THE MAXIMAL ANGULAR GAP AMONG RECTANGULAR GRID POINTS

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Abstract. Let $\mathcal{A} = \{a_1, \dots, a_m\}$ and $\mathcal{B} = \{b_1, \dots, b_n\}$ be two sets of real numbers. Consider the (at most) mn rays from the origin to the points (a_i, b_j), and define the aperture $\mathrm{Ap}(\mathcal{A}, \mathcal{B})$ to be the largest angular gap between consecutive rays. Clearly, $\mathrm{Ap}(\mathcal{A}, \mathcal{B}) \ge \frac{2\pi}{mn}$. Let $f(m,n)$ denote the minimum aperture of any $m \times n$ rectangular array, as defined above. In this paper, we show, that for sufficiently large n, $f(n,n) < \frac{220}{n^2}$, so that $f(n,n) = \Omega(n^{-2})$. We also show that $f(m,n) = \frac{2\pi}{mn}$ only when $m = 2$, or $n = 2$ or $(m,n) = (4,4), (4,6)$ or $(6,4)$.

1. Introduction and overview

Let us define the *aperture* of an arbitrary point set $\mathcal{P} \subset \mathbf{R}^2$, $\mathrm{Ap}(\mathcal{P})$, to be the supremum of the angles of the empty open sectors defined by $\mathcal{P}$ and centered at the origin. If $\mathcal{P}$ is finite, then $\mathrm{Ap}(\mathcal{P})$ can be viewed as the maximum gap between consecutive rays from the origin to the points of $\mathcal{P}$. It is possible, of course, that some rays overlap, but the Pigeonhole Principle clearly implies that

$$\mathrm{Ap}(\mathcal{P}) \ge \frac{2\pi}{|\mathcal{P}|}. \tag{1.1}$$

In this paper we investigate the aperture of Cartesian products of finite sets.

Let $\mathcal{A} = \{a_1 < \cdots < a_m\}$ and $\mathcal{B} = \{b_1 < \cdots < b_n\}$ be two sets of real numbers and consider the rays from the origin to the mn points (a_i, b_j). The *aperture* of

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$\mathcal{A}$ and $\mathcal{B}$, $\text{Ap}(\mathcal{A},\mathcal{B})$, is the largest angle between *consecutive* rays. For example, if $\mathcal{A}=\mathcal{B}=\{-1,0,1\}$, then there are 8 rays, equally spaced, and $\text{Ap}(\mathcal{A},\mathcal{B})=\pi/4$. Let

$$f(m,n)=\inf\{\text{Ap}(\mathcal{A},\mathcal{B}) : |\mathcal{A}|=m, |\mathcal{B}|=n\}.$$

It is easy to see that $f(3,3)=\pi/4$. Suppose otherwise, and $|\mathcal{A}|=|\mathcal{B}|=3$ but $\text{Ap}(\mathcal{A},\mathcal{B})<\pi/4$. Then each of the open quadrants $k\pi/2<\theta<(k+1)\pi/2$ must contain at least two points from $\mathcal{A}\times\mathcal{B}$, and hence at most one point from $\mathcal{A}\times\mathcal{B}$ can have a coordinate equal to zero. This implies that $0\notin\mathcal{A},\mathcal{B}$, so $\mathcal{A}$ and $\mathcal{B}$ each have either one positive and two negative elements or one negative and two positive elements, and this implies that one open quadrant has only one point.

In section two we improve the trivial bound $f(m,n)\ge\frac{2\pi}{mn}$ slightly, based on the parity of m and n and using arguments similar to those of the last paragraph:

Theorem 1. *The following lower bounds hold:*

$$\begin{gathered} f(2r,2s)\ge\frac{2\pi}{4rs}; \\ f(2r,2s+1), f(2r+1,2s)\ge\frac{2\pi}{4rs+2}; \\ f(2r+1,2s+1)\ge\frac{2\pi}{4rs+4}. \end{gathered} \tag{1.2}$$

We say that $(\mathcal{A},\mathcal{B})$ is a *perfect* (m,n)*-grid* if its aperture is equal to the bound given in Theorem 1. It is shown in Lemma 3 that in this case, the angles of the rays are completely specified and are either $\{\frac{j\pi}{N}\}_{0\le j<N}$ or $\{\frac{(2j+1)\pi}{N}\}_{0\le j<N}$ for suitable N.

Observe that in any array, if $\tan\theta_1=a/b$, $\tan\theta_2=a/b'$, $\tan\theta_3=a'/b$ and $\tan\theta_4=a'/b'$, then $\tan\theta_1\tan\theta_4=\tan\theta_2\tan\theta_3$. Thus, any subrectangle of the array induces an identity in the products of the slopes of the rays. Using a theorem of Myerson [M] on the solutions to the Diophantine equation

$$\sin(\tfrac{a\pi}{n})\sin(\tfrac{b\pi}{n})=\sin(\tfrac{c\pi}{n})\sin(\tfrac{d\pi}{n}),$$

and a considerable amount of case-analysis, we are able to determine the perfect grids.

Theorem 2. *Perfect* (m,n)*-grids exist for* $m\ge n$ *precisely when*

$$(m,n)\in\{(m,2),(m,3),(4,4),(5,5),(6,4)\}.$$

In section three we examine the special case $m=n$, and even using the restrictive assumption $\mathcal{A}=\mathcal{B}$, we give an explicit construction showing that the lower bound given in Theorem 1 is the correct order of magnitude:

Theorem 3.

$$f(n,n) < \frac{220}{n^2}. \tag{1.3}$$

Further, if n is sufficiently large ($n \geq 14,000$, say), then "220" can be replaced by "168". Theorem 3 is done by first considering the angular sector $1/n^2 \leq \theta \leq \pi/4$, and then reflecting the construction eight-fold.

We conclude the paper in section four with a list of open questions.

The authors come to this problem from two directions. For the first author, this work fits into the context of the subject of discrepancy theory. In the last two decades discrepancy theory, which originates mainly from the theory of Diophantine approximations in number theory, has found interest and applications in geometry, probability theory, ergodic theory, computer science, combinatorics. See the book of Beck and Chen [BC], the chapter from the Handbook of Combinatorics [BS], or Vera Sós' survey [S].

One of the basic problems in geometric and number theoretic discrepancy theory is to find *regular* distributions of discrete objects, like a finite set of numbers or points in a region of the Euclidean plane. A large number of classical results can be formulated in this language. For example, it is obvious that an equidistant sequence of fractions with gaps exactly $1/n$ supplies the least irregular n-set in the unit interval. But as conjectured by van der Corput and proved by Aardenne-Ehrenfest and subsequently improved by K. Roth, by Davenport and others, there is no such *well-distributed* n-sequence, s, in the unit square; one can always find an aligned rectangle R which contains more than $\mu(R)n + c\sqrt{\log n}$ or less than $\mu(R)n - c\sqrt{\log n}$ points of s, where c is an absolute constant, and μ denotes Lebesgue measure (see [B] for details).

The motivations of the second author lie far from discrepancy. In the course of solving Waring's Problem in 1909, Hilbert proved the existence, for all positive integers n and k of rationals $\lambda_j > 0$ and $\alpha_j \in \mathbf{Z}^n$ so that

$$(x_1^2 + \cdots + x_n^2)^k = \sum_j \lambda_j(\alpha_{j1}x_1 + \cdots + \alpha_{jn}x_n)^{2k}. \tag{1.4}$$

Hausdorff [H] almost immediately gave a constructive proof of these identities using Hermite polynomials; in his construction, there are only $k+2$ distinct $\alpha_{j\ell}$'s, or $k+1$, if the restriction to $\mathbf{Z}^n$ is relaxed. (These bounds do not depend on n.) Hausdorff's construction, however, is not best possible in this regard; for example there is a representation of $(x_1^2 + \cdots + x_n^2)^3$ in which $\alpha_{j\ell} \in \{-1, 0, 1\}$. One way to find an absolute lower bound is to set $x_\ell = 0$ in (1.4) for $\ell \geq 3$. Consider the following special case:

$$(x^2 + y^2)^k = \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_{ij}(a_i x + a_j y)^{2k}. \tag{1.5}$$

It is not hard to show (see eg [R,p.108]) that (1.5) is equivalent to a quadrature formula for *homogeneous* polynomials $h(x,y)$ of degree $2k$:

$$\frac{1}{2\pi}\int_0^{2\pi} h(\cos\theta,\sin\theta)\ d\theta = 2^{-2k}\binom{2k}{k}\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_{ij}h(a_i,a_j). \tag{1.6}$$

Write $(a_i,a_j) = r_{ij}(\cos\theta_{ij},\sin\theta_{ij})$ and $\nu_{ij} = \lambda_{ij}r_{ij}^{2k}$, so (1.6) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} h(\cos\theta,\sin\theta)\ d\theta = 2^{-2k}\binom{2k}{k}\sum_{i=1}^{r}\sum_{j=1}^{r}\nu_{ij}h(\cos\theta_{ij},\sin\theta_{ij}). \tag{1.7}$$

There exist homogeneous polynomials h which are spherical harmonics, so that the integral in (1.7) vanishes, but which are positive on "most" of the interval $[0,2\pi]$. This quadrature formula is contradicted unless, in every case, we can be assured that at least one θ_{ij} belongs to the small intervals of negativity for h. For more details, see the forthcoming [R2]. Unfortunately, Theorem 3 shows that we cannot automatically assume a larger discrepancy than expected, and so the original purpose of the investigation has proved to be unfruitful in its application.

2. Sharpening the lower bound, and perfect grids.

We shall say that a set of real numbers $\mathcal{A}$ is *balanced* if it contains as many positive as negative numbers (and, so, $0 \in \mathcal{A}$ if and only if $|\mathcal{A}|$ is odd.) For positive integers m and n, define $L(m,n)$ by:

(2.1) $L(2r,2s) = 4rs,\ L(2r,2s{+}1) = L(2r{+}1,2s) = 4rs{+}2,\ L(2r{+}1,2s{+}1) = 4rs{+}4,$

Theorem 1. *For all integers m and n,*

$$f(m,n) \geq \frac{2\pi}{L(m,n)}.$$

If $\mathrm{Ap}(\mathcal{A},\mathcal{B}) = \frac{2\pi}{L(m,n)}$, *then there are exactly $L(m,n)$ distinct and evenly spaced rays from the origin to $\mathcal{A}\times\mathcal{B}$ and both $\mathcal{A}$ and $\mathcal{B}$ are balanced.*

Proof. First suppose that $m = 2r$ and $n = 2s$. There are at most $4rs$ rays, and a total angle of 2π, so the lower bound is immediate. If $\mathrm{Ap}(\mathcal{A},\mathcal{B}) = \frac{2\pi}{4rs}$, then the angle between any pair of consecutive rays must equal the aperture. Suppose now without loss of generality, that $\mathcal{A}$ and $\mathcal{B}$ have at least as many negative elements as positive elements. To be specific, suppose $\mathcal{A}$ has $k \leq r$ positive elements and $\mathcal{B}$ has $\ell \leq s$ positive elements. Then there will be $k\ell$ rays in the first quadrant which form the boundary for $k\ell + 1$ sectors. These sectors cover the quadrant and possibly more, hence $(k\ell+1)\frac{2\pi}{4rs} \geq \frac{\pi}{2}$. This implies $rs \leq k\ell+1$. But $k \leq r$ and $\ell \leq s$, so $k = r$ and $\ell = s$. Thus, $\mathcal{A}$ and $\mathcal{B}$ must be balanced.

Now suppose $m = 2r$ and $n = 2s+1$ and first suppose that $0 \notin \mathcal{B}$. Again, we may assume without loss of generality that $\mathcal{B}$ has $\ell \leq s$ positive elements. The $2r\ell$ rays with positive y component bound $2r\ell+1$ sectors which cover the upper half plane and more, and so $(2r\ell+1)Ap(\mathcal{A},\mathcal{B}) > \pi$, hence $\mathrm{Ap}(\mathcal{A},\mathcal{B}) > \frac{2\pi}{4r\ell+2} \geq \frac{2\pi}{4rs+2}$. If $0 \in \mathcal{B}$, then we may again assume that $\mathcal{B}$ has $\ell \leq s$ positive elements. The $2r\ell$ rays with positive component now bound $2r\ell+1$ sectors which cover the upper half plane (precisely if $\mathcal{A}$ has both positive and negative elements – if not, then $\mathrm{Ap}(\mathcal{A},\mathcal{B}) \geq \pi$!), and as before, we obtain $\mathrm{Ap}(\mathcal{A},\mathcal{B}) \geq \frac{2\pi}{4r\ell+2} \geq \frac{2\pi}{4rs+2}$. Thus $\mathrm{Ap}(\mathcal{A},\mathcal{B}) \geq \frac{2\pi}{L(2r,2s+1)}$. If equality occurs, then $0 \in \mathcal{B}$ and there are precisely $L(2r,2s+1) = 4rs+2$ rays (those to the points in $\mathcal{A}\times\mathcal{B}$ with non-zero coordinates, plus rays in the positive and negative x-directions), and they must therefore be evenly spaced.

Since $f(m,n) = f(n,m)$, the only remaining case is $(m,n) = (2r+1, 2s+1)$. This is handled similarly to the above. Suppose as before that $\mathcal{A}$ has $k \leq r$ and $\mathcal{B}$ has $\ell \leq s$ positive elements. If 0 is missing from either $\mathcal{A}$ or $\mathcal{B}$, then the $k\ell$ rays in the first quadrant are edges of bound $k\ell+1$ sectors with angle greater than $\frac{\pi}{2}$, so $(rs+1)\,\mathrm{Ap}(\mathcal{A},\mathcal{B}) \geq k\ell\,\mathrm{Ap}(A,B) > \frac{\pi}{2}$. If $0 \in \mathcal{A}$ and $0 \in \mathcal{B}$, then $\mathrm{Ap}(\mathcal{A},\mathcal{B}) \geq \frac{2\pi}{L(2r+1,2s+1)}$, with equality only when $k = r$ and $\ell = s$. □

We shall say that $(\mathcal{A},\mathcal{B})$ is a *perfect (m,n)-grid* if $|\mathcal{A}| = m$, $|\mathcal{B}| = n$ and $\mathrm{Ap}(\mathcal{A},\mathcal{B}) = \frac{2\pi}{L(m,n)}$. We show below that the angles of the rays in a perfect grid are determined by (m,n) and that these induce identities among products of the tangents of rational multiples of π. For this reason, we need to study the equation

$$\tan\alpha\tan\beta = \tan\gamma\tan\delta < 1. \tag{2.2}$$

We shall assume without loss of generality that

$$0 < \alpha < \gamma \leq \delta < \beta < \tfrac{\pi}{2}. \tag{2.3}$$

Since $\tan\theta\tan(\frac{\pi}{2}-\theta) = 1$, it follows immediately from (2.2) that

$$\alpha+\beta, \quad \gamma+\delta < \tfrac{\pi}{2}$$

Lemma 1. *If* (2.2) *and* (2.3) *hold, then in fact*

$$\gamma+\delta < \alpha+\beta < \tfrac{\pi}{2}. \tag{2.4}$$

Proof. If $\tan t\tan u = \lambda$ with $t \leq u$, then $\tan t \leq \sqrt{\lambda}$ and $u = \arctan(\lambda\cot t)$, and if we let $\Phi(t) = t+u = t+\arctan(\lambda\cot t)$, then

$$\Phi'(t) = -(1-\lambda)\frac{\lambda-\tan^2 t}{\lambda^2+\tan^2 t} < 0,$$

hence $\Phi(\alpha) > \Phi(\gamma)$. □

Lemma 2. *If* (2.2) *holds, then*

$$\sin(\tfrac{\pi}{2} - (\beta+\alpha))\sin(\tfrac{\pi}{2} - (\delta-\gamma)) = \sin(\tfrac{\pi}{2} - (\beta-\alpha))\sin(\tfrac{\pi}{2} - (\delta+\gamma)). \tag{2.5}$$

where

$$0 < \tfrac{\pi}{2} - (\beta+\alpha) < \tfrac{\pi}{2} - (\delta+\gamma), \quad \tfrac{\pi}{2} - (\beta-\alpha) < \tfrac{\pi}{2} - (\delta-\gamma) < \tfrac{\pi}{2}. \tag{2.6}$$

Proof. We have

$$\tan\alpha\tan\beta = \frac{2\sin\alpha\sin\beta}{2\cos\alpha\cos\beta} = \frac{\cos(\beta-\alpha) - \cos(\alpha+\beta)}{\cos(\beta-\alpha) + \cos(\alpha+\beta)}$$

and similarly,

$$\tan\gamma\tan\delta = \frac{\cos(\delta-\gamma) - \cos(\gamma+\delta)}{\cos(\delta-\gamma) + \cos(\gamma+\delta)}.$$

After cross-multiplying and cancelling like terms, we obtain

$$\cos(\alpha+\beta)\cos(\delta-\gamma) = \cos(\beta-\alpha)\cos(\gamma+\delta).$$

We now reduce using $\cos x = \sin(\frac{\pi}{2} - x)$ to obtain (2.5). The inequalities in (2.6) follow from (2.3) and (2.4). □

The reason we have recast our problem in this way is that (2.5) was completely analyzed by Myerson [M,p.80] when the arguments are rational multiples of π. First, consider the identity

$$\sin\tfrac{\pi}{6}\sin\varphi = \sin\tfrac{\varphi}{2}\sin\left(\tfrac{\pi}{2} - \tfrac{\varphi}{2}\right). \tag{2.7}$$

Myerson's Theorem. *Suppose*

$$\sin\pi x_1 \sin\pi x_2 = \sin\pi x_3 \sin\pi x_4. \tag{2.8}$$

There are only finitely many solutions to (2.8) *other than* (2.7), *in which the* x_i *are rational numbers satisfying* $0 < x_1 < x_3 \le x_4 < x_2 < 1/2$ *– in the following numbered list,* x_1, x_2, x_3 *and* x_4 *are given in order:*

1.	1/21	8/21	1/14	3/14
2.	1/14	5/14	2/21	5/21
3.	4/21	10/21	3/14	5/14
4.	1/20	9/20	1/15	4/15
5.	2/15	7/15	3/20	7/20
6.	1/30	3/10	1/15	2/15
7.	1/15	7/15	1/10	7/30
8.	1/10	13/30	2/15	4/15
9.	4/15	7/15	3/10	11/30
10.	1/30	11/30	1/10	1/10
11.	7/30	13/30	3/10	3/10
12.	1/15	4/15	1/10	1/6
13.	2/15	7/15	1/6	3/10
14.	1/12	5/12	1/10	3/10
15.	1/10	3/10	1/6	1/6

For reasons that will become clear later, we make the following Corollary:

Corollary 1. *Suppose N is a positive integer. The only solutions to the equations*

$$\tan\tfrac{\pi}{N}\tan\tfrac{a\pi}{N} = \tan\tfrac{2\pi}{N}\tan\tfrac{b\pi}{N} \tag{2.9}$$

and

$$\tan\tfrac{\pi}{N}\tan\tfrac{a\pi}{N} = \tan\tfrac{3\pi}{N}\tan\tfrac{b\pi}{N}, \tag{2.10}$$

where $N/2 > a > b > 1$, are given by

$$\tan\tfrac{\pi}{30}\tan\tfrac{7\pi}{30} = \tan\tfrac{2\pi}{30}\tan\tfrac{4\pi}{30}, \tag{2.11(i)}$$

$$\tan\tfrac{\pi}{30}\tan\tfrac{11\pi}{30} = \tan\tfrac{2\pi}{30}\tan\tfrac{8\pi}{30}, \tag{2.11(ii)}$$

and

$$\tan\tfrac{\pi}{6k}\tan\tfrac{(2k\pm1)\pi}{6k} = \tan\tfrac{3\pi}{6k}\tan\tfrac{(k\pm1)\pi}{6k}, \tag{2.12(i)}$$

$$\tan\tfrac{\pi}{60}\tan\tfrac{13\pi}{60} = \tan\tfrac{3\pi}{60}\tan\tfrac{5\pi}{60}, \tag{2.12(ii)}$$

$$\tan\tfrac{\pi}{60}\tan\tfrac{25\pi}{60} = \tan\tfrac{3\pi}{60}\tan\tfrac{17\pi}{60}. \tag{2.12(iii)}$$

Proof. Let $j = 2$ or 3. Then (2.9) or (2.10) is an equation of the form (2.2) with

$$\alpha = \tfrac{\pi}{N}, \quad \beta = \tfrac{a\pi}{N}, \quad \gamma = \tfrac{j\pi}{N}, \quad \delta = \tfrac{b\pi}{N}.$$

Hence by (2.5),

$$\sin(\tfrac{\pi}{2} - \tfrac{(a+1)\pi}{N}) \sin(\tfrac{\pi}{2} - \tfrac{(b-j)\pi}{N}) = \sin(\tfrac{\pi}{2} - \tfrac{(a-1)\pi}{N}) \sin(\tfrac{\pi}{2} - \tfrac{(b+j)\pi}{N}). \tag{2.13}$$

We now apply Myerson's Theorem. The equation (2.7), which is a disguised version of the double-angle formula for sines, is equivalent to

$$\tan x \tan\left(\frac{\pi}{3} + x\right) = \tan 3x \tan\left(\frac{\pi}{6} + x\right), \tag{2.14}$$

which is also amusing to prove directly. We immediately obtain a solution to (2.9) when $x = \pm\frac{1}{N}$, provided 6 divides N. (If x is negative, we cancel the minus signs in $\tan x$ and $\tan 3x$.) Letting $N = 6k$, we obtain (2.12)(i). For completeness, we must actually check all 16 cases where the ratio of arguments on opposite sides of (2.14) is equal to 2 or 3, but we never get a "new" solution to (2.9) or (2.10) with $1 < b < a < N/2 - 1$.

Any other solutions to (2.13) must appear in the list of singular cases. By Lemma 2, we may make the following identifications:

$$x_1 = \tfrac{1}{2} - \tfrac{(a+1)}{N}, \quad x_2 = \tfrac{1}{2} - \tfrac{(b-j)}{N}, \quad \{x_3, x_4\} = \left\{\tfrac{1}{2} - \tfrac{(a-1)}{N}, \tfrac{1}{2} - \tfrac{(b+j)}{N}\right\}.$$

Hence $\frac{2}{N} = x_3 - x_1$ or $x_4 - x_1$ and $\frac{2j}{N} = x_2 - x_4$ or $x_2 - x_3$ respectively. Hence it suffices to look at the list of singular solutions for those in which $\frac{x_2 - x_4}{x_3 - x_1}$ or $\frac{x_2 - x_3}{x_4 - x_1}$ are equal to j, which is 2 or 3. This occurs in equations 9, 11, 12 and 15, leading to the singular solutions shown in (2.11) and (2.12)(ii) and (iii). □

We now give some constructions of perfect (m, n)-grids, adopting in this section the convention that $m \ge n$. These will turn out to be the only examples.

If $n = 2$, then $f(2r, 2) = \frac{2\pi}{4r}$ and $f(2r+1, 2) = \frac{2\pi}{4r+2}$, hence $f(m, 2) = \frac{2\pi}{2m}$ in either case. Let $\mathcal{A} = \{\cot \frac{(2j-1)\pi}{2m} : 1 \le j \le m\}$ and $\mathcal{B} = \{-1, 1\}$. Then there will be rays with argument $\frac{(2j-1)\pi}{2m}$ above the x-axis and $\pi + \frac{(2j-1)\pi}{2m}$ below the x-axis, and these constitute $2m = mn$ rays with common angular separation $\frac{2\pi}{2m}$.

If $n = 3$, then $f(2r, 3) = \frac{2\pi}{4r+2}$ and $f(2r+1, 3) = \frac{2\pi}{4r+4}$, hence $f(m, 2) = \frac{2\pi}{2m+2}$ in either case. Now let $\mathcal{A} = \{\cot \frac{j\pi}{m+1} : 1 \le j \le m\}$ and $\mathcal{B} = \{-1, 0, 1\}$. Then the rays above the x-axis have argument $\frac{j\pi}{m+1}$, $1 \le j \le m$, and those below the x-axis have the same arguments plus π. There are two rays on the x-axis, which have arguments 0 and π, and together these give a complete system of angles $\frac{j\pi}{m+1}$, $0 \le j \le 2m+1$.

We have $f(4, 4) = \frac{\pi}{8}$. Let $\mathcal{A} = \{\pm 1, \pm \tan \frac{\pi}{16} \tan \frac{3\pi}{16}\}$ and $\mathcal{B} = \{\pm \tan \frac{\pi}{16}, \pm \tan \frac{3\pi}{16}\}$. Then the four rays in the first quadrant have slopes $\tan \frac{\pi}{16}$, $\tan \frac{3\pi}{16}$ and their reciprocals, $\tan \frac{7\pi}{16}$ and $\tan \frac{5\pi}{16}$. Repeating in the other four quadrants, we see that the rays have angles $\frac{(2j+1)\pi}{16}$, $0 \le j \le 15$, and so $(\mathcal{A}, \mathcal{B})$ is perfect.

Similarly, $f(5,5) = \frac{\pi}{10}$. Let

$$\mathcal{A} = \{0, \pm 1, \pm \tan \tfrac{\pi}{10} \tan \tfrac{2\pi}{10}\}$$

and

$$\mathcal{B} = \{0, \pm \tan \tfrac{\pi}{10}, \pm \tan \tfrac{2\pi}{10}\}.$$

Then the four rays in the first quadrant have slopes $\tan \frac{\pi}{10}$, $\tan \frac{2\pi}{10}$ and their reciprocals, $\tan \frac{4\pi}{10}$ and $\tan \frac{3\pi}{10}$. There are rays in the four directions of the axes, and after reflection, we see that there are 20 rays evenly spaced with angle $\frac{\pi}{10}$.

Finally, $f(6,4) = \frac{\pi}{12}$, and let

$$\mathcal{A} = \left\{ \pm \frac{\tan \frac{3\pi}{24}}{\tan \frac{\pi}{24}}, \pm 1, \pm \frac{\tan \frac{\pi}{24}}{\tan \frac{3\pi}{24}} \right\}, \qquad \mathcal{B} = \{\pm \tan \tfrac{3\pi}{24}, \pm \tan \tfrac{9\pi}{24}\}.$$

Once again, by symmetry, it suffices to look at the first quadrant. Using the identity $\tan(\frac{\pi}{2} - \theta) = (\tan\theta)^{-1}$, we see that the rays to the points (a_i, b_j) have slopes

$$\tan \tfrac{\pi}{24}, \quad \tan \tfrac{3\pi}{24}, \quad \frac{\tan^2 \frac{3\pi}{24}}{\tan \frac{\pi}{24}}, \quad \frac{\tan \frac{\pi}{24}}{\tan^2 \frac{3\pi}{24}}, \quad \tan \tfrac{9\pi}{24}, \quad \tan \tfrac{11\pi}{24}. \tag{2.15}$$

But upon taking (2.12)(i) with the positive sign and $k = 4$, we see that $\tan \frac{\pi}{24} \tan \frac{9\pi}{24} = \tan \frac{3\pi}{24} \tan \frac{5\pi}{6k}$, and since $\tan \frac{9\pi}{24} \tan \frac{3\pi}{24} = 1$, the middle two numbers in (2.15) are $\tan \frac{5\pi}{24}$ and $\tan \frac{7\pi}{24}$, as desired.

We now show that the pattern of angles illustrated by these examples is the only possibility for a perfect grid. For convenience, we assume that if $\mathcal{A}, \mathcal{B}$ is a perfect (m,n)-grid, and one of m and n is odd, then n is odd, temporarily suspending the assumption $m \le n$.

Lemma 3. *If $(\mathcal{A}, \mathcal{B})$ is a perfect (m,n) grid and n is odd, then the arguments of the rays from the origin to $\mathcal{A} \times \mathcal{B}$ are contained in $\{j \frac{2\pi}{L(m,n)} : 0 \le j \le L(m,n) - 1\}$.*

If $m = 2r$ and $n = 2s$, then the arguments of a perfect (m,n)-grid consist of $\{\frac{(2j+1)\pi}{4rs} : 0 \le j \le 4rs - 1\}$.

Proof. The first assertion is immediate from the fact that in a perfect grid, if n is odd, then $0 \in \mathcal{B}$, so one argument is 0, and they are separated by $\frac{2\pi}{L(m,n)}$.

For the second statement, suppose $m = 2r$, $n = 2s$, and $\mathcal{A} = \{a_i, -c_i : 1 \le i \le r\}$ and $\mathcal{B} = \{b_j, -d_j : 1 \le j \le s\}$, where $a_i, b_j, c_i, d_j > 0$. Further, let $A = \prod_i a_i$, $B = \prod_j b_j$, $C = \prod_i c_i$, $D = \prod_j d_j$. Suppose the ray with the smallest argument has argument θ; then $0 < \theta < \frac{\pi}{2rs}$. we wish to show that $\theta = \frac{\pi}{4rs}$. If we take the product of the slopes of all the rays in the first quadrant, we obtain the formula

$$\prod_{\ell=0}^{rs-1} \tan(\theta + \tfrac{\ell\pi}{2rs}) = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{b_j}{a_i} = \frac{B^r}{A^s}$$

There are three other formulas, found from the other three quadrants:

$$\prod_{\ell=rs}^{2rs-1} |\tan(\theta + \tfrac{\ell\pi}{2rs})| = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{b_j}{c_i} = \frac{B^r}{C^s}$$

$$\prod_{\ell=2rs}^{3rs-1} \tan(\theta + \tfrac{\ell\pi}{2rs}) = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{d_j}{c_i} = \frac{D^r}{C^s}$$

$$\prod_{\ell=3rs}^{4rs-1} |\tan(\theta + t\frac{\ell\pi}{2rs})| = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{d_j}{a_i} = \frac{D^r}{A^s}$$

Since $|\tan\alpha + \pi/2| = (\tan\alpha)^{-1}$, it follows from the foregoing that

$$\frac{B^r}{A^s} = \frac{C^s}{B^r} = \frac{D^r}{C^s} = \frac{A^s}{D^r}.$$

Since the product of these four fractions is 1, it follows that each fraction is itself equal to 1, hence

$$F(\theta) := \prod_{\ell=0}^{rs-1} \tan(\theta + \tfrac{\ell\pi}{2rs}) = 1.$$

Observe that F is strictly increasing on $(0, \frac{\pi}{2rs})$. By pairing off the terms for ℓ and $rs - 1 - \ell$, we see that $F(\frac{\pi}{4rs}) = 1$. Thus $\theta = \frac{\pi}{4rs}$. $\square$

Theorem 2. *There are no perfect (m,n)-grids with $m \ge n$ unless $m = 2$ or 3, or $(m,n) \in \{(4,4),(5,5),(6,4)\}$.*

Proof. Suppose (m,n) is given and $(\mathcal{A},\mathcal{B})$ is a perfect (m,n)-grid with $m \le n$. We have already shown that perfect (m,n)-grids exist in the listed cases.

We first show that there are no other perfect (m,n)-grids with $m,n \le 6$.

Suppose $(\mathcal{A},\mathcal{B})$ were a perfect $(4,5)$-grid, and write $\mathcal{A} = \{a_1 > a_2 > 0 > a_3 > a_4\}$ and $\mathcal{B} = \{b_1 > b_2 > b_3 = 0 > b_4 > b_5\}$. By Lemma 3, the angles of the rays in the first quadrant are $\{j\frac{\pi}{9} : 1 \le j \le 4\}$, hence

$$\left\{\tfrac{b_2}{a_1} < \tfrac{b_1}{a_2}, \tfrac{b_2}{a_2} < \tfrac{b_1}{a_2}\right\} = \left\{\tan\tfrac{\pi}{9}, \tan\tfrac{2\pi}{9}, \tan\tfrac{3\pi}{9}, \tan\tfrac{4\pi}{9}\right\}. \tag{2.16}$$

This implies that $\tan\frac{\pi}{9}\cdot\tan\frac{4\pi}{9} = \tan\frac{2\pi}{9}\cdot\tan\frac{3\pi}{9}$, but this is numerically false ($2.064.. \ne 1.453..$) (We cannot apply the Corollary because $\frac{\pi}{9} + \frac{4\pi}{9} > \frac{\pi}{2}$.)

Similarly suppose $(\mathcal{A},\mathcal{B})$ were a perfect $(6,5)$-grid, and write $\mathcal{A} = \{a_1 > a_2 > a_3 > 0 > a_4 > a_5 > a_6\}$ and $\mathcal{B} = \{b_1 > b_2 > b_3 = 0 > b_4 > b_5\}$. By Lemma 3, the angles of the rays in the first quadrant are $\{j\frac{\pi}{13} : 1 \le j \le 4\}$ and

$$\left\{\tfrac{b_j}{a_i} : 1 \le i \le 3, 1 \le j \le 2\right\} = \left\{\tan\tfrac{\pi}{13}, \tan\tfrac{2\pi}{13}, \cdots, \tan\tfrac{6\pi}{13}\right\}. \tag{2.17}$$

Of all the fractions $\frac{b_j}{a_i}$, clearly the smallest two are $\frac{b_3}{a_1}$ and either $\frac{b_3}{a_2}$ or $\frac{b_2}{a_1}$, and the largest is $\frac{b_1}{a_3}$. Hence the algebraic identity $\frac{b_3}{a_1}\frac{b_2}{a_2} = \frac{b_3}{a_2}\frac{b_2}{a_1}$ implies that

$$\tan\tfrac{\pi}{13}\tan\tfrac{a\pi}{13} = \tan\tfrac{2\pi}{13}\tan\tfrac{b\pi}{13}, \tag{2.18}$$

with $a \le 5$. Since $\frac{(a+1)\pi}{13} < \frac{\pi}{2}$, the Corollary states that no solution to (2.18) exists, so no perfect $(6,5)$-grid exists.

Finally, suppose $(\mathcal{A},\mathcal{B})$ were a perfect $(6,6)$-grid, and write $\mathcal{A} = \{a_1 > a_2 > a_3 > 0 > a_4 > a_5 > a_6\}$ and $\mathcal{B} = \{b_1 > b_2 > b_3 > 0 > b_4 > b_5 > b_6\}$. By Lemma 3, we have

$$\left\{\tfrac{b_j}{a_i} : 1 \le i \le 3, 1 \le j \le 3\right\} = \left\{\tan\tfrac{\pi}{36}, \tan\tfrac{3\pi}{36}, \cdots, \tan\tfrac{17\pi}{36}\right\}. \tag{2.19}$$

We must argue more carefully here than before because of (2.11)(i). Again, we can conclude that $\frac{b_3}{a_1} = \tan\frac{\pi}{36}$ and either $\frac{b_3}{a_2}$ or $\frac{b_2}{a_1}$ is equal to $\tan\frac{3\pi}{36}$, hence

$$\frac{b_3}{a_1}\frac{b_2}{a_2} = \frac{b_3}{a_2}\frac{b_2}{a_1} \implies \tan\tfrac{\pi}{36}\tan\tfrac{a\pi}{36} = \tan\tfrac{3\pi}{36}\tan\tfrac{b\pi}{36}.$$

This implies that $(a,b) = (11,5)$ or $(a,b) = (13,7)$. Since $\frac{b_2}{a_2} < \frac{b_2}{a_3}, \frac{b_1}{a_2}, \frac{b_1}{a_3}$, we must have $\frac{b_2}{a_2} \le \tan\frac{11\pi}{36}$, so the latter is impossible. But if $\frac{b_2}{a_2} = \tan\frac{11\pi}{36}$, then we must have

$$\frac{b_1}{a_3}\frac{b_2}{a_2} = \frac{b_1}{a_2}\frac{b_2}{a_3} \implies \tan\tfrac{17\pi}{36}\tan\tfrac{11\pi}{36} = \tan\tfrac{13\pi}{36}\tan\tfrac{15\pi}{36},$$

which is numerically false ($16.324 \ne 8.003$). Hence no perfect (6,6)-grid exists.

The general argument follows the pattern of the last two examples. Suppose first $(\mathcal{A},\mathcal{B})$ is a perfect (m,n)-grid, where $n = 2s+1$ is odd and $m = 2r$ or $2r+1$ is either even or odd, $r,s \ge 2$, but $(r,s) \ne (2,2)$. Write $\mathcal{A} = \{a_1 > \cdots > a_r > 0 \ldots\}$ and $\mathcal{B} = \{b_1 > \cdots > b_s > b_{s+1} = 0 > \ldots\}$. Observe that $L(m,n) = 2t$ is even in any case. By Lemma 3, we have

$$\left\{\tfrac{b_j}{a_i} : 1 \le i \le r, 1 \le j \le s\right\} = \left\{\tan\tfrac{\pi}{t}, \tan\tfrac{2\pi}{t}, \cdots, \tan\tfrac{\lfloor (t-1)/2\rfloor\pi}{t}\right\}. \tag{2.20}$$

As before, the smallest fractions $\frac{b_j}{a_i}$ are $\frac{b_s}{a_1}$ and either $\frac{b_s}{a_2}$ or $\frac{b_{s-1}}{a_1}$, and the largest is $\frac{b_1}{a_r}$. Hence

$$\frac{b_s}{a_1}\frac{b_{s-1}}{a_2} = \frac{b_s}{a_2}\frac{b_{s-1}}{a_1} \implies \tan\tfrac{\pi}{t}\tan\tfrac{a\pi}{t} = \tan\tfrac{2\pi}{t}\tan\tfrac{b\pi}{t}. \tag{2.21}$$

Since $(r,s) \ne (2,2)$, $a \le \lfloor (t-1)/2\rfloor - 1$, hence $a+1 < t/2$, and the Corollary may be invoked. There is no solution at all to (2.21), except in the special case where $t = 30$, hence $L(m,n)=60$. Since $4 \mid 60$, this means that m and n are odd, and $4rs+4=60$ implies that $rs=14$. Since $r,s \geq 2$, this gives essentially one case – $(m,n)=(5,15)$.

Suppose $(\mathcal{A},\mathcal{B})$ were a perfect (5,15)-grid. With the notation as above, (2.20) and (2.21) become

$$\left\{\tfrac{b_j}{a_i} : 1 \leq i \leq 2, 1 \leq 7 \leq s\right\} = \left\{\tan\tfrac{\pi}{30}, \tan\tfrac{2\pi}{30}, \cdots, \tan\tfrac{14\pi}{30}\right\}, \tag{2.22}$$

$$\frac{b_7}{a_1}\frac{b_6}{a_2} = \frac{b_7}{a_2}\frac{b_6}{a_1} \implies \tan\tfrac{\pi}{30}\tan\tfrac{a\pi}{30} = \tan\tfrac{2\pi}{30}\tan\tfrac{b\pi}{30}, \tag{2.23}$$

where $(a,b)=(7,4)$ or $(11,8)$. Now $\frac{b_6}{a_2} < \frac{b_i}{a_2}$ for $1 \leq i \leq 5$, hence $a \leq 9$. Thus

$$\frac{b_7}{a_1} = \tan\tfrac{\pi}{30}, \quad \frac{b_6}{a_2} = \tan\tfrac{7\pi}{30}, \quad \left\{\frac{b_7}{a_2}, \frac{b_6}{a_1}\right\} = \{\tan\tfrac{2\pi}{30}, \tan\tfrac{4\pi}{30}\}.$$

But, if $\tan\frac{2\pi}{30} = \frac{b_7}{a_2}$, then there will be five other equations

$$\frac{b_7}{a_1}\frac{b_j}{a_2} = \frac{b_7}{a_2}\frac{b_j}{a_1} \implies \tan\tfrac{\pi}{30}\tan\tfrac{a_j\pi}{30} = \tan\tfrac{2\pi}{30}\tan\tfrac{b_j\pi}{30} \tag{2.24}$$

for $1 \leq j \leq 5$, and the Corollary says that at most two other identities such as (2.24) may exist – with $(a_j,b_j)=(11,8),(14,13)$, the latter coming from the violation of the $\alpha+\beta<\frac{\pi}{2}$ constraint. This is impossible, so we must have

$$\tan\tfrac{2\pi}{30} = \frac{b_6}{a_1}, \quad \tan\tfrac{4\pi}{30} = \frac{b_7}{a_2}.$$

Since $\tan\frac{3\pi}{30} < \frac{b_7}{a_2} \leq \frac{b_j}{a_2}$, we must have $\tan\frac{3\pi}{30} = \frac{b_5}{a_1}$. But then,

$$\frac{b_7}{a_1}\frac{b_5}{a_2} = \frac{b_7}{a_2}\frac{b_5}{a_1} \implies \tan\tfrac{\pi}{30}\tan\tfrac{a\pi}{30} = \tan\tfrac{4\pi}{30}\tan\tfrac{3\pi}{30}, \tag{2.25}$$

After taking (2.11)(i) with the negative sign and $k=5$, we see that (2.25) implies that $a=9$. To recapitulate, at this point, we have identified $\frac{b_j}{a_i}$ as $\tan\frac{k\pi}{30}$ for $(a_i,b_j) = (7,1),(6,1),(5,1),(7,2),(6,2),(5,2)$ with $k=1,2,3,4,7,9$ respectively. The next ray has argument $\tan\frac{5\pi}{30}$ and must be the smallest $\frac{b_j}{a_i}$ with $j \leq 4$; hence it is $\frac{b_4}{a_1}$. Finally,

$$\frac{b_7}{a_1}\frac{b_4}{a_2} = \frac{b_7}{a_2}\frac{b_4}{a_1} \implies \tan\tfrac{\pi}{30}\tan\tfrac{a\pi}{30} = \tan\tfrac{4\pi}{30}\tan\tfrac{5\pi}{30}, \tag{2.26}$$

But a computation shows that (2.26) implies $a=11.29..$ is not an integer, and at long last, we've shown that there is no perfect (15,5)-grid.

Suppose now that $(\mathcal{A},\mathcal{B})$ is a perfect (m,n)-grid, where $m=2r$, $n=2s$, $m\geq n\geq 2$, but $(m,n)\neq(2,2),(3,2)$. Write $\mathcal{A}=\{a_1>\cdots>a_r>0\dots\}$ and $\mathcal{B}=\{b_1>\cdots>b_s>0>\dots\}$. Observe that $L(m,n)=4rs$ and by Lemma 3, we have

$$\left\{\tfrac{b_j}{a_i}:1\leq i\leq r,1\leq j\leq s\right\}=\left\{\tan\tfrac{\pi}{4rs},\tan\tfrac{3\pi}{4rs},\cdots,\tan\tfrac{(2rs-1)\pi}{4rs}\right\}. \tag{2.27}$$

As before, we can argue that $\tan\frac{\pi}{4rs}=\frac{b_s}{a_1}$ and $\tan\frac{3\pi}{4rs}=\frac{b_{s-1}}{a_1}$ or $\frac{b_{s-1}}{a_2}$, and

$$\frac{b_s}{a_1}\frac{b_{s-1}}{a_2}=\frac{b_s}{a_2}\frac{b_{s-1}}{a_1}\implies\tan\tfrac{\pi}{4rs}\tan\tfrac{a\pi}{4rs}=\tan\tfrac{3\pi}{4rs}\tan\tfrac{b\pi}{4rs}. \tag{2.28}$$

Observe that $\frac{b_j}{a_i}\geq\frac{b_{s-1}}{a_2}$ when $1\leq j\leq s-1$ and $2\leq i\leq r$. Thus, there are at least $(r-1)(s-1)-1=rs-r-s$ rays in the first quadrant with angle greater than $\frac{b_{s-1}}{a_2}$, and so

$$a\leq 2r+2s-1. \tag{2.29}$$

If (2.12)(i) holds with (2.29), then $6k=4rs$ and $2k\pm1\leq 2r+2s-1$, hence

$$4rs=6k\leq 6r+6s\mp3\implies(2r-3)(2s-3)\leq 9\mp3 \tag{2.30}$$

It is easy to check that the integer solutions to (2.30) for $r\geq s\geq 2$ are (2,2), (3,2), (4,2), (5,2), (6,2), (7,2), (3,3). Since $4rs=6k$, rs is a multiple of 3, and the only undiscussed case is $(r,s)=(6,2)$, for which we must have $k=8$ and the negative sign. The identity is

$$\tan\tfrac{\pi}{48}\tan\tfrac{15\pi}{48}=\tan\tfrac{3\pi}{48}\tan\tfrac{7\pi}{48},$$

and $\frac{b_6}{a_1}=\tan\frac{\pi}{48}$, $\frac{b_5}{a_2}=\tan\frac{15\pi}{48}$ and either $\frac{b_6}{a_2}$ or $\frac{b_5}{a_1}$ is equal to $\tan\frac{3\pi}{48}$. If $\frac{b_6}{a_2}=\tan\frac{3\pi}{48}$, then an argument identical to (2.24) shows that there are too many identities of the form (2.10), hence $\frac{b_5}{a_1}=\tan\frac{3\pi}{48}$. But now the same argument shows that we must have $\frac{b_4}{a_1}=\tan\frac{5\pi}{48}$, but

$$\frac{b_6}{a_1}\frac{b_4}{a_2}=\frac{b_6}{a_2}\frac{b_4}{a_1}\implies\tan\tfrac{\pi}{48}\tan\tfrac{a\pi}{48}=\tan\tfrac{7\pi}{48}\tan\tfrac{5\pi}{48}, \tag{2.31}$$

But a computation shows that (2.31) implies $a=18.29..$ is not an integer, and so there is no perfect (12,4)-grid using (2.11)(i).

In the final case, (2.28) holds as a consequence of (2.11)(ii) or (iii), hence $4rs=60$, so $(r,s)=(5,3)$ and (2.29) implies that $a\leq 15$. This rules out (2.11)(iii) and we are left with one more calculation. We are considering a perfect (10,6)-grid, and as before, $\frac{b_5}{a_1}=\tan\frac{\pi}{60}$, $\frac{b_4}{a_2}=\tan\frac{13\pi}{60}$, with $\{\frac{b_5}{a_2},\frac{b_4}{a_1}\}=\{\tan\frac{3\pi}{60},\tan\frac{5\pi}{60}\}$. Regardless of which angle is which, we have $\tan\frac{7\pi}{60} = \frac{b_3}{a_1}$ or $\frac{b_1}{a_3}$, and the appropriate rectangle leads to one of two possible equations:

$$\tan\tfrac{\pi}{60}\tan\tfrac{a\pi}{60} = \tan\tfrac{7\pi}{60}\tan\tfrac{3\pi}{60}, \qquad \tan\tfrac{\pi}{60}\tan\tfrac{b\pi}{60} = \tan\tfrac{7\pi}{60}\tan\tfrac{5\pi}{60}. \tag{2.32}$$

But a final computation shows that (2.32) implies $a = 21.205, b = 16.66$, and so there are no perfect (10,6)-grids. □

Let us conclude this section with a conjecture. Call an $(\mathcal{A},\mathcal{B})$ grid *equiangular* if the angular gaps between any consecutive rays are exactly $2\pi/N$. Beside the equiangular grids described in Theorem 2 one can have the following example $\mathcal{A} = \mathcal{B} := \{0, \pm 1, \pm\tan\frac{\pi}{8}\}$ where $N = 16$. We **conjecture** that these are the only equiangular grids, (or at least there are only finitely many more).

3. An asymptotically optimal upper bound when $m = n$.

The key to our construction is the following Lemma.

Lemma 4. *For each integer $n \ge 2$, there exist sets $\mathcal{C}$ and $\mathcal{D}$, with $|\mathcal{C}| < 3.613n - 1$ and $|\mathcal{D}| < 2.861n - 1$ such that the rays from the origin to $\{c_i, d_j\}$ lie in the sector* $\arctan(1/n) \le \theta \le \pi/4$ *with consecutive angle at most $1/n^2$.*

Proof. We define $\mathcal{C}$ and $\mathcal{D}$ as unions of short arithmetic progressions. Let

$$\{a, a+d, \dots; b\} := \{a, a+d, \dots, a+rd\}, \quad \text{where} \quad a + rd < b \le a + (r+1)d;$$

this contains $\lceil\frac{b-a}{d}\rceil$ elements. Now let

$$\mathcal{C} = \bigcup_{i=1}^{n-1} \{in^2, in^2 + i^{3/2}n, \dots; (i+1)n^2\}, \tag{3.1}$$

$$\mathcal{D} = \bigcup_{i=1}^{n-1} \{n^2 + n\sqrt{i-1}, n^2 + n\sqrt{i-1} + i, \dots; n^2 + n\sqrt{i}\}. \tag{3.2}$$

We have

$$|\mathcal{C}| = \sum_{i=1}^{n-1} \lceil\tfrac{n^2}{i^{3/2}n}\rceil < n - 1 + n\sum_{i=1}^{\infty} i^{-3/2} = n - 1 + 2.612375..n < 3.613n - 1.$$

Furthermore,

$$|\mathcal{D}| = \sum_{i=1}^{n-1} \lceil\tfrac{n(\sqrt{i}-\sqrt{i-1})}{i}\rceil < n - 1 + n\sum_{i=1}^{\infty} \frac{1}{i(\sqrt{i}+\sqrt{i-1})}$$
$$= n - 1 + 1.8600251...n < 2.861n - 1.$$

Next observe that, if $0 < u < v \le 1$, then by the Mean Value Theorem, $v - u > \arctan v - \arctan u$. It therefore suffices to show that if $1/n \le w \le 1$, then there exist $c \in \mathcal{C}$ and $d, d' \in \mathcal{D}$ so that

$$\frac{d}{c} > w > \frac{d'}{c}, \qquad \frac{d}{c} - \frac{d'}{c} < \frac{1}{n^2},$$

because this implies that the sector containing the ray with angle $\arctan w$ has angular measure less than $1/n^2$.

For $w < 1$, define i by $i + 1 \ge 1/w > i$, with $1 \le i \le n - 1$. Then $in^2 \le n^2/w < (i + 1)n^2$, hence there exists $c \in \mathcal{C}$ so that

$$c \ge \tfrac{n^2}{w} \ge c - i^{3/2}n,$$

hence

$$n^2 + \sqrt{i}n \ge wc \ge n^2.$$

Hence there exist $d, d' \in \mathcal{D}$ with $d - d' \le i$ so that $d \ge wc \ge d'$. Thus,

$$\frac{d}{c} \ge w \ge \frac{d'}{c}, \qquad \frac{d}{c} - \frac{d'}{c} \le \frac{i}{c} \le \frac{i}{in^2} = \frac{1}{n^2}. \quad \square$$

Proof of Theorem 3. We show that for all n

(3.3) $$f(n, n) \le \frac{1}{(\lfloor \frac{n+1}{12.95} \rfloor)^2}.$$

Here the right hand side is less than $\frac{220}{n^2}$ for $n \ge 200$, and less than $\frac{168}{n^2}$ for $n \ge 14{,}000$. For $n < 200$ the $n \times n$ integer grid $\{(a, b) : n/2 < a, b \le n/2\}$ has aperture $1/\frac{n-1}{2}$ which is less than $220/n^2$.

To prove (3.3) let

$$\mathcal{A}_n = \mathcal{B}_n = \mathcal{C}_n \cup \mathcal{D}_n \cup -\mathcal{C}_n \cup -\mathcal{D}_n \cup \{0, n, n^3\}.$$

Then $|\mathcal{A}_n| = |\mathcal{B}_n| < 12.95n - 1$. We shall show that $\mathrm{Ap}(\mathcal{A}_n, \mathcal{B}_n) \le 1/n^2$. Thus, if we let $m = \lfloor \frac{n+1}{12.95} \rfloor$, then $f(n, n) \le 1/m^2$. For n sufficiently large, we obtain the bound announced in the abstract.

By considering the eight sets of rays to the points $(\pm c_i, \pm d_j)$ and $(\pm d_j, \pm c_i)$, we see that the full range of angles has been covered with the exception of four narrow sectors of width $2\arctan(1/n) < 2/n$, centered in the direction of the axes. By symmetry, it suffices to consider $0 \le \theta \le 1/n$, and once again, it suffices to show that there are rays whose slopes differ by at most $1/n^2$ covering the sector.

There are rays to $(n, 0)$, (n^3, n) and to (c, n) for every $c \in \mathcal{C}$. This hits the positive x-axis and the smallest slope which appears is $n/n^3 = 1/n^2$. The difference between slopes for consecutive c_i is

$$\frac{n}{c_{i+1}} - \frac{n}{c_i} = \frac{n(c_{i+1} - c_i)}{c_i c_{i+1}} < \frac{n i^{3/2} n}{(in^2)^2} < \frac{1}{n^2}. \quad \square$$

4. Concluding remarks, further distribution problems

The natural extensions of the work of Section 2 would be to compute $f(m,n)$ for other values of (m,n) and perhaps find characterizations of $(\mathcal{A},\mathcal{B})$ so that its aperture is at least locally minimized with respect to perturbations of the elements. In terms of the original problem, it would be interesting to determine the dimension of the vector space over $\mathbf{Q}$ spanned by $\{\tan \frac{j\pi}{N} : 1 \le j \le N/2\}$.

The natural extension of Theorem 3 would be to extend this upper bound by constructing an $m \times n$ grid, $\mathcal{P}$, with $\mathrm{Ap}(\mathcal{P}) = \mathcal{O}(1/mn)$. In [FR] we shall do this for $m^{1/3} << n \le m$. We think that such an upper bound does not exist when the ratio of m and n is extremely large.

We were not able to establish any significant improvement on the trivial lower bound. One can conjecture, that, indeed, for large n, $f(n,n) > (1+\varepsilon)2\pi/n^2$ for some absolute constant $\varepsilon > 0$.

We do not have any information on the *global* distribution of the n^2 directions. We conjecture, that its (classical) discrepancy is very large, maybe even $\Omega(n^2)$. This means that for some fixed $\varepsilon > 0$, for every grid one can find a small sector with angle α, such that the number of directions from $\mathcal{P}$ in that sector differs from $(\alpha/2\pi)n^2$ by at least εn^2.

The obvious example, the integer lattice $\{-n, -(n-1), \ldots, -1, 0, 1, \ldots n\}^2$, has many huge gaps, of order $O(1/n)$, not only between the direction 0 and $\tan^{-1}(1/n)$, but near $\tan^{-1}(a/b)$ for small b. It has very dense and relatively thin sectors, and its discrepancy is really $\Omega(n^2)$.

Diamond and Pomerance [DP] have studied the *minimum* non-zero gap between the directions of the set $S(r)$, the set of lattice points on the plane with distance from the origin at most r. It is very close to $1/(r^2 - r)$, though the number of directions is $\Omega(r^2)$.

We cannot even prove that the ratio of the aperture (maximum gap) to the minimum gap goes to ∞ when m,n both tend to ∞.

A general method in discrepancy theory is to obtain an estimation from the discrepancy of the (of a) random structure. Indeed, if we select a random n-set (for $\mathcal{C}$ and $\mathcal{D}$) from the members of the geometric sequence $1, q, q^2, q^3, \ldots, q^{2n^2 \log n}$, where $q = 1 + (1/n^2)$, then one can generate a grid of discrepancy about $O(\log n/n^2)$. This example, with a more careful combination of random and intentional selection leads to another example showing $f(n,n) = O(1/n^2)$.

There has been much research on the structure of the set $A + A$, where A is a (frequently, a finite) set of reals (vectors), and $A + A$ is the set of numbers of the form $a + a'$ with $a, a' \in A$, see e.g., the books of Freiman [F] and Ruzsa [Ru]. Our theorem yields an example for an n-set such that the maximum gap in $\tan^{-1}\exp(A+A)$ is at most $O(1/n^2)$. One can think that the machinery developed for the $A + A$ type problems can be used here. Also, instead of considering the transformation $\tan^{-1}\exp(\cdot)$, it would be interesting to investigate other elementary functions.

We also think, that further study of this problem can lead to an explicit construction of a problem of Erdős and Purdy, solved by Alon [A], proving that the size of the smallest subset S of the $n \times n$ lattice L such that the lines through at least two of the points of S cover L is between $\Omega(n^{2/3})$ and $O(n^{2/3} \log n)$.

All of the above problems have natural generalizations for higher dimensional Euclidean spaces, and also for other normed spaces, whose study also could lead to interesting phenomena.

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