2,T}(w) + \lambda_{4,T}(w) = r$.

If T is not empty and w is an interior point, then the relations $0 < \lambda_{j,T}(w) < 1$ and $\sum_j \lambda_{j,T}(w) = 1$ show that no sum of two can be integral. □

The geometric interpretation of this result (see [7]) is that in the fundamental region of $\mathbb{Z}^3/\Lambda$, every point in $\mathbb{Z}^3$ lies in one of the three interior diagonal parallelograms which avoid T. Khan points out the surprising fact that there is no purely geometric proof of this result. Alex Kasprzyk has pointed out to the author that clean tetrahedra correspond to a cone which, in toric geometry, is a terminal quotient singularity, and remarks that this situation is discussed in [14, p.379].

4. 1-POINT LATTICE TETRAHEDRA

The goal of this section is to prove the following classification theorem, which has been proved in a somewhat different way by A. Kasprzyk [6].

Theorem 7. *If T is a 1-point lattice tetrahedron, then T is equivalent to $T_{3,3,4}$, $T_{2,2,5}$, $T_{2,4,7}$, $T_{2,6,11}$, $T_{2,7,13}$, $T_{2,9,17}$, $T_{2,13,19}$ or $T_{3,7,20}$.*

The proof of this result will come from combining Theorems 13 and 14 below. We begin our discussion with a derivation of the possible barycentric coordinates for the interior lattice point. This was done in [15] in a less transparent way. Our proof relies on two simple observations about a clean 1-point lattice tetrahedron T with interior lattice point w. The first is that $BC_T(w)$ must have special arithmetic properties (via Corollary 6). The second is that w subdivides T into four *empty* tetrahedra.

First, suppose $T = T(v_1, v_2, v_3, v_4)$ is a clean tetrahedron and $w \in int(T)$, where

$$BC_T(w) = \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{d_1}{N}, \frac{d_2}{N}, \frac{d_3}{N}, \frac{d_4}{N}\right). \tag{10}$$

Since $T_{0,0,1}$ is empty, Theorem 3 implies that T is equivalent to some $T_{a,b,n}$, satisfying (3), and we may assume $T = T_{a,b,n}$, where, by Theorem 4(ii), $n = gN$. (Caveat: in the proof, we might subsequently apply one of the maps from Lemma 1 to permute the vertices and replace $T_{a,b,n}$ with an equivalent $T_{a',b',n}$. This, of course, permutes the d_j's.)

The following is a slight restatement of Theorem 4.

Theorem 8. *(i) If T is a 1-point lattice tetrahedron with interior point w satisfying* (10)*, then*

$$2 \le s \le N-1 \implies \sum_{j=1}^{4} \left\{\frac{sd_j}{N}\right\} > 1. \tag{11}$$

(ii) If $g = 1$ and (11) *holds, then T is a 1-point lattice tetrahedron.*

Proof. (i) Suppose T is a 1-point lattice tetrahedron, but (11) fails for $s = s_0 \ge 2$. Then

$$w' := \sum_{j=1}^{4} \{s_0\lambda_j\} v_j = \sum_{j=1}^{4} (s_0\lambda_j - \lfloor s_0\lambda_j \rfloor) v_j = s_0 w - \sum_{j=1}^{4} \lfloor s_0\lambda_j \rfloor v_j \in \mathbb{Z}^3$$

is also an interior lattice point in T, hence $w' = w$, and by the uniqueness of barycentric coordinates, $\{s_0\lambda_j\} = \lambda_j$. It follows that $(s_0-1)\lambda_j \in \mathbb{Z}$ and $N \mid (s_0-1)d_j$, so N divides $s_0 - 1$, a contradiction.

(ii) Let $w = (r, s, t_0)$. Then by (6) and (10),

$$\left(\frac{d_1}{N}, \frac{d_2}{N}, \frac{d_3}{N}, \frac{d_4}{N}\right) = \left(\left\{\frac{t_0(n-c)}{n}\right\}, \left\{\frac{t_0(n-a)}{n}\right\}, \left\{\frac{t_0(n-b)}{n}\right\}, \left\{\frac{t_0}{n}\right\}\right). \tag{12}$$

As noted earlier, since $\gcd(t_0, N) = 1$, multiplication by t_0 permutes the non-zero residue classes mod N. Thus, (11) and (12) imply that $A_t = 1$ only for $t = t_0$, hence $k = 1$ and T is a 1-point lattice tetrahedron. $\square$

The condition $g = 1$ is essential in Theorem 8(ii). For example, consider $T_{7,7,8}$ and $w = (2,2,2)$. Then $BC(w) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, which clearly satisfies (11) with $g = 2$. However, $(1,1,1) = \frac{1}{2}(v_1 + w)$ is another interior point in $T_{7,7,8}$.

The second observation we make is simpler. The interior point w subdivides T into four tetrahedra: $T_1 = T(v_2, v_3, v_4, w)$, $T_2 = T(v_1, v_3, v_4, w)$, $T_3 = T(v_1, v_2, v_4, w)$ and $T_4 = T(v_1, v_2, v_3, w)$. (We fix these notations for the rest of this section.) The following lemma was used in [9], though without the simplified implications of Lemma 6.

Lemma 9. *The four lattice tetrahedra T_1, T_2, T_3 and T_4 are empty.*

Proof. This is immediate, because $T_j \cap \mathbb{Z}^3 \subseteq T \cap \mathbb{Z}^3 = \{v_1, v_2, v_3, v_4, w\}$. $\square$

Lemma 10. *Suppose T is a 1-point tetrahedron, then in the notation of* (10)*, each d_i divides one of $d_j + d_k$, $d_j + d_\ell$ or $d_k + d_\ell$, where $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$.*

Proof. Suppose $i = 4$ for clarity. Then since $w = \sum_{j=1}^4 \lambda_j v_j$, we have

$$v_4 = \left(\frac{-d_1}{d_4}\right) v_1 + \left(\frac{-d_2}{d_4}\right) v_2 + \left(\frac{-d_3}{d_4}\right) v_3 + \left(\frac{N}{d_4}\right) w, \tag{13}$$

so $BC_{T_4}(v_4) = (-d_1/d_4, -d_2/d_4, -d_3/d_4, N/d_4)$. By Lemma 9, T_4 is empty, and hence by Corollary 6, d_4 divides one of $\{d_1 + d_2, d_1 + d_3, d_2 + d_3\}$. $\square$

We shall say that $(d_1, d_2, d_3, d_4) \in \mathbb{N}^4$ is *ripe* if, for $N = \sum d_i$, we have:

(i) For each j, $\gcd(d_j, N) = 1$.
(ii) Each d_i divides one of $d_j + d_k$, $d_j + d_\ell$ or $d_k + d_\ell$.
(iii) If $d_j = d_i$ or $d_j = 2d_i$, then $d_i = 1$.

Lemma 11. *If T is a 1-point lattice tetrahedron, then (d_1, d_2, d_3, d_4) is ripe.*

Proof. Conditions (i) and (ii) follow from Theorem 4(ii) and Lemma 10. For (iii), suppose for concreteness that $d_1 > 1$ and $d_2 = \epsilon d_1$, where $\epsilon \in \{1, 2\}$. Let $1 \neq s \equiv d_1^{-1} \pmod{N}$. We claim that for $j = 3, 4$, $sd_j \not\equiv N - 1 \pmod{N}$. Suppose otherwise. Then $sd_j \equiv N - 1 \pmod{N}$ implies $d_j \equiv N - d_1 \pmod{N}$, so $d_j = N - d_1$, hence $d_1 + d_2 + d_3 + d_4 > N$. It follows that $\{sd_j/N\} \le (N-2)/N$ for $j = 3, 4$, thus

$$\sum_{j=1}^{4} \left\{\frac{sd_j}{N}\right\} = \frac{1}{N} + \frac{\epsilon}{N} + \sum_{j=3}^{4} \left\{\frac{sd_j}{N}\right\} \le \frac{1 + \epsilon + 2(N-2)}{N} < 2.$$

Since this sum is an integer, it must equal 1, which contradicts Theorem 8(i). $\square$

We now need a tedious case-analysis, which is nevertheless shorter than the tedious case-analysis in the corresponding proof in [15].

Theorem 12. *If* (d_1, d_2, d_3, d_4) *is ripe and* $d_1 \le d_2 \le d_3 \le d_4$*, then* (d_1, d_2, d_3, d_4) *is* $(1,1,1,1)$*,* $(1,1,1,2)$*,* $(1,1,2,3)$*,* $(1,2,3,5)$*,* $(1,3,4,5)$*,* $(2,3,5,7)$ *or* $(3,4,5,7)$*.*

Proof. It is easy to verify that each of the quadruples given in the statement is ripe. Note also that if $d_j = \alpha_j t + \beta_j u$ for integers α_j, β_j, t, u, $1 \le j \le 4$, then $\gcd(t,u) = 1$; otherwise we would have $\gcd(d_j, N) > 1$, violating (i).

We first consider the cases in which at least two d_j's are equal, so $d_1 = d_2 = 1$ by (iii). If $(1,1,1,u)$ is ripe, then $u \mid 2$ by (ii), so $u = 1$ or 2. These give the first two examples. If $(1,1,t,u)$ is ripe, with $t, u \ge 2$, then u divides 2 or $t+1$ by (ii). But $u = 2$ implies $t = 2$ by monotonicity, violating (iii). Therefore, $u \mid t+1$, and $u \ge t$ implies $u = t+1$. But if $(1,1,t,t+1)$ is ripe, then $t > 1$ divides 2 or $t+2$, so $t = 2$, giving $(1,1,2,3)$, the third example.

We now assume

$$1 \le d_1 < d_2 < d_3 < d_4. \tag{14}$$

Since d_4 divides a pair-sum less than $2d_4$, it must equal that pair-sum. There are three cases, which we consider in turn: (a) $d_4 = d_2 + d_3$, (b) $d_4 = d_1 + d_3$, (c) $d_4 = d_1 + d_2$.

If $(d_1, d_2, d_3, d_2 + d_3)$ is ripe, then d_3 divides $d_1 + d_2, d_1 + d_2 + d_3$ or $2d_2 + d_3$ and hence $d_1 + d_2$ or $2d_2$. Both are $< 2d_3$, so either $d_3 = d_1 + d_2$ or $d_3 = 2d_2$. The latter implies that $d_2 = 1$ by (iii), violating (14), so $d_3 = d_1 + d_2$.

After writing $d_1 = t$ and $d_2 = u$, we now suppose that $(t, u, t+u, t+2u)$ is ripe, where $u \ge 2$. Then u divides $2t+u$, $2t+2u$ or $2t+3u$, and so $u \mid 2t$. Since $\gcd(t,u) = 1$, $u = 2$ and $(t, 2, t+2, t+4)$ is ripe. It follows from (14) that $2 = t+1$ by (14), and we obtain $(1,2,3,5)$, the fourth example. This completes case (a).

Next, suppose that $(d_1, d_2, d_3, d_1 + d_3)$ is ripe, so d_3 divides $d_1 + d_2$, $2d_1 + d_3$ or $d_1 + d_2 + d_3$, and hence either $d_1 + d_2$ or $2d_1$. Both are $< 2d_3$, so either $d_3 = d_1 + d_2$ or $d_3 = 2d_1$. Again, $d_3 = 2d_1$ implies $d_1 = 1$ by (iii), so $d_3 = 2$, violating (14). Thus, $d_3 = d_1 + d_2$ and $(t, u, t+u, 2t+u)$ is ripe, so again, $u \ge 2$. We have $N = 4t + 3u$, so (i) implies that u is odd. Since t divides $t + 2u$, $2t + 2u$ or $3t + 2u$, we have $t \mid 2u$, so $t \in \{1, 2\}$. If $t = 1$ and $(1, u, 1+u, 2+u)$ is ripe, then u divides $2+u$, $3+u$ or $3+2u$, forcing $u = 3$ and $(1,3,4,5)$, the fifth example. If $t = 2$ and $(2, u, 2+u, 4+u)$ is ripe, then u divides $4+u$, $6+u$ or $6+2u$. Again, $u = 3$ is the only choice, giving $(2,3,5,7)$, the sixth example. This completes case (b).

Finally, suppose $(d_1, d_2, d_3, d_1 + d_2)$ is ripe, so d_3 divides $d_1 + d_2$, $2d_1 + d_2$ or $d_1 + 2d_2$. Since $d_1 + d_2 < 2d_3$, these are each $< 3d_3$. If $d_3 = d_1 + d_2$, $2d_1 + d_2$ or $d_1 + 2d_2$, then $d_3 \ge d_4$, violating (14). Thus, $2d_3$ equals $d_1 + d_2, 2d_1 + d_2$ or $d_1 + 2d_2$, and the first is impossible, so either $2d_3 = 2d_1 + d_2$ or $2d_3 = d_1 + 2d_2$, so either d_2 or d_1 is even and either $(t, 2v, t+v, t+2v)$ or $(2v, t, t+v, t+2v)$ is ripe. Except for the order of the first two terms, these are the same case. Since $2v$ divides $2t+v$, $2t+2v$ or $2t+3v$, $v \mid 2t$, so $v \in \{1, 2\}$. Suppose $v = 1$ and $(t, 2, t+1, t+2)$ or $(2, t, t+1, t+2)$ is ripe. The former is impossible by (14); the latter implies that $t \ge 3$ divides $t+3$, $t+4$ or $2t+3$, which implies $t = 3$ (and $t+1 = 4$) or $t = 4$. In either case, (iii) is violated. In the final case, $v = 2$ and $(t, 4, t+2, t+4)$ or $(4, t, t+2, t+4)$ is ripe. The former implies that $4 = t+1$, yielding $(3,4,5,7)$, the final example. The latter implies that $t > 4$ is odd and divides $t+6$, $t+8$ or $2t+6$, which is impossible. This completes case (c) and the proof. ☐

The concept of ripeness seems worth considering in its own right, especially if (iii) is jettisoned. Without (iii), there are several additional families of ripe quadruples satisfying (i) and (ii): $(1,t,t,t)$, $(2,t,t,t,)$ (where $\gcd(t,2) = 1$), $(1,t,t,2t)$, $(3,t,t,2t)$ (where $\gcd(t,3) = 1$), $(1,t,2t,3t)$ and $(5,t,2t,3t)$ (where $\gcd(t,5) = 1$). These families generalize the first four of the examples given. It would also be interesting to study ripeness in n-tuples for $n \geq 5$.

We now use ripeness to give a new proof of Theorem 5.9 in [15].

Theorem 13. *(i) If T is a 1-point lattice tetrahedron with interior point w, then, up to a permutation of the coordinates, $BC_T(w)$ is one of the following quadruples:* $\lambda^{(1)} = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$, $\lambda^{(2)} = \left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{2}{5}\right)$, $\lambda^{(3)} = \left(\frac{1}{7},\frac{1}{7},\frac{2}{7},\frac{3}{7}\right)$, $\lambda^{(4)} = \left(\frac{2}{11},\frac{1}{11},\frac{3}{11},\frac{5}{11}\right)$, $\lambda^{(5)} = \left(\frac{1}{13},\frac{3}{13},\frac{4}{13},\frac{5}{13}\right)$, $\lambda^{(6)} = \left(\frac{2}{17},\frac{3}{17},\frac{5}{17},\frac{7}{17}\right)$, $\lambda^{(7)} = \left(\frac{3}{19},\frac{5}{19},\frac{4}{19},\frac{7}{19}\right)$.

(ii) If T is a 1-point lattice tetrahedron and $g = 1$, then T is equivalent to $T_{3,3,4}$, $T_{2,2,5}$, $T_{2,4,7}$, $T_{2,6,11}$, $T_{2,7,13}$, $T_{2,9,17}$ or $T_{2,13,19}$.

Proof. (i) By Lemma 10, $BC_T(w)$ must be ripe. It is easily checked that every ripe quadruple in Theorem 12 satisfies (11), and so is $BC_T(w)$ for some 1-point tetrahedron by Theorem 8(ii).

(ii) Suppose $g = 1$. Then $n = N$ and $\lambda = \lambda^{(\ell)}$ up to some permutation of the coordinates. Take $f \in \mathcal{L}$ (if necessary) to permute the vertices of T so that $\lambda = \lambda^{(\ell)}$ and T is a clean $T_{a',b',n} (\approx T_{a,b,n})$ via Lemma 1. Then $a', b' \in \mathbb{Z}$ and

$$w = \left(\frac{d_1}{n}\right) \cdot (0,0,0) + \left(\frac{d_2}{n}\right) \cdot (1,0,0) + \left(\frac{d_3}{n}\right) \cdot (0,1,0) + \left(\frac{d_4}{n}\right) \cdot (a',b',n).$$

Note that $w \in \mathbb{Z}^3$ if and only if $d_2 + d_4 a' \equiv d_3 + d_4 b' \equiv 0 \pmod{n}$; that is, $a' \equiv -d_2 d_4^{-1} \pmod{n}$ and $b' \equiv -d_3 d_4^{-1} \pmod{n}$. Using (d_2, d_3, d_4) from the seven cases in (i), we obtain the seven tetrahedra in (ii). ☐

An application of Lemma 1 yields the equivalent clean 1-point tetrahedra of the form $T_{a,b,n}$ with $0 \leq a, b \leq n-1$. (In case parameters are repeated, treat the triples as multi-sets.)

(1) $T_{3,3,4}$;
(2) $T_{a,b,5}$, $\{a,b\} \in \{2,2,2\}, \{3,4,4\}$;
(3) $T_{a,b,7}$, $\{a,b\} \in \{2,2,4\}, \{2,3,3\}, \{4,5,6\}$;
(4) $T_{a,b,11}$, $\{a,b\} \in \{2,3,7\}, \{2,4,6\}, \{3,4,5\}, \{6,8,9\}$;
(5) $T_{a,b,13}$, $\{a,b\} \in \{2,3,9\}, \{2,5,7\}, \{3,4,7\}, \{8,9,10\}$;
(6) $T_{a,b,17}$, $\{a,b\} \in \{2,3,13\}, \{2,7,9\}, \{5,6,7\}, \{4,5,9\}$;
(7) $T_{a,b,19}$, $\{a,b\} \in \{2,5,13\}, \{3,4,13\}, \{3,7,10\}, \{4,5,11\}$.

We complete our characterization of 1-point lattice tetrahedra. Suppose now that T is a 1-point lattice tetrahedron and $g = n/N > 1$, where $BC_T(w) = \lambda^{(\ell)}$ for some

$1 \le \ell \le 7$ and, without loss of generality, $\lambda_4^{(\ell)}$ is the largest component of $\lambda^{(\ell)}$. Since $vol(T_4) = \lambda_4 \cdot vol(T) = \lambda_4 n/6 = gd_4/6 > 1/6$, the (empty) tetrahedron T_4 is equivalent to T_{1,e,gd_4}, where

$$1 \le e \le gd_4 - 1, \qquad \gcd(e, gd_4) = 1. \tag{15}$$

We apply a unimodular map so that

$$v_1 = (0,0,0),\ v_2 = (1,0,0),\ v_3 = (0,1,0),\ w = (1, e, gd_4). \tag{16}$$

In doing so, the vertices v_1, v_2, v_3 may be permuted, and so $BC_T(w) = \lambda^{(\ell)}$, with some uncertainty about the order of the first three cooordinates. It follows from (13) that

$$v_4 = \left(\frac{N - d_2}{d_4}, \frac{eN - d_3}{d_4}, gN \right). \tag{17}$$

Since $v_4 \in \mathbb{Z}^3$, we see that d_4 must divide $N - d_2$ and $eN - d_3$. This first condition reduces to $d_4 \mid d_1 + d_3$, which has already been accommodated in Theorem 13(i); in fact, it is easy to check that the first coordinate of v_4 is 3 (if $\ell = 1$), and 2 (if $\ell \ge 2$). The second condition is that $eN \equiv d_3 \bmod d_4$. Further, since T is clean, Theorem 3 implies that

$$\gcd\left(\frac{N - d_2}{d_4}, gN \right) = \gcd\left(\frac{eN - d_3}{d_4}, gN \right) = \gcd\left(\frac{eN + d_1}{d_4}, gN \right) = 1. \tag{18}$$

(We have used here that the numerator of $-c = a + b - 1$ is $N - d_2 + eN - d_3 - d_4$.)

We discuss the seven cases in turn.

First, suppose $\ell = 1$, so that $d_1 = d_2 = d_3 = d_4 = 1, N = 4$, and

$$v_4 = (3, 4e - 1, 4g), \qquad w = (1, e, g),$$

where $1 \le e \le g - 1$. Observe that the unimodular map $(x, y, z) \mapsto (x, 1 - x - y + z, z)$ permutes v_1 and v_3, fixes v_2 and sends $v_4 \mapsto (3, 4(g - e) - 1, 4g)$ and $w \mapsto (1, g - e, g)$, hence we may assume without loss of generality that $e \le g/2$. Further, if g is even, then $\frac{1}{2}(v_2 + w)$ or $\frac{1}{2}(v_4 + w)$ is a lattice point, depending on whether e is even or odd; thus, g is odd and $1 \le e \le (g-1)/2$. Here, (15) and (18) imply that $\gcd(e, g) = \gcd(3, 4g) = \gcd(4e - 1, g) = \gcd(4e + 1, 4g) = 1$. In particular, $g \ne 3$.

If $g = 5$, then the possible values for e are 1 or 2, and the first is ruled out by $gcd(5, 20) > 1$, hence $e = 2$. In this case $T = T_{3,7,20}$, which has already been identified as another 1-point tetrahedron. Now suppose $g \ge 7$. We compute $BC_{T_j}(0,0,1)$ for $j = 2, 3$:

$$(0,0,1) = \left(\frac{e + g + 1}{g} \right) \cdot v_1 + \left(\frac{-e + 1}{g} \right) \cdot v_3 + \left(\frac{1}{g} \right) \cdot v_4 + \left(\frac{-3}{g} \right) \cdot w;$$
$$(0,0,1) = \left(\frac{2e + g}{g} \right) \cdot v_1 + \left(\frac{e - 1}{g} \right) \cdot v_2 + \left(\frac{e}{g} \right) \cdot v_4 + \left(\frac{-4e + 1}{g} \right) \cdot w.$$

Since T_2 and T_3 are empty, Corollary 6 implies that g divides 2, $e - 2$ or $e + 2$ and g divides $2e - 1$, $3e - 1$ or $3e$. Since $e + 2 \le \frac{g+3}{2} < g$, the first condition implies that $e = 2$, and this means that the second condition is impossible, and there are no "new" 1-point tetrahedra with $\lambda = \lambda^{(1)}$. This reproduces the result of Mazur [9].

We now consider $\ell \ge 2$. In this case, we have $\frac{N-d_2}{d_4} = 2$, and so by (18), $\gcd(2, gN) = 1$; thus, $g \ge 3$ is odd.

Suppose $\ell = 2$, so $d_1 = d_2 = d_3 = 1, d_4 = 2, N = 5$. It follows from (17) that $5e - 1$ is even, hence $e = 2k + 1$ is odd. Then

$$w = (1, 2k + 1, 2g), \qquad v_4 = (2, 5k + 2, 5g),$$

with $0 \le k \le g - 1$ and $\gcd(2k + 1, 2g) = \gcd(5k + 2, 5g) = \gcd(5k + 3, 5g) = 1$. If $g = 3$, note that $k = 1, 2, 0$ (in that order) are ruled out by the gcd conditions, so we must have $g \ge 5$. Again, we compute $BC_{T_j}(0, 0, 1)$ for $j = 2, 3$:

$$(0,0,1) = \left(\frac{g+k+1}{g}\right) \cdot v_1 + \left(\frac{-k}{g}\right) \cdot v_3 + \left(\frac{1}{g}\right) \cdot v_4 + \left(\frac{-2}{g}\right) \cdot w;$$

$$(0,0,1) = \left(\frac{g+2k+1}{g}\right) \cdot v_1 + \left(\frac{k}{g}\right) \cdot v_2 + \left(\frac{2k+1}{g}\right) \cdot v_4 + \left(\frac{-5k-2}{g}\right) \cdot w.$$

Corollary 6 implies that g must divide $k + 2$, $k - 1$ or 1 and g must divide $3k + 1$ or $4k + 2$. The first implies that $k = 1$ or $k = g - 2$. Since odd $g \ge 5$ cannot divide 4 or 6, we must have $k = g - 2$, so g divides $3g - 5$ or $4g - 6$. This implies that $g = 5$, so $k = 3$ and $T = T_{2,17,25}$. However, T contains the interior point $(1, 5, 7)$, as well as $w = (1, 7, 10)$, so is not a 1-point lattice tetrahedron.

Suppose $\ell = 3$, so $d_1 = d_2 = 1$, $d_3 = 2$, $d_4 = 3$ and $N = 7$. It follows from (17) that $3 \mid 7e - 2$, hence $e = 3k + 2$ and

$$w = (1, 3k + 2, 3g), \qquad v_4 = (2, 7k + 4, 7g),$$

with $0 \le k \le g - 1$ and $\gcd(3k + 2, 3g) = \gcd(7k + 4, 7g) = \gcd(7k + 5, 7g) = 1$. If $g = 3$, the gcd conditions rule out $k = 1, 2$, so $k = 0$. But $T = T_{2,4,21}$ contains $(1, 1, 5)$ and $(1, 2, 10)$ as well as $w = (1, 2, 9)$, and so is not a 1-point lattice tetrahedron. Otherwise, $g \ge 5$, and we once again compute $BC_{T_j}(0, 0, 1)$ for $j = 2, 3$:

$$(0,0,1) = \left(\frac{g+k+1}{g}\right) \cdot v_1 + \left(\frac{-k}{g}\right) \cdot v_3 + \left(\frac{1}{g}\right) \cdot v_4 + \left(\frac{-2}{g}\right) \cdot w;$$

$$(0,0,1) = \left(\frac{2g+3k+2}{2g}\right) \cdot v_1 + \left(\frac{k}{2g}\right) \cdot v_2 + \left(\frac{3k+2}{2g}\right) \cdot v_4 + \left(\frac{-7k-4}{2g}\right) \cdot w.$$

Again, g divides 1, $k - 1$ or $k + 2$, so $k = 1$ or $k = g - 2$, from the first equation. If $k = 1$, then $2g$ divides 6 or 10, so $g = 5$ and $k = 3$, but $gcd(7 \cdot 3 + 4, 7 \cdot 5) > 1$. If $k = g - 2$, then $2g$ divides 6 or 8, which is impossible.

Suppose $\ell = 4$, so $d_1 = 2$, $d_2 = 1$, $d_3 = 3$, $d_4 = 5$ and $N = 11$. It follows from (17) that $5 \mid 11e - 3$, hence $e = 5k + 3$ and

$$w = (1, 5k + 3, 5g), \qquad v_4 = (2, 11k + 6, 11g),$$

with $0 \le k \le g-1$ and $\gcd(5k+3, 5g) = \gcd(11k+6, 11g) = \gcd(11k+7, 11g) = 1$. If $g = 3$, the gcd conditions rule out $k = 0, 1$, so $k = 2$. But $T = T_{2,28,33}$ contains $(1, 6, 7)$

and $(1,7,8)$ as well as $w = (1,13,15)$ and so is not a 1-point lattice tetrahedron. Otherwise, $g \geq 5$, and we once again compute $BC_{T_j}(0,0,1)$ for $j = 2,3$:

$$(0,0,1) = \left(\frac{g+k+1}{g}\right) \cdot v_1 + \left(\frac{-k}{g}\right) \cdot v_3 + \left(\frac{1}{g}\right) \cdot v_4 + \left(\frac{-2}{g}\right) \cdot w;$$

$$(0,0,1) = \left(\frac{3g+5k+3}{3g}\right) \cdot v_1 + \left(\frac{k}{3g}\right) \cdot v_2 + \left(\frac{5k+3}{3g}\right) \cdot v_4 + \left(\frac{-11k-6}{g}\right) \cdot w.$$

Again, g divides 1, $k-1$ or $k+2$, so $k = 1$ or $k = g-2$, from the first equation. If $k = 1$, then by the second equation, $3g$ divides 9 or 16, neither one of which is possible. If $k = g-2$, then $3g$ divides $6g-9$ or $10g-14$, which are also both impossible.

Suppose $\ell = 5$, so $d_1 = 1$, $d_2 = 3$, $d_3 = 4$, $d_4 = 5$ and $N = 13$. It follows from (17) that $5 \mid 13e - 4$, hence once again $e = 5k+3$ and

$$w = (1, 5k+3, 5g), \qquad v_4 = (2, 13k+7, 13g),$$

with $0 \leq k \leq g-1$ and $\gcd(5k+3, 5g) = \gcd(13k+7, 13g) = \gcd(13k+8, 13g) = 1$. If $g = 3$, the gcd conditions rule out $k = 0, 2, 1$, in that order, so $g \geq 5$. As before,

$$(0,0,1) = \left(\frac{3g+3k+2}{3g}\right) \cdot v_1 + \left(\frac{-3k-1}{3g}\right) \cdot v_3 + \left(\frac{1}{3g}\right) \cdot v_4 + \left(\frac{-2}{3g}\right) \cdot w;$$

$$(0,0,1) = \left(\frac{4g+5k+3}{4g}\right) \cdot v_1 + \left(\frac{3k+1}{4g}\right) \cdot v_2 + \left(\frac{5k+3}{4g}\right) \cdot v_4 + \left(\frac{-13k-7}{4g}\right) \cdot w.$$

From the first equation, $3g$ divides 1, $3k$ or $3k+3$, so $k = 0$ or $k = g-1$. If $k = 0$, then the second equation implies that $4g$ divides 4 or 6. If $k = g-1$, then the second equation implies that $4g$ divides $8g-4$ or $10g-4$. None of these is possible.

Suppose $\ell = 6$, so $d_1 = 2$, $d_2 = 3$, $d_3 = 5$, $d_4 = 7$ and $N = 17$. It follows from (17) that $7 \mid 17e - 5$, hence $e = 7k+4$ and

$$w = (1, 7k+4, 7g), \qquad v_4 = (2, 17k+9, 17g),$$

with $0 \leq k \leq g-1$ and $\gcd(7k+4, 7g) = \gcd(17k+9, 17g) = \gcd(17k+10, 17g) = 1$. If $g = 3$, the gcd conditions rule out $k = 2, 0, 1$, in that order, so $g \geq 5$, and

$$(0,0,1) = \left(\frac{3g+3k+2}{3g}\right) \cdot v_1 + \left(\frac{-3k-1}{3g}\right) \cdot v_3 + \left(\frac{1}{3g}\right) \cdot v_4 + \left(\frac{-2}{3g}\right) \cdot w;$$

$$(0,0,1) = \left(\frac{5g+7k+4}{5g}\right) \cdot v_1 + \left(\frac{3k+1}{5g}\right) \cdot v_2 + \left(\frac{7k+4}{5g}\right) \cdot v_4 + \left(\frac{-17k-9}{5g}\right) \cdot w.$$

From the first equation, $3g$ again divides 1, $3k$ or $3k+3$, so $k = 0$ or $k = g-1$. If $k = 0$, the second equation implies that $5g$ divides 5 or 8. If $k = g-1$, then the second equation implies that $5g$ divides $10g-5$ or $14g-6$. Again, none of these is possible.

Finally, suppose $\ell = 7$, so $d_1 = 3$, $d_2 = 5$, $d_3 = 4$, $d_4 = 7$ and $N = 19$. It follows from (17) that $7 \mid 19e - 4$, hence $e = 7k + 5$ and

$$w = (1, 7k + 5, 7g), \qquad v_4 = (2, 19k + 13, 19g),$$

with $0 \le k \le g-1$ and $\gcd(7k+5, 7g) = \gcd(19k+13, 19g) = \gcd(19k+14, 19g) = 1$. If $g = 3$, the gcd conditions rule out $k = 1, 2$, so $k = 0$. But $T = T_{2,13,57}$, which contains at least $(1, 3, 13)$ and $(1, 4, 17)$ as well as $w = (1, 5, 21)$, and so is not a 1-point lattice tetrahedron. Otherwise, $g \ge 5$, and, one last time:

$$(0,0,1) = \left(\frac{5g+5k+4}{5g}\right) \cdot v_1 + \left(\frac{-5k-3}{5g}\right) \cdot v_3 + \left(\frac{1}{5g}\right) \cdot v_4 + \left(\frac{-2}{5g}\right) \cdot w;$$

$$(0,0,1) = \left(\frac{4g+7k+5}{4g}\right) \cdot v_1 + \left(\frac{5k+3}{4g}\right) \cdot v_2 + \left(\frac{7k+5}{4g}\right) \cdot v_4 + \left(\frac{-19k-13}{4g}\right) \cdot w.$$

The first equation implies that $5g$ divides one of 1, $5k + 2$ or $5k + 5$, so $k = g - 1$, and the second equation implies that $4g$ divides $12g - 4$ or $14g - 4$, neither of which is possible.

We have at long last completed a detailed proof of the following theorem, which completes the classification of the 1-point tetrahedra.

Theorem 14. *If T is a 1-point tetrahedron with $BC_T(w) = \lambda^{(\ell)}$ and $g > 1$, then $\ell = 1$, $g = 5$ and $T \approx T_{3,7,20} \approx T_{3,11,20} \approx T_{7,11,20}$.*

5. Lattice widths and other questions

If $S \in \mathbb{R}^n$ is a lattice polytope and $u \in \mathbb{Z}^n$ then its *u-width* is defined to be $\max\{u \cdot x : x \in S\} - \min\{u \cdot x : x \in S\}$, and its *lattice width* is the minimum of its u-widths, taken over $u \in \mathbb{Z}^n \setminus 0$. (Without loss of generality, we may always assume that the components of u have no common factor.) Since $u \cdot x = (uM^{-1}) \cdot (Mx)$, lattice width is preserved by unimodular maps. If S has lattice width w, then $S \cap \mathbb{Z}^n$ lies in $w+1$ consecutive "lattice hyperplanes" $\pi_{j_0}, \dots \pi_{j_0+w}$, where $\pi_j = \{x : u \cdot x = j\}$. The Euclidean distance between π_j and π_k is $|k-j|/|u|$, so a small geometric distance may correspond to a large lattice width if u has large components.

Theorem 5 shows that an empty lattice tetrahedron must have lattice width 1. This does not hold for simplices in dimension $d \ge 4$; see [3]. We present a possibly sporadic result for 1-point lattice tetrahedra; it is proved using Theorem 7, rather than by an *a priori* argument.

Corollary 15. *If T is a 1-point lattice tetrahedron, then T has lattice width 2.*

Proof. This is immediately true for any $T \approx T_{2,b,n}$, which is contained in $0 \le x \le 2$; take $u = (1, 0, 0)$. The two remaining cases are $T_{3,3,4}$ and $T_{3,7,20}$, which are contained in $0 \le x + y - z \le 2$ and $0 \le 2x + 2y - z \le 2$, respectively. $\square$

More generally, suppose $T = T_{a,b,n}$ and $u = (r, s, t)$. Then we see that the u-width of $T_{a,b,n}$ is equal to

$$\max\{0, r, s, ar + bs + nt\} - \min\{0, r, s, ar + bs + nt\}. \tag{19}$$

We present, without proof, the directions $\pm u$ in which the 1-point tetrahedra have width 2. Up to sign, the planes must be $\{\pi_0, \pi_1, \pi_2\}$ if $v_1 = (0, 0, 0)$ is on an outer plane, or $\{\pi_{-1}, \pi_0, \pi_1\}$ if v_1 is on the middle plane. This gives us a small finite set of (r, s) to check. Let $\ell_j = |T \cap \pi_j|$ and let $\ell_u(T) := (\ell_0, \ell_1, \ell_2)$ or $\ell_u(T) := (\ell_{-1}, \ell_0, \ell_1)$, respectively. Since the interior point must lie in the middle plane, the four possibilities for $\ell_u(T)$ are $(3, 1, 1)$ (or $(1, 1, 3)$, $(2, 1, 2)$, $(2, 2, 1)$ (or (1,2,2)) and $(1, 3, 1)$. The first pair is impossible: suppose a face of T lies on a plane, then after a unimodular map we have $u = (0, 0, 1)$, the face can be placed on $z = 0$, and has area $1/2$ by Pick's Theorem. By hypothesis, T has altitude 2, and so volume $\frac{2}{6}$, which is too small for a 1-point tetrahedron. It turns out that $\ell_u(T)$ is *not* an invariant; several of the smaller 1-point tetrahedra have different configurations in different directions.

Somewhat surprisingly, $T_{3,3,4}$ has width two in nine directions: $\ell_u(T) = (1, 3, 1)$ for $u = (1, 0, -1), (0, 1, -1), (1, -1, 0), (2, 1, -2), (1, 2, -2)$ and $(1, 1, -1)$ and $\ell_u(T) = (2, 1, 2)$ for $u = (2, 0, -1), (0, 2, -1)$ or $(2, 2, -3)$. The next larger tetrahedron, $T_{2,2,5}$, has width two in six directions: $\ell_u(T) = (1, 3, 1)$ for $u = (2, 1, -1), (1, 2, -1)$ and $(1, -1, 0)$ and $\ell_u(T) = (2, 2, 1)$ for $u = (1, 0, 0), (0, 1, 0)$ and $(1, 1, -1)$. Next, $T_{2,4,7}$ has width two in four directions: $\ell_u(T) = (1, 3, 1)$ for $u = (2, 1, -1)$ and $\ell_u(T) = (2, 2, 1)$ for $u = (1, 0, 0), (0, 2, -1)$ and $(1, 1, -1)$. Each of $T_{2,6,11}, T_{2,7,13}$ and $T_{2,9,17}$ has width two in two directions: $\ell_u(T) = (2, 2, 1)$ for $u = (1, 0, 0)$ and $u = (0, 2, -1)$. Finally, $T_{2,13,19}$ and $T_{3,7,20}$ each have width two in one direction: $\ell_u(T) = (2, 2, 1)$ for $u = (1, 0, 0)$ and $u = (2, 2, -1)$ respectively.

A check of 2-point lattice tetrahedra shows that most have lattice width 2; however $T_{3,5,23}$ is one with lattice width 3.

We make some elementary remarks about the width of lattice k-tetrahedra for $k \geq 2$. As noted above, $T_{3k,3k,3k+1}$ is a lattice k-tetrahedron; it also has width 2, considering $u = (1, -1, 0), (1, 0, -1)$ or $(0, 1, -1)$. Thus, width need not go to infinity with $i(T)$. The following result applies to all $T_{a,b,n}$, whether clean or not.

Theorem 16. *The lattice width of $T_{a,b,n}$ is $\leq 2\lceil n^{1/3} \rceil$.*

Proof. This is a simple pigeonhole principle argument. Let $m = \lceil n^{1/3} \rceil$ and consider $\{ra + sb \pmod{n} : 0 \leq r, s \leq m\}$. There are $(m+1)^2$ residues, and so two must differ by at most $(n-1)/(m+1)^2 < m$. Thus, $(r_1 - r_2)a + (s_1 - s_2)b \equiv j \pmod{n}$ with $0 \leq r_1, r_2, s_1, s_2, j \leq m$. Now let $r = r_1 - r_2$ and $s = s_1 - s_2$ and choose t so that $ra + sb + tn = j$ and let $u = (r, s, t)$. Since $-m \leq r, s \leq m$, (19) implies the lattice width of T in the u-direction is at most $2m$. $\square$

The following example shows that the bound in Theorem 16 has the correct order of magnitude. Let $T = T_{m,m^2,m^3+1}$. We first check that this is clean. Clearly, $\gcd(m, m^3 + 1) = \gcd(m^2, m^3 + 1) = 1$. Observe that $c(m, m^2, m^3 + 1) = 1 - m - m^2$

and let $h = \gcd(c, m^3+1)$. Since c is odd, so is h, and since h is odd, and since $(m-1)c + (m^3+1) = 2m$, we have that $h \mid m$. But h divides m^3+1, so $h = 1$.

We claim that T has lattice width m. This bound is achieved for $u = (1,0,0)$ or $u = (1, m, -1)$. Suppose otherwise that there exists $u = (r, s, t)$ so that

$$\max\{0, r, s, mr + m^2 s + (m^3+1)t\} - \min\{0, r, s, mr + m^2 s + (m^3+1)t\} \le m - 1.$$

Then in particular, $|r|, |s| \le m-1$, and so $|mr + m^2 s| \le (m+m^2)(m-1) = m^3 - m$. If $|t| \ge 1$, then

$$|mr + m^2 s + (m^3+1)t| \ge |(m^3+1)t| - (m^3 - m) \ge m + 1.$$

If $t = 0$, then $|mr + m^2 s| = m|r + ms| \ge m$ unless $r + ms = 0$. In this case, $m \mid r$ implies that $r = 0$, so $s = 0$ and $u = (0,0,0)$. Therefore, the lattice width equals m.

We make the following conjecture.

Conjecture 17. *If T is a k-point lattice tetrahedron, then its lattice width is $\le k+1$, and there is at least one interior lattice point on each of the consecutive lattice planes in any minimal direction.*

If T is a 2-point lattice tetrahedron with interior points w_1 and w_2, then each interior point subdivides T into 4 tetrahedra. The other interior point will be on an edge, a face or interior to one of the subtetrahedra, and each case can occur. For example, $T_{5,5,7}$ has two interior points: $w_1 = (1,1,1)$ and $w_2 = (3,3,4)$, and $w_2 = \frac{1}{2}(v_4 + w_1)$ is on an edge, whereas $w_1 = \frac{1}{4}(v_1 + v_2 + v_3 + w_2)$ is interior. Another 2-point lattice tetrahedron is $T_{5,5,8}$, with interior points $w_1 = (1,1,1)$ and $w_2 = (2,2,3)$. Here, each is on a face determined by the other: $w_1 = \frac{1}{3}(v_2 + v_3 + w_2)$, $w_2 = \frac{1}{3}(v_1 + v_4 + w_1)$. It is possible for both to be interior; for example if $T = T_{11,13,16}$ with $w_1 = (1,1,1)$ and $w_2 = (5,6,7)$, then

$$BC_{T(v_1,v_2,v_3,w_2)}(w_1) = \left(\tfrac{3}{7}, \tfrac{2}{7}, \tfrac{1}{7}, \tfrac{1}{7}\right), \qquad BC_{T(v_2,v_3,v_4,w_1)}(w_2) = \left(\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{3}{7}, \tfrac{1}{7}\right).$$

Another worthwhile project would be the classification of lattice tetrahedra with one interior point and a positive number of boundary points. Lemma 3 could be used in the special case that one of the four faces has no non-vertex lattice points; the arguments of Theorem 4 can be adapted to count the number of lattice points in $T_{a,b,n}$, when (3) does not hold. It is not clear how to proceed if no face is relatively empty. In view of White's Theorem, these questions become considerably more difficult, even in four dimensions.

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