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Algebraic Geometry

A new approach to Hilbert's theorem on ternary quartics

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Abstract

Hilbert proved that a non-negative real quartic form $f(x, y, z)$ is the sum of three squares of quadratic forms. We give a new proof which shows that if the plane curve Q defined by f is smooth, then f has exactly 8 such representations, up to equivalence. They correspond to those real 2-torsion points of the Jacobian of Q which are not represented by a conjugation-invariant divisor on Q. ***To cite this article: V. Powers et al., C. R. Acad. Sci. Paris, Ser. I ●●● (●●●●).***

Résumé

Une nouvelle approche du théorème de Hilbert sur les quartiques ternaires. Hilbert a démontré qu'une forme réelle non négative $f(x, y, z)$ de degré 4 est la somme de trois carrés de formes quadratiques. Nous donnons une nouvelle démonstration qui montre que si la courbe plane Q definie par f est non singulière, alors f a exactement 8 telles représentations, à equivalence près. Elles correspondent aux points de 2- torsion du jacobien de Q qui ne sont pas représentés par un diviseur de Q invariant par conjugaison. ***Pour citer cet article : V. Powers et al., C. R. Acad. Sci. Paris, Ser. I ●●● (●●●●).***

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1. Introduction

A *ternary quartic form* is a homogeneous polynomial $f(x,y,z)$ of degree 4 in three variables. If f has real coefficients, then f is *non-negative* if $f(x,y,z) \geqslant 0$ for all real x, y, z. Hilbert [5] showed that every non-negative real ternary quartic form is a sum of three squares of quadratic forms. His proof (see [8,9] for modern expositions) was non-constructive: The map

$$\pi : (p,q,r) \longmapsto p^2 + q^2 + r^2$$

from triples of real quadratic forms to non-negative quartic forms is surjective, as it is both open and closed when restricted to the preimage of the (dense) connected set of non-negative quartic forms which define a smooth complex plane curve. An elementary and constructive approach to Hilbert's theorem was recently begun by Pfister [6].

A *quadratic representation* of a complex ternary quartic form $f = f(x,y,z)$ is an expression

$$f = p^2 + q^2 + r^2 \tag{1}$$

where p, q, r are complex quadratic forms. A representation $f = (p')^2 + (q')^2 + (r')^2$ is *equivalent* to this if p,q,r and p',q',r' have the same linear span in the space of quadratic forms.

Powers and Reznick [7] investigated quadratic representations computationally, using the Gram matrix method of [1]. In several examples of non-negative real ternary quartics, they always found 63 inequivalent representations as a sum of three squares of complex quadratic forms; 15 of these were sums or differences of squares of real forms. We explain these numbers, in particular the number 15, and show that precisely 8 of the 15 are sums of squares.

If the complex plane curve Q defined by $f = 0$ is smooth, it has genus 3, and so the Jacobian J of Q has $2^6 - 1 = 63$ non-zero 2-torsion points. Coble [2, Chapter 1, §14] showed that these are in one-to-one correspondence with equivalence classes of quadratic representations of f. If f is real, then Q and J are defined over $\mathbb{R}$. The non-zero 2-torsion points of $J(\mathbb{R})$ correspond to *signed quadratic representations* $f = \pm p_1^2 \pm p_2^2 \pm p_3^2$, where p_i are real quadratic forms. If f is also non-negative, the real Lie group $J(\mathbb{R})$ has two connected components, and hence has $2^4 - 1 = 15$ non-zero 2-torsion points. We use Galois cohomology to determine which 2-torsion points give rise to sum of squares representations over $\mathbb{R}$.

Theorem 1.1. *Suppose that $f(x,y,z)$ is a non-negative real quartic form which defines a smooth plane curve Q. Then the inequivalent representations of f as a sum of three squares are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of $J(\mathbb{R})$, where J is the Jacobian of Q.*

2. Quadratic representations of smooth ternary quartics

Let $f(x,y,z)$ be an irreducible quartic form over $\mathbb{C}$, and let Q be the curve $f = 0$ in the complex projective plane. Assume that Q is smooth. The Picard group $\mathrm{Pic}(Q)$ of Q is the group of Weil divisors on Q, modulo divisors of rational functions. Let J be the Jacobian of Q, so that J is the identity component of $\mathrm{Pic}(Q)$. The following proposition is due to Coble [2, Chapter 1, §14].

Proposition 2.1. *The non-trivial 2-torsion points of J are in one-to-one correspondence with the equivalence classes of quadratic representations of f.*

Proof. Given a quadratic representation (1), consider the map

$$\varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad x \mapsto \bigl(p(x) : q(x) : r(x)\bigr).$$

The image of Q under φ is the conic C defined by the equation $y_0^2 + y_1^2 + y_2^2 = 0$. Let y be any point in C, then $\varphi^*(y)$ is an effective divisor of degree 4 that is not the divisor of a linear form. Indeed, after a linear change of

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coordinates we can assume $y=(0:1:\mathrm{i})$. A linear form vanishing on $\varphi^*(y)$ would divide each conic $\alpha p+\beta(q+\mathrm{i}r)$ through $\varphi^*(y)$, and thus would divide

$$f=p^2+(q+\mathrm{i}r)(q-\mathrm{i}r),$$

contradicting the irreducibility of f.

Fix a linear form ℓ, then $L:=\mathrm{div}(\ell)$ is an effective divisor of degree 4 on Q. Let $\zeta=[\varphi^*(y)-L]$. Since $2y$ is the divisor of a linear form (the tangent line to C at y), $\varphi^*(2y)$ is the divisor on Q of a quadratic form. Thus $2\zeta=0$. Moreover, $\zeta\neq 0$ as $\varphi^*(y)$ is not the divisor of a linear form. The 2-torsion point ζ of J depends only upon the map φ.

Conversely, suppose that $\zeta\in J(\mathbb{C})$ is a non-zero 2-torsion point. Let $D\neq D'$ be effective divisors which represent the class $\zeta+[L]$ in $\mathrm{Pic}(Q)$. As Q has genus 3, the Riemann–Roch Theorem implies that there is a pencil of such divisors. Then $2D$, $2D'$ and $D+D'$ are effective divisors of degree 8, and are all linearly equivalent to $2L$, the divisor of a conic. Again from the Riemann–Roch Theorem it follows that there are quadratic forms q_0, q_1 and q_2 such that

$$\mathrm{div}(q_0)=2D,\quad \mathrm{div}(q_1)=2D'\quad\text{and}\quad \mathrm{div}(q_2)=D+D'.$$

Therefore, the rational function $g:=q_0q_1/q_2^2$ on Q is constant. Scaling q_1 and q_2 appropriately, we may assume that $g\equiv 1$ on Q and also that $f=q_0q_1-q_2^2$. Diagonalizing the quadratic form $q_0q_1-q_2^2$ gives a quadratic representation for f. This defines the inverse of the previous map. □

3. Quadratic representations of real quartics

Suppose now that f is a non-negative real quartic form defining a smooth real plane curve Q with complexification $Q_{\mathbb{C}}=Q\otimes_{\mathbb{R}}\mathbb{C}$. The elements of $\mathrm{Pic}(Q)$ can be identified with those divisor classes in $\mathrm{Pic}(Q_{\mathbb{C}})$ that are represented by a conjugation-invariant divisor. Let J be the Jacobian of Q.

If $\zeta\in J(\mathbb{C})$ is the 2-torsion point corresponding to a signed quadratic representation

$$f=\pm p^2\pm q^2\pm r^2$$

consisting of real polynomials p,q,r, then $\zeta=\bar\zeta$, i.e., $\zeta\in J(\mathbb{R})$.

Conversely, let $0\neq\zeta\in J(\mathbb{R})$ with $2\zeta=0$, and let L be the divisor on Q of a linear form ℓ. We can choose an effective divisor $D\neq\overline{D}$ on $Q_{\mathbb{C}}$ representing the class $\zeta+[L]$. Then $2D$, $2\overline{D}$ and $D+\overline{D}$ are each equivalent to $2L$. Let r be a real quadratic form with divisor $D+\overline{D}$, and let g be a complex quadratic form with divisor $2D$ (both divisors taken on $Q_{\mathbb{C}}$).

Since $D\sim\overline{D}$, there is a rational function h on $Q_{\mathbb{C}}$ with $\mathrm{div}(h)=\overline{D}-D$. Let $c=h\bar h$, a nonzero real constant on Q. Since $\mathrm{div}(r)=\mathrm{div}(g)+\mathrm{div}(h)$, there is a complex number $\alpha\neq 0$ with $\frac{r}{g}=\alpha h$ on Q, which implies that

$$c|\alpha|^2=\frac{r}{g}\cdot\frac{\bar r}{\bar g}=\frac{r^2}{p^2+q^2}$$

on Q, where p and q are the real and imaginary parts of $g=p+\mathrm{i}q$. So the quartic form

$$u:=r^2-c\,|\alpha|^2(p^2+q^2)$$

vanishes identically on Q. Since $u\neq 0$, f is a constant multiple of u. If $c>0$, we get a signed quadratic representation of f, with both signs $\pm$ occurring. If $c<0$, f must be a positive multiple of u since f is non-negative, and we get a representation of f as a sum of three squares of real forms.

We now calculate the sign of c. For this we use the well-known exact sequence

$$0\to\mathrm{Pic}(Q)\to\mathrm{Pic}(Q_{\mathbb{C}})^G\xrightarrow{\partial}\mathrm{Br}(\mathbb{R})\to\mathrm{Br}(Q).$$

It arises from the Hochschild–Serre spectral sequence for étale cohomology with coefficients $\mathbb{G}_m$. Here $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathrm{Pic}(Q_{\mathbb{C}})$ by conjugation, and $\mathrm{Pic}(Q_{\mathbb{C}})^G$ is the group of G-invariant divisor classes. Moreover, $\mathrm{Br}(\mathbb{R})$ is the Brauer group of $\mathbb{R}$, which is of order 2, and $\mathrm{Br}(Q)$, the Brauer group of Q, can be identified with the subgroup of $\mathrm{Br}\,\mathbb{R}(Q)$ consisting of all Brauer classes which are everywhere unramified. The map $\mathrm{Br}(\mathbb{R}) \to \mathrm{Br}(Q)$ is the restriction map.

It is easy to see that $c < 0$ if and only if $\partial(\zeta)$ is the non-trivial class in $\mathrm{Br}(\mathbb{R})$.

By a classical theorem of Witt [12], every non-negative rational function on a smooth projective curve over $\mathbb{R}$ is a sum of two squares of rational functions. Since Q is smooth and f is non-negative, this forces $Q(\mathbb{R}) = \emptyset$. Hence -1 is a sum of two squares in $\mathbb{R}(Q)$. This means $(-1, -1) = 0$ in $\mathrm{Br}(Q)$, and hence the map ∂ is surjective.

Since the genus of Q is odd (equal to 3), it follows from a classical theorem of Weichold [11,3] that all classes in $\mathrm{Pic}(Q_{\mathbb{C}})^G$ have even degree, and the real Lie group $J(\mathbb{R})$ has exactly two connected components. This implies that the sequence

$$0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \xrightarrow{\partial} \mathrm{Br}(\mathbb{R}) \to 0$$

is (split) exact. Since $J(\mathbb{R})^0 \cong (S^1)^3$ as a real Lie group, there exist $2^4 - 1 = 15$ non-zero 2-torsion classes in $J(\mathbb{R})$. The 8 that do not lie in $J(\mathbb{R})^0$, or equivalently, which cannot be represented by a conjugation-invariant divisor on $Q_{\mathbb{C}}$, are precisely those that give rise to sums of squares representations of f. This completes the proof of Theorem 1.1.

We close with a few remarks about the singular case. Wall [10] studies quadratic representations of (possibly singular) complex ternary quartic forms f. If f is irreducible, the non-trivial 2-torsion points on the generalized Jacobian of the curve $Q = \{f = 0\}$ again give equivalence classes of quadratic representations of f. These representations are special in that they have no basepoints.

Quadratic representations with a given base locus $B \neq \emptyset$ are in one-to-one correspondence with all 2-torsion points on the Jacobian of a curve $\widetilde{Q}$, which is the image of Q under the complete linear series of quadrics through B. By classifying all possibilities for B one arrives at the number of inequivalent quadratic representations of f. If the form f is real and non-negative, this classification, together with arguments from Galois cohomology, gives all inequivalent representations of f as a sum of squares. If f is reducible, different methods can be applied to complete the picture. This complete analysis will appear in an unabridged version.

Uncited references

[4]

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