Uniform denominators in Hilbert’s Seventeenth Problem

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1 Introduction and overview

In 1888, Hilbert [H4] proved that if $n \geq 3$, then there exist real positive semidefinite forms $p = \rho(x_1, \ldots, x_n)$ which cannot be written as a sum of squares of forms. He also proved that for $n = 3$ such a form is always a sum of squares of rational functions. Hilbert’s Seventeenth Problem asked whether a positive semidefinite form in any number of variables must be a sum of squares of rational functions. In the 1920s, Artin solved Hilbert’s Seventeenth Problem in the affirmative by using the Artin-Schreier theory of real fields. This proof was not constructive.

A few years later, Pólya [P1] presented a concrete proof in one special case: if $p$ is both positive definite and even, then for sufficiently large $r$, $p \cdot \left(\sum x_i^2\right)^r$ has positive coefficients, and so is per se a sum of squares of monomials. This fact implies that $p$ is a sum of squares of rational functions with common denominator $\left(\sum x_i^2\right)^{r/2}$. In 1940, Habicht [H1] used Pólya’s result to write an arbitrary positive definite form $p$ as a quotient of two sums of squares of monomials. It follows from his proof that $p$ is a sum of squares of rational functions with positive definite denominator. In each case, if $p$ has rational coefficients, then so do the monomials. (See [H2, pp. 35–59, 300–304]). These results are stronger than Artin’s in specifying the nature of the denominators, but weaker in that they only apply to real positive definite forms.

The restriction to positive definite forms is necessary. There exist positive semidefinite forms $\rho$ which have the remarkable property that, in any representation $\rho = \sum a_i x_i^2$, where $a_i = f_i/g_i$ is a rational function, each form $g_i$ must have a specified non-trivial zero. The existence of these so-called “bad points” ensures that $p \cdot \left(\sum x_i^2\right)^r$ can never be a sum of squares of forms for any $r$. Habicht’s theorem implies that no positive definite form can have a bad point. According
to Delzell's thesis [D3], bad points were first noted by E. G. Straus in an unpublished 1956 letter to G. Kreisel. Delzell's thesis contains an extensive history of the subject (see chapter V); this contains much material unpublished elsewhere. Another subject with an extensive history is that of "continuous" solutions to Hilbert's 17th problem over a real closed field $R$. These are forms of the shape $p(x) = \sum \lambda_j(q_j(x))^2$, where $f_j = \sum \lambda_j(q_j(x))$, which hold for every positive semidefinite form $p$ of fixed degree, under the assumptions that $0 \leq \lambda_j(p) \in R$, $f_j(q_j(x))$ and $g_j(q_j(x))$ belong to $R[x_1, \ldots, x_n]$ and $\lambda_j(p) = \sum f_j$ depend continuously on $p$. Such formulas exist in which the coefficients of the parameters have the forms $\sup \inf h_{j,k}$ for polynomials $h_{j,k}$ in the coefficients of $p$. See [D5, D6, D7] for recent results on this subject.

In 1951, Becker [B1, B2] extended the Artin theory, giving necessary and sufficient conditions for a rational function $p$ over a formally real field to be a sum of $2k$-th powers of rational functions. Roughly speaking, $p$ must be a sum of multiple of 2$k$ and all "zeroes" must have "2k-th order". A concrete application [B2, p.144] is that for all $k \geq 1$, there exist $0 < \lambda_j \in Q$ and polynomials $f_j$ and $g_j$ in $Q[x]$ such that

$$B(q) = 1 + \frac{t^2}{2 + t^2} = \sum \lambda_j \frac{f_j(q_j)}{g_j(q_j)} \quad (1.1)$$

Explicit versions of (1.1) are known for $k = 1, 2$, and there has been interest in finding them for all $k$. Formula (1.3) is such an expression, but one in which $f_j, g_j \in R[x]$. (1.5)

One can deduce from recent work of Becker and Powers [B3] that there is an identity (1.1) in which each $q_j(x)$ is positive definite. Schmid has also recently shown [S1, Cor. 4.1] that if $f$ and $g$ are real positive definite polynomials in one variable with the same degree, then an expression like (1.1) holds with $(f/g)(t)$ replacing $B(q)$, in which $f_j$ and $g_j$ are polynomials of the same degree.

Here is an overview of the paper.

The author's memoir [R1] developed a set of notations for forms in several variables over a field $K$ of characteristic 0; in this paper, $K \subseteq R$. A brief review is given in section two. Let $H_{\text{p}}(X^d)$ denote the vector space of forms in $K[x_1, \ldots, x_n]$ of degree $d$. This usage of $d$ and $d$ is fixed and $m$ will always denote an even integer. There are three closed convex cones in $H_{\text{p}}(\mathbb{R}^d)$ of particular interest: the cone of positive semidefinite forms, denoted $Q_{\text{p}}$, the cone of sums of squares of forms, denoted $\Sigma_{\text{q}}$, and the cone of sums of $m$-th powers of linear forms, denoted $Q_{\text{m}}$. The solution to Hilbert's Seventeenth Problem amounts to the assertion that, if $p \in P_{Q_{\text{p}}}$, then there exists $h \in H_{\text{p}}(\mathbb{R}^d)$ so that $h^2 \in \Sigma_{Q_{\text{p}} \Sigma_{\text{q}}}$. The main result of this paper is that, if $p \in P_{Q_{\text{p}}}$ is positive definite, then for sufficiently large $d$, we can take $h = \sum_{j} x_j^2$ and $h^2 \in Q_{\text{p}} \Sigma_{Q_{\text{q}}}$. Let $Q_{\text{p}}(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. As part of his solution of Waring's Problem, Hilbert [H15] established that $G_{\text{p}} \in Q_{\text{p}}$, for all $(n, r)$; in fact, $G_{\text{p}}$ can be written as a sum of $2r$-th powers of linear forms with rational coefficients (see Proposition

2.6 and Theorems 6.9 and 6.12). Families of such "rational" representations are known explicitly for $r = 1, 2$, and for a few other $(n, r)$. No general formula is known which applies to all $(n, r)$, although it is not difficult to find specific examples; see for example Dickson [D9, pp.717-724], or [R1, §§8]. (Hausdorff [H3] gave an explicit representation of $G_{\text{p}}$ as a sum of $2r$-th powers of linear forms with real coefficients including the roots of the Hermite polynomials.) Our analytical analysis of (1.1) will require a trigonometric representation of $(x^2 + y^2)^r$ as a sum of $2r$-th powers (see (2.9)), which is not rational (except accidentally, for $r = 1, 3$).

Associated to a form $g \in H_{\text{p}}(\mathbb{R}^d)$ is the $d$-th order differential operator $d(g)$, defined by replacing each occurrence of $x_j$ by $\frac{\partial}{\partial x_j}$. For example, $d(G_{\text{p}})$ is the Laplacian $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$. In the 19th century, Sylvester developed (and Clifford exploited [C2, p.119]) the method of "contravariant differentiation". We use this to give a simple representation for $d(g)$ applied to a sum of $r$-th powers (Proposition 2.8):

$$d(g) \sum_{x_j} (a_1x_1 + \cdots + a_nx_n) \frac{\partial}{\partial x_j} g = \frac{r!}{(r-1)!} \sum_{x_j} g(a_1x_1 + \cdots + a_nx_n) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} g - d(g)$$

An immediate consequence of these ideas is the first main result.

First Theorem (see Theorem 3.1) If $g \in Q_{\text{p}}, m_{\text{a}}$, and $f \in P_{\text{p, q}}$, then $d(f \cdot g) \in Q_{\text{p, m}_{\text{a}}, r}$.

We apply the first Theorem for $g = G_{\text{p}} \in Q_{\text{p}, 2n}$. It is easy to see that if $f \in H_{\text{p}}(\mathbb{R}^d)$ and $g \in H_{\text{p}}(\mathbb{R}^d)$, then $G_{\text{p}}^{2r} - d(g)$ must be a factor of $d(G_{\text{p}})^G_{\text{p}}$ and the quotient must also have degree $d$. Define $d(f)(g)$ in $H_{\text{p}}(\mathbb{R}^d)$ by $d(f)(g) = \frac{G_{\text{p}}}{G_{\text{p}}^{2r} - d(g)}$. If $g \in H_{\text{p}}(\mathbb{R}^d)$ is psd, then by the First Theorem, $d(f)(G_{\text{p}}) \in Q_{\text{p}, 2n}$; in fact, it is a non-negative $K$-linear combination of the $(2r - m)$-th powers of linear forms whose $2r$-th powers sum to $G_{\text{p}}$.

We give explicit formulas for $d(f)(g)$ and $d(f)(g)$ for $p \in H_{\text{p}}(\mathbb{R}^d)$ (see Theorems 3.7 and 3.9), written as linear combinations of $(\Delta G_{\text{p}})^G_{\text{p}}$, where $\Delta G_{\text{p}}$ denotes the $d$-th iterated Laplacian and $\Delta G_{\text{p}}$ is $p$. (These are finite sums, because $\Delta G_{\text{p}}(p) = 0$ for $d > 2$.) It turns out that a suitably normalized version of $d(f)(g)$ converges to the identity. If $p$ is positive definite, then for sufficiently large $r$, $d(f)(g)$ is also positive definite by Theorem 3.11. The proofs of Theorems 3.7, 3.9 and 3.11 are deferred until the fourth section. For $p \neq P_{\text{p}, m}$, we need a measure of how close $p$ is to having a non-trivial zero. Let

$$d(p) = \inf\{p(a) : a \in S_{\text{n} - 1}\}$$

$$\sup\{p(a) : a \in S_{\text{n} - 1}\}$$

Second Theorem (see Theorem 3.12) Suppose $p \in H_{\text{p}}(\mathbb{R}^d)$ is positive definite. If $r > \frac{\min(n - 1)}{2} - 2n$, then $G_{\text{p}}^{2r}$ is a non-negative $K$-linear combination of a set of $(n + 2r)$-th powers of linear forms in $Q_{\text{p}}(x_1, \ldots, x_n)$. This set depends only on $n, m_{\text{a}}$ and $r$.\]
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As noted earlier, Pólya proved a similar but weaker result: if \( p \) is an even positive definite form, then \( pG_r^G \) has positive coefficients for \( r \) sufficiently large. The condition on \( K \) also carries over; see [H2, p.304] for \( K = \mathbb{Q} \). Pólya did not provide estimates for \( r \). This has been done very recently by de Loera and Santos [D2], who also describe some new algorithms for implementing the representations. Interestingly, their bound on \( r \) also depends on \( (p^2) \).

Let \( p_{n,r}^k \) be the set of \( p \in P_n \) so that \( (p) \geq r \); \( p_{n,r} = \bigcup_{k \geq r} p_{n,r}^k \). For each \( \epsilon > 0 \), the Second Theorem implies that if \( p \in P_n \) so \( r \geq \frac{\log(1/\epsilon)}{\log 2} = \frac{\log 2^{2n}}{\log 2} \) and \( G_{2n} = \sum_{0 \leq j \leq n} (2j)^{2n} \), then \( pG_{2n} = \sum 2^a(p_{2j})(2j)^{2n+j} \), where \( a(p) \geq 0 \) is linear in \( p \). Such a representation clearly cannot hold over all of \( \mathbb{F}_r \); see Corollary 3.18 below.

The Second Theorem gives new, concrete information about representations as a sum of \( 2k \)-th powers of rational functions. The following result (without the specification of the denominators) can be given an abstract proof using Becker’s theory.

**Third Theorem (see Theorems 3.15 and 3.16)** If \( p \in K[x_1, \ldots, x_n] \) is a positive definite form of degree \( m = 2k \), then \( p \) is a non-negative \( K \)-linear combination of \( 2k \)-th powers of rational functions in \( K[x_1, \ldots, x_n] \) whose denominators are powers of \( G_r \). If \( p \) and \( q \) are positive definite forms in \( K[x_1, \ldots, x_n] \) and the degree of the rational function \( p/q \) is a multiple of \( 2k \), then \( p/q \) is a non-negative \( K \)-linear combination of \( 2k \)-th powers of rational functions whose numerators are in \( K[x_1, \ldots, x_n] \) and whose denominators are powers of \( G_r \) and \( G_q \).

The Second Theorem also contains the hard part of a curious equivalence (Corollary 3.18): \( p \in P_n \) is positive definite if and only if \( pG_{2n} \in G_qm^{2n} \) for some \( r \geq 1 \). This is not a trivial consequence of the existence of bad points: there exist positive semidefinite forms \( p \) which are not definite and for which \( pG_r \) is a sum of squares of forms. (See for example [R3, p.275] and [C1, p.579].) For that matter, we show in section five that there exist positive definite forms \( p \) so that \( \Phi_r^{-1}(p) \) is not psd, but \( pG_{2n} \in G_{2n}m^{2n} \).

The fourth section contains proofs of the formulas needed in section three. The formula for \( \Phi_k(f) \) follows from an old theorem used by Hobson in his studies of spherical harmonics. The formula for \( \Phi_k^+(f) \) requires some tricky but elementary computations. In order to give the estimate on \( x \) for which \( \Phi_k^+(f) \) is psd, we need a simple, but apparently new, bound on the \( \|x^\circ \| \) norm of \( \Delta : H_0(k^2 \to H_0(k^2) \).

Section five gives applications of the Second and Third Theorems to quadratic forms. We generalize \( B(t) \) by considering quadratic forms in \( n \) variables and their degenerations. We show (Corollary 5.3) that if \( a > 0, b > 0 \) and \( 2k + 1 \geq r \geq n \), then \( a(x^2 + b) + b(x^2 + y) \) is a sum of \( (a+b)^n \) \( 2k \)-th powers of rational functions with denominator \( b^2 + 1 + \cdots + 1 \).

Fix \( k \geq 1 \) and let \( L_0(x,y) = (x^2 + y^2)(x + y) \) and \( \lambda_j = (k + 1)(c_j + 1) \) for \( 0 \leq j \leq k + 1 \). Using the known trigonometric identity (5.9) for \( (x^2 + y^2) \) and the proof of Corollary 5.3, we obtain:

\[
(1.3) \quad B(t) = \frac{1 + t^2}{2 + t^2} = \sum_{k=0}^{2k-2} \sum_{j=0}^{k-1} \frac{a_j (\sqrt{2} \cdot t) (\sqrt{2} \cdot t)}{2 + t^2} \int_{0}^{\infty} \lambda_j \frac{dx}{dx^2 + y^2}.
\]

Although (1.3) gives \( B(t) \) as a sum of \( 2k \)-th powers in \( \mathbb{R}(t) \), the summands are not in \( \mathbb{Q}(t) \). Such a representation cannot yet be found by our methods, because there is no known analogue to (5.9) with coefficients in \( \mathbb{Q} \).

Section six presents a self-contained proof of Hilbert’s theorem about \( G_r^G \) and more, mostly using the ideas of [R1]. We give an integral representation for \( U \) which implies that \( G_r^G \) is interior to \( Q_{2n,2r} \) (a result essentially due to Hilbert). In fact, \( \Phi_k(f) \) can be given an integral interpretation, and if \( 0 \leq f \leq \mathbb{P} \), then \( \Phi_k(f) \) is interior to \( Q_{2n,2r} \) (see Theorem 6.9). Finally, we show that any \( h \in H_0(k^2) \) in the interior of \( Q_{2n,2r} \) is a non-negative \( K \)-linear combination of \( (n^2)^2 - 2k \)-th powers of linear forms with coefficients in \( \mathbb{Q} \) (Theorem 6.12). Theorem 6.12 may fail for \( h \) on the boundary of \( Q_{n,2r} \); we have shown that \( r \geq 1 \), \( (x + \sqrt{2}y)^n (x - \sqrt{2}y)^m \) is in \( \mathbb{Q}(y) \) is not a non-negative \( K \)-linear combination of \( 2k \)-th powers of linear forms in \( \mathbb{Q}(y) \).

I am happy to acknowledge much advice and many suggestions. Vicki P. Scott asked me about \( B(t) \) until I finally realized how interesting it is; the methods in this paper were originally developed for the study of (1.1). Chip DeZell patiently answered my questions about the continuous versions of Hilbert’s Seventeenth Problem, and reminded me about bad points, both grammatical and algebraic. Alex Prestel saved me from some rashly incorrect historical assertions. Eberhard Becker helped sharpen the focus of the paper on its main results and simplified some of the proofs. These colleagues also suffered through an earlier (and much more oblique) version of Theorem 3.11. Finally, Norm Levenberg and Doron Zeilberger gave me the valuable references [D8] and [S2], respectively.

### 2 Preliminary material

In this section we present some basic notations and facts about real forms in several variables. For the most part, this material can also be found in [R1, p.125].

Let \( K \) be a field of characteristic \( 0 \), and let \( H_0(k^2) \) denote the set of homogeneous polynomials in \( K[x_1, \ldots, x_n] \) with degree \( d \). In this paper, \( K \) will always be a subfield of \( \mathbb{R} \). The index set for monomials in \( H_0(k^2) \) consists of \( n \)-tuples of non-negative integers:

\[
\mathcal{F}(n,d) = \left\{ (i_1, \ldots, i_n) : \sum_{k=1}^{n} i_k = d \right\}.
\]

Write \( N(n,d) = \binom{n+d}{n+1} = \left| \mathcal{F}(n,d) \right| \) and for \( i \in \mathcal{F}(n,d) \), let \( c(i) = \prod_{j=1}^{d} u_j \) be the associated multinomial coefficient. The multinomial abbreviation \( u \) means \( u_1 \ldots u_d \), where \( u \) may be an \( n \)-tuple of constants or variables. Every \( f \in H_0(k^2) \) can be written as
The identification of \( f \) with the \( N(n, d) \)-tuple \( (a(f); i) \) shows that \( H_d(K^*) \cong K^*(n, d) \) as a vector space. For \( \alpha \in K^* \), we define \( (\alpha)\gamma^d(x) \) in \( H_d(K^*) \) by
\[
(\alpha)\gamma^d(x) = \sum_{i\in\mathcal{I}(a, d)} c(i)\alpha^d(i)\gamma^d.
\]
We define a fundamental symmetric bilinear form. For \( p \) and \( q \) in \( H_d(K^*) \), let
\[
[p, q] = \sum_{i\in\mathcal{I}(a, d)} c(i)\alpha(p; i)\alpha(q; i).
\]
For \( i \in \mathcal{I}(a, d) \), let \( D' = \prod_{j=1}^{d} \alpha_j^{1/d} \) and define \( f(D') = \sum_{i\in\mathcal{I}(a, d)} c(i)\alpha^d(i)\gamma^d \) to be the \( d \)-th order differential operator associated to \( f \in H_d(K^*) \). For any forms \( f \) and \( g \) (possibly of different degree), \( f(D')g(D') = g(D')f(D') \) is commutative.

**Lemma 2.4** If \( p, q, g \in H_d(K^*) \), \( g \in H_d(K^*) \), \( h \in H_{a+n}(K^*) \), \( \alpha \in K^* \) and \( i, j \in \mathcal{I}(a, d) \), then: (i) \( [p, (\alpha)\gamma^d] = p(\alpha) \); (ii) \( [p, i] = a(p; i) \); (iii) \( D'[p, q] = p(D')q \); (iv) \( [p, (\alpha)\gamma^d] = (d + e)[gp, h] \).

**Proof.** The first two formulas follow from (2.1), (2.2) and (2.3), since \( a((\alpha)\gamma^d) \) is \( \gamma^d \) and \( a(\gamma^d) = \gamma^d \), even if \( i = 0 \) or \( i \neq 0 \). Since both sides of (iii) are bilinear in \( p \) and \( g \), it suffices to prove (iii) for monomials \( p = c(i)\gamma^d \). In this case, \( [p, q] = 0 \) if \( i \neq j \) and \( [c(i)\gamma^d, c(j)\gamma^d] = 1 \) by (i). On the other hand,
\[
c(i)\gamma^d(j) = c(i)\sum_{k=1}^{n} \frac{(d)^{k-1} \gamma^d}{\alpha_k^k} = d! \prod_{k=1}^{\alpha_k} (j_k - (j_k - 1))x_k^{k-1}.
\]
(2.5)

It is easy to see from (2.5) that \( c(i)\gamma^d(j) = d! \). If \( i \neq j \), then \( \sum_{k=1}^{\alpha_k} j_k \neq k \) and so \( c(i)\gamma^d(j) = 0 \). Thus (iii) is true. Finally, two applications of (iii) give \( (d + e)[gp, h] = (gp)(D')h = p(D)(g(D)h) = [p, (\alpha)\gamma^d] \), proving (iv).

Note that Lemma 2.4(iii) implies that the bilinear product is non-degenerate: if \( [p_1, q] = [p_2, q] \) for \( p_1, p_2 \in H_d(K^*) \) and all \( q \in H_d(K^*) \), then \( p_1 = p_2 \).

A set \( C \subset K^* \) is a closed convex cone if it is closed under addition, multiplication by positive reals and is also closed topologically. Similarly, a set of forms \( \mathcal{B} \subset H_d(K^*) \) is a closed convex cone in \( H_d(K^*) \) if it is closed under addition, multiplication by positive reals and is also closed topologically. Here, \( f(0^d, i) \to a(i) \gamma^d \) for all \( i \in \mathcal{I}(a, d) \). If \( \mathcal{B} \) is a closed convex cone in \( H_d(K^*) \), then \( \{a(f; i) : f \in \mathcal{B}\} \) is a closed convex cone in \( R^{K^*(n, d)} \).
3 The main theorems

We begin this section with an application of Proposition 2.8.

Theorem 3.1. Suppose \( h \in Q_{a,b} \) and \( q \in P_{a,m} \). Then \( q(D)h \in Q_{a,b} \).

Proof. If \( h = \sum_k (a_k x)^k \), then \( q(D)h = \sum_{k,j} q(a_k) a_j x^{k+j} \) by Proposition 2.8. Since \( q \) is psd by hypothesis, \( q(a_k) \geq 0 \) for all \( k \), and we are done.

Theorem 3.1 suggests an inverse problem: given a \( p \), can we find suitable \( q \) and \( h \) so that \( p = q(D)h \)? Fortunately, \( G_j \in Q_{a,b} \) for all \((n,s)\) by Proposition 2.6, and it turns out that \( q(D)G_j \) can always be explicitly computed. Before we state the general formulas, we consider low degrees. Note that

\[
\frac{\partial}{\partial x_k}(G_j) = \frac{\partial}{\partial x_k}(x_1^2 + \ldots + x_k^2 + \ldots + x_n^2)^{j-1}.
\]

It follows easily from (3.2) that

\[
\frac{\partial}{\partial x_k}(G_j) = 2(x_2 - 2x_1 x_k^{j-1} + x_k^2)^{j-2}, \quad j \neq 1.
\]

More generally, if \( f \in H_o(R^n) \) and \( s \geq d \), then every monomial term in \( f(D) \) involves \( d \) differentiations, hence \( G_j \) divides \( f(D)G_j \), with quotient of degree \( 2s - d - 2(s-d) \). Accordingly, for \( f \in H_o(R^n) \) and \( s \geq d \), we define \( \Phi_f(p) \in H_o(R^n) \) by

\[
f(D)G_j = \Phi_f(p)G_j.
\]

It is clear from (3.4) that \( \Phi_f = \Phi_{f,x_1,x_2} \) is linear. By (3.2) and (3.3), \( \Phi_f(x_1 x_2) = 2x_1 x_2 \), \( \Phi_f(x_1 x_2) = 4s(s-1)x_1 x_2 \). If \( f \neq \Phi_g \), recall that \( D = G_j(D) \). Since the \( x_i \) are not isolated, we have by linearity:

\[
\Phi_f(f) = 2f, \quad \Phi_f(f) = 4s(s-1)f + s(D)f G_n, \quad f \in H_o(R^n).
\]

Since \( \Delta(G_n) = 2n \) and \( \Phi_f(G_n) = (4s-1)G_n + 2nG_n = 4s(s-1)G_n - 2nG_n \), we can invert:

\[
\Phi_f^{-1}(f) = \frac{1}{2s} f, \quad f \in H_o(R^n).
\]

It turns out that, if \( p \) is psd, then \( \Phi_f^{-1}(p) \) is psd for \( s \) sufficiently large. This result follows from three theorems whose proofs are deferred to the next section. We first obtain closed forms for \( \Phi_f \) and \( \Phi_f^{-1} \); compare (3.8) to (3.5) and (3.10) to (3.6) for \( d = 1, 2 \). Also observe from (3.7) that \( \frac{\partial}{\partial x_k}\Phi_f(p) \to p \) as \( s \to \infty \), since \( \frac{\partial}{\partial x_k}\Phi_f(p) \to 0 \) for \( k \geq 1 \).

Theorem 3.7. If \( f \in H_o(R^n) \) and \( s \geq d \), then

\[
\Phi_f(p) = \sum_{i=0}^d \frac{(s-1+i)^2}{2i+1} \Delta_i(p) G_i.
\]

Theorem 3.9. If \( f \in H_o(R^n) \) and \( s \geq d \), then

\[
\Phi_f^{-1}(p) = \frac{1}{(s-1)^2} \sum_{i=0}^d \frac{s-1+i}{2i+1} \Delta_i(p) G_i.
\]

Theorem 3.11. Suppose \( p \in P_{a,m} \) is positive definite and recall the definition of \( e(p) \) from (1.2). If \( s \geq \frac{\max(a-1)}{\max(a-1)} - 1 \), then \( \Phi_f^{-1}(p) \in P_{a,m} \).

Theorem 3.12. If \( p \in H_o(K^n) \) is positive definite and \( s \geq \frac{\max(a-1)}{\max(a-1)} - 1 \), then \( pG_j \in Q_{a,b} \); in fact, \( pG_j \) is a non-negative \( K \)-linear combination of \((m+2)r\)-th powers of linear forms in \( \{x_1, \ldots, x_n\} \).

Proof. Let \( h = G_j^{2m} \) and \( q = \Phi_f^{-1}(p) \). Taking \( s = r + m \) in Theorem 3.11, we see that \( q \) is psd, and by Theorem 3.11, \( q = (G_j(p) G_j)^{2m} = q_{re}(p) G_j \in Q_{a,b} \). In fact, Proposition 2.6 states that for suitable \( 0 \leq \lambda_1 \in Q(x_1) \), \( x_1, \ldots, x_n \),

\[
\sum_{i=1}^{N(x_1, \ldots, x_n)} \lambda_i (x_1 x_1 + \ldots + x_n x_n)^{2r+2m}.
\]

Apply \( q(D) \) to both sides of (3.13) and use Proposition 2.8:

\[
\sum_{i=1}^{N(x_1, \ldots, x_n)} \lambda_i (x_1 x_1 + \ldots + x_n x_n)^{2r+2m}.
\]

Since the coefficients in (3.10) are rational, \( p \in H_o(K^n) \) implies \( q = \Phi_f^{-1}(p) \in H_o(K^n) \), so \( \lambda_i(x_1) \in K \), and (3.14) gives the desired representation.

Theorem 3.15. If \( p \in H_o(K^n) \) is positive definite and \( 2k j \), then \( p = \sum_{i=1}^{N(x_1, \ldots, x_n)} \lambda_i (f G_n)^{2m} \), where \( 0 \leq \lambda_i \in K \) and \( f(x_1, \ldots, x_n) \). Proof. Let \( m = 2k \). By Theorem 3.12, for sufficiently large \( u \), there exist linear forms \( L_i \) over \( Q \) and \( 0 \leq \lambda_i \in K \) so that \( pG_j^{2m} = \sum \lambda_i L_i^{2m} G_j^{2m} \), or \( p = \sum \lambda_i (G_j G_n)^{2m} \).
Theorem 3.16 Suppose \( p \in H_N(K^n) \) and \( q \in H_N(K^n) \) are both positive definite, and \( 2k|m_1 - m_2 \). Then for sufficiently large \( v \),
\[
\frac{P}{q} = \sum_{I} \lambda_I \left( \frac{f_I}{q G_{x}^{2k}} \right)
\]
for suitable \( 0 \leq \lambda_I \in K \) and \( f_I \in Q[x_1, \ldots, x_n] \).

Proof. Let \( m_1 - m_2 = 2k \) and pick a large enough that \( p G_{x}^{m_1/2} \) and \( q G_{x}^{m_2/2} \) are in \( Q_{2k}, \) and \( Q_{2k} \), respectively. Then by Theorem 3.12, there are linear forms \( L_{x} \) and \( L_{y} \) in \( Q[x_1, \ldots, x_n] \) and \( \mu_x, \mu_y > 0 \) in \( K \) so that:
\[
\frac{P}{q} = \frac{p G_{x}^{m_1/2}}{q G_{x}^{m_2/2}} = \sum_{I} \mu_x (L_{x}^{2k} x_I^{2k}) \left( \sum_{I} \nu_y (L_{y}^{2k} y_I^{2k}) \right)^{-1} \left( \sum_{I} \nu_y (L_{y}^{2k} y_I^{2k}) \right)^{2k - 1}
\]

The numerator in (3.17) is the product of \( 2k \) terms, each of which is a sum of \( 2k \)-th powers, and so is a sum of \( 2k \)-th powers. The (common) denominator in (3.17) is \( (G_{x}^{m_1/2})^{2k} \). \( \frac{P}{q} \) is a sum of \( 2k \)-th powers of rational functions with denominator \( G_{x}^{m_2/2} \).

One might hope that Theorem 3.12 could be extended to pad forms, since Theorem 3.1 only requires that \( \theta_x^{i}(x) \geq 0 \) at any \( n \)-tuple \( (a_1, \ldots, a_n) \) used in (2.7). We shall see in the next section that it is possible for \( p G_{x}^{m} \) to be in \( Q_{2k} \), even though \( \theta_x^{i}(x) \) is not pad. However, the requirement that \( p \) be pad (not merely pad) in Theorem 3.12 is essential.

Corollary 3.18 A form \( p \in H_d(R^n) \) is \( pd \) if and only if \( p G_{x}^{r} \in Q_{d,m+2r} \) for some \( r \geq 1 \).

Proof. One direction is Theorem 3.12. For the other, suppose \( p G_{x}^{r} \in Q_{d,m+2r} \); clearly, \( p \) is pd; must it be \( pd \)? Suppose \( p(u) = 0 \) for \( u \in S^{n-1} \); after an orthogonal change of variables (fixing \( G_{x} \)), assume \( u = (1, 0, \ldots, 0) \). Suppose there exists \( \alpha \in K \) so that
\[
(p(x_1, \ldots, x_n)x_1^2 + \ldots + x_n^2)^r \geq \alpha (x_1^2(x_1^2 + \ldots + x_n^2))^{m+2r}.
\]

Setting \( x = \alpha \) in (3.19), we see that \( 0 = \sum_{i=1}^{n} \alpha_i^{m+2r} \), hence \( \alpha_i = 0 \) for all \( i \). Thus \( x_1 \) does not appear at all on the right-hand side of (3.19). Let \( x_1 \beta \) be the terms in \( p \) which contain \( x_1 \) to the maximum exponent; possibly, \( \beta = 0 \). Then \( x_1^{m+2r} \beta \) appears uncancelled in the left-hand side, a contradiction.

\[\Box\]

4 Proofs of Theorems 3.7, 3.9 and 3.11

In this section we give combinatorial proofs of Theorems 3.7 and 3.9 and an analytic proof of Theorem 3.11. First, we prove Hobson's Theorem, which dates to 1982 [H1, H2] and can be found in [H2]. Another recent proof of Hobson's Theorem is given by Strasburger [S2]. We will need a lemma on the iterated Laplacian.

Lemma 4.1 (i) If \( q \in H_{d}(R^n) \) and \( k \geq 0 \), then
\[
\Delta^k(xq) = 2k \frac{\partial (\Delta^{k-1}(q))}{\partial x_k} + x_k \Delta^k(q).
\]
(ii) If \( 0 \leq k \leq s \), then
\[
\Delta^k(G_{x}^{2k}) = 2^{2k} \binom{n}{2} \frac{r}{2} + s - 1 \left( G_{x}^{2k-1} \right)^k.
\]

Proof. If \( 2k > d + 1 \), then degree considerations imply that both sides of (4.2) vanish. Otherwise, (4.2) is valid provided both sides have the same inner product with every \( p \in H_{d+1-2k}(R^n) \). But by the product rule and repeated applications of Lemma 2.4(iii),
\[d - 2k + 1] \left( \Delta^k(xq) \right) = (d + 1) [x_k G_{x}^{2k}] + \sum_{j=1}^{d} \frac{\partial (G_{x}^{2k})_{x_j}}{\partial x_j} \left( \Delta^k(xq) \right) + \sum_{j=1}^{d} \frac{\partial (G_{x}^{2k})_{x_j}}{\partial x_j} \left( \Delta^k(xq) \right)
\]

Thus both sides of (4.2) have the same inner product with \( (d - 2k + 1)p \). For (ii), (3.3) implies that \( \Delta^k(G_{x}^{2k}) = 4s(\frac{n}{2} + s - 1)G_{x}^{2k-1} \), so (4.3) holds for \( k = 1 \). If (4.3) holds for \( k \), then
\[\Delta^{k+1}(G_{x}^{2k}) = \Delta^k(\Delta(G_{x}^{2k})) = 4s(\frac{n}{2} + s - 1)G_{x}^{2k-1} \]

so (4.3) holds for \( k + 1 \) as well, and the induction is complete.

Proposition 4.4 (Hobson) If \( p \in H_{d}(R^n) \), and \( F \) is sufficiently differentiable, then
\[
p(D)F(G_{x}) = \sum_{i=0}^{[d/2]} \frac{d!}{2^k k!} \Delta^k(p)(xq)_{x_i}.
\]
Proof. It suffices by linearity to prove (4.5) for monomials \( p \in H_d(\mathbb{R}^n) \). It is trivial for \( d = 0 \); if \( d = 1 \), let \( p = x_i \), and \( \frac{\partial}{\partial x_i} F(G_0) = F'(G_0) \frac{\partial G_0}{\partial x_i} = 2 x_i F'(G_0) = \frac{2 x_i}{2} \Delta^1(p) F^{(1-0)}(G_0) \), as desired. Now suppose (4.5) holds for monomials of degree \( d \). If \( x' \in H_d(\mathbb{R}^n) \), then \( x' = x x_i \) for some \( i \), where \( x' \in H_d(\mathbb{R}^n) \). Then we have the following sequence of identities, identified after the fact:

\[
D^1 F(G_n) = \sum_{k=0}^d \frac{d^k}{2k!} \frac{\partial}{\partial x_i} \Delta^k(x') F^{(d-k)}(G_n) = \sum_{k=0}^d \frac{2^d}{2k!} \Delta^k(x') F^{(d-k)}(G_n) = \sum_{k=0}^d \frac{2^d}{2k!} \Delta^k(x') F^{(d-k)}(G_n).
\]

The induction hypothesis is used in the first line, and the product rule is used to get to the second line. In the third line, the chain rule is used in the first sum, and the second sum is written in terms of \( k + 1 \), rather than \( k \). In the fourth equation, the index of the second sum is shifted from \( k + 1 \) to \( k \). The summation should now be taken over \( k \geq 1 \), but the summation is \( 0 \) for \( k = 0 \), so it can be taken over \( k \geq 0 \). Terms are combined in the fifth equation, and Lemma 4.1(i) is used in the last line to complete the induction. □

Proof of Theorem 3.7 Take \( F(z) = z^i \) above; \( F^{(d-k)}(z) = (z)_k e^{-z} e^{-(d-k)} \), so by (4.5),

\[
\Phi_k(p) G_n^{d-k} = p(D) G_n^{d-k} = \sum_{i=0}^d (\frac{i}{i}) \frac{1}{i!} \frac{2^d}{2k!} \Delta^1(p) G_n^{d-k}.
\]

After cancellation of \( G_n^{d-k} \), (4.6) becomes (3.8). □

Proof of Theorem 3.9 By applying \( \Phi_k \) to (3.10) and scaling, we find:

\[
\Phi_k(p) G_n^{d-k} = \sum_{k=0}^d \frac{2^d}{2k!} \Delta^k(p) G_n^{d-k}.
\]

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(4.7) \[ (s)_k 2^d p = \sum_{l=0} \frac{(-1)^l}{2^d \Gamma(\frac{3}{2} + k - l)} \Phi_k(D(G_n)^{d-k}). \]

Since \( \Delta^k(p) G_n^{d-k} \in H_d(\mathbb{R}^n) \) for each \( k \leq d/2 \), by (3.4) and Lemma 4.1(ii),

\[
\Phi_k(D(p) G_n^{d-k}) = (\Delta^i(p) G_n^{d-k})(D(G_n)^{d-k}) = (\Delta^i(p)(G_n^{d-k})) = \sum_{k=0}^d \frac{2^d}{2k!} \Delta^k(p) G_n^{d-k}.
\]

Upon cancellation of \( G_n^{d-k} \) from both sides of (4.8), we obtain the formula

\[
\Phi_k(\Delta^i(p) G_n^{d-k}) = (s)_k 2^d (s + k - 1) ! \Phi_k(D(G_n)^{d-k}).
\]

We now plug (4.9) into the right-hand side of (4.7) and then use (3.8), keeping in mind that \( \Delta^i(p) \in H_{2d}(\mathbb{R}^n) \):

\[
\sum_{l=0} \frac{(-1)^l}{2^d \Gamma(\frac{3}{2} + s + k - l)} \Phi_k(D(p) G_n^{d-k}) = \sum_{l=0} \frac{(-1)^l (s)_k 2^d (s + k - l) ! \Phi_k(D(G_n)^{d-k})}{2^d \Gamma(\frac{3}{2} + s + k - l)} = \sum_{l=0} \frac{(-1)^l (s)_k 2^d (s + k - l) ! \Phi_k(D(G_n)^{d-k})}{\Gamma(\frac{3}{2} + s + k - l)}
\]

(4.10)

Since \( (s)_k 2^d (s + k - 1) ! \) is the same for all \( k \), we can stratify the final sum in (4.10) by \( s+k \).

(4.11)

The inner sum in (4.11) vanishes unless \( s = 0 \), in which case it equals 1, so the entire right-hand side of (4.7) reduces to \( (s)_k 2^d p \), which was the assertion. □

The proof of Theorem 3.11 requires more interesting in its own right, but perhaps for a different paper! A key step requires a 1928 inequality due to Szegö which can be found in [D8, p.97], and is a sharpening of Bernstein’s Inequality.

Proposition 4.12 (Szegö) If \( H(\theta) = \sum_{\theta=0}^d \cos(\theta) + \sum_{\theta=0}^d b_k \sin(\theta) \) is a trigonometric polynomial of degree \( d \) and \( |H(\theta)| \leq M \) for \( \theta \in [0, 2\pi] \), then

\[
d^2(H(\theta)^2 + H'(\theta)^2) \leq d^2 M^2 \quad \text{for} \quad \theta \in [0, 2\pi].
\]

Theorem 4.14 If \( f \in H_d(\mathbb{R}^n) \) and \( f(u) \leq M \) for \( u \in S^{n-1} \), then \( |D^d f(u)| \leq n^d d! M(d) \) for \( u \in S^{n-1} \).
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(4.19)
\[
(p_{\rho}(p))(u) \geq \epsilon - \sum_{i \leq 5} \frac{n_i(m)\epsilon_i - 1}{2}k.
\]

The sum in (4.19) is finite because \(\Delta^i(p) = 0 \text{ for } i < \frac{n}{2}\). We have \((m)_k \leq (m(m-1))^{\frac{n}{2}}\) and \(\left( \frac{2}{n} + \frac{1}{2} \right) \geq 2\left( \frac{2}{n} + \frac{1}{4} \right)\). Thus, (4.19) implies

(4.20)
\[
(p_{\rho}(p))(u) \geq - \sum_{i \leq 5} \frac{n_i(m-m-1)}{2} \epsilon + \frac{n_i}{2}
\]

Since \(\epsilon \geq \frac{\epsilon}{4(m-m-1)} - \frac{\epsilon}{2}\), we have \(\frac{\epsilon}{4(m-m-1)-4\epsilon} \leq \epsilon \log 2\), and (4.20) implies

(4.21)
\[
(p_{\rho}(p))(u) \geq \epsilon - \sum_{i=1}^{\infty} \frac{\epsilon \log 2}{i} = \epsilon - \epsilon \log 3 + 1 = 1 + \epsilon - 2^\epsilon.
\]

Let \(F(t) = 1 + t - 2^t\). Then \(F'(t) < 0\) and \(F(0) = 1\), hence \(F(t) \geq 0\) on \([0, 1]\). In particular, \(\epsilon \in [0, 1]\), so \(1 + \epsilon - 2^\epsilon \geq 0\) and (4.21) completes the proof.

\[\square\]

5 Quadratic examples

This section on quadratic polynomials does not use Theorems 3.7, 3.9 and 3.11.

\[\text{Lemma } 5.1 \text{ If } \sum_{k=1}^{n} a_k a_k = \sum_{k=1}^{n} a_k^2, \text{ then } \Phi_{\rho}(p) \text{ is psd if and only if }\]

(2s + n - 2) \rho_k^2 \leq \sum_{k=1}^{n} a_k \rho_k^2 \text{ for all } k, 1 \leq k \leq n.

\[\text{Proof. } \text{Observe that } |\Delta_f \rho_k | = 2\sum_{j=1}^{n} a_{ij}, \text{ hence by (3.6)(ii),}\]

\[4s(-1)^{s-1} \Phi_{\rho}(f) - \frac{1}{4(\frac{2}{n} + \frac{1}{2})} |\Delta_f| \rho_\kappa = \sum_{k=1}^{n} a_{k}^2 - \frac{2}{4(\frac{2}{n} + \frac{1}{2})} \sum_{j=1}^{n} a_{j}^2 \rho_j^2 = \frac{1}{2(\frac{2}{n} + \frac{1}{2})} \sum_{k=1}^{n} (2s + n - 2) a_k \rho_k^2.\]

\[\text{Theorem } 5.2 \text{ If } 2r + 1 \geq \lambda \geq \frac{n-1}{2n-1} \text{ and } \lambda \rho_k = \lambda \rho_k^2 + \sum_{k=1}^{n} a_k \rho_k^2, \text{ then } \rho_k \in Q_{\rho_k,2}.\]

\[\text{Proof. } \text{If } \Phi_{\rho}(p) \text{ is psd, then } \Phi_{\rho}^{-1}(p)(D)D\Phi_{\rho}^{-1} = \rho_k \rho_k - 1 \in \mathbb{Q}_{\rho_k,2}, \text{ by Theorem } 3.1. \text{ Since } \sum_{j=1}^{n} a_{ij} = \lambda + n - 1 \text{ and } a_{ij} \text{ takes the values } 1 \text{ and } \lambda, \text{ Lemma } 5.1 \text{ implies that } \Phi_{\rho}^{-1}(p) \text{ is psd if and only if } (n+2) \lambda \geq n+1 - \lambda \text{ and } n+2r \geq n+1 - \lambda.\]

\[\square\]
Let's compare Theorem 5.2 with Theorem 3.11. If $0 \leq \lambda \leq 1$, then clearly $c(p\lambda) = \lambda$. By Theorem 3.11, $\Phi^{-1}(p\lambda)$ is psd provided $r + 1 \geq \frac{2^4k}{4k + 3} - \frac{n-1}{2}$. By Theorem 5.2, $\Phi^{-1}(p\lambda)$ is psd precisely when $\lambda \geq \frac{r - 1}{r + 1}$, or $\lambda \geq \frac{n-1}{2}$. Theorem 3.11 is only off by a factor of $\log 2$. We show later that $p_3G_2 \in Q_{2,4}$ does not imply $\Phi^{-1}(\lambda) \in P_{3,2}$.

**Corollary 5.3** Suppose $a > 0, b > 0$ and $2k + 1 \geq \frac{n}{b}\frac{n}{a}$. Then there exist quadratic polynomials $\{\xi_i(t_1, \ldots, t_n)\}, i \leq i \leq N(n+1,4k)$, such that

$$a + t_1^2 + \cdots + t_2^2 = \sum_{i=1}^{N(n+1,4k)} \left( \frac{\xi_i(t_1, \ldots, t_n)}{b + t_1^2 + \cdots + t_2^2} \right)^{2k}. $$

**Proof.** Suppose $2k + 1 \geq \frac{n}{b}\frac{n}{a}$. Then by Proposition 2.6 and Theorem 5.2, there exist linear forms $\{\beta(t_1, \ldots, t_n)\}$ and $\{\gamma(t_1, \ldots, t_n)\}$ so that

$$a + t_1^2 + \cdots + t_2^2 = \sum_{i=1}^{N(n+1,2k)} \gamma_i^2(t_1, \ldots, t_n);$$

$$a + t_1^2 + \cdots + t_2^2 = \sum_{i=1}^{N(n+1,2k+2)} \beta_i^2(t_1, \ldots, t_n).$$

Multiply (5.5)(i) and (5.5)(ii):

$$a + t_1^2 + \cdots + t_2^2 = \sum_{i=1}^{N(n+1,2k)} \gamma_i^2(t_1, \ldots, t_n).$$

Let $L(x,y) = \cos \left( \frac{\pi}{k+2} x \right) \sin \left( \frac{\pi}{k+2} y \right)$. Then by Lemma 5.1

$$\Phi^{-1}(\frac{1}{2}x^2 + y^2) = \frac{1}{16k(k+1)^2} \cos \left( \frac{\pi}{k+2} x + \frac{\pi}{k+2} y \right).$$

Now multiply (5.10)(i) and (5.12) and use $\frac{1}{2}x^2 + y^2$ to obtain (5.4).
The identity (1.3) is also fundamentally defective, in that $B(t)$ is not given as a sum of $2k$-th powers in $Q(t)$. The need for $\sqrt{2}$ could be eliminated by writing $(2x^2 + y^2)^{k/2}$ as in (2.7); this is possible by Theorem 6.12. We can find $h(x, y) = ax^2 + by^2$ so that $h \left( 2x^2 + y^2 \right)^{k/2} = (2x^2 + y^2)^{k/2}$ and proceed as above. Unfortunately, there are no known general explicit formulas for writing any $(x^2 + y^2)^k$, $0 < \lambda \in \mathbb{Q}$, as a $Q$-linear combination of $2k$-th linear forms over $\mathbb{Q}$.

We can use the methods of [R1] (see (1.23), Corollary 3.17 and Theorem 5.1) to find necessary and sufficient conditions under which $(a^2 + b^2)(x^2 + y^2)^{k-1} \in \mathcal{Q}_{1,2}$. In the interest of space, we shall summarize the results at the end of this section.

The case $r = 2$ can be done from first principles. By scaling, we may assume that $(a, b) = (1, 1)$. If $(a^2 + b^2)(x^2 + y^2)^{k-1} = \lambda^* = (\lambda + b + \lambda + a)^{k-1}$, then $\lambda = \sum c_k^2 d_k^2 = \sum c_k^2 d_k^2 + 1 = \sum d_k^2$, hence, by the Cauchy-Schwarz inequality, $\lambda \geq \left( \lambda \right)^{\frac{k}{2}}$, so $\lambda^* = 34 + \lambda + 1 = 0$, or $17 - 12\sqrt{2} \leq \lambda \leq 17 + 12\sqrt{2}$.

On the other hand,

$$\frac{1}{2} \left( (\sqrt{2} \pm 1) x + y \right)^k + (\sqrt{2} \pm 1) x - y \right)^k = \left( (17 \pm 12\sqrt{2}) x^2 + y^2 \right)^{k-1}.$$

Thus, by convexity, $(a^2 + b^2)(x^2 + y^2)^{k-1} \in \mathcal{Q}_{1,2}$ precisely when $17 - 12\sqrt{2} \leq \lambda \leq 17 + 12\sqrt{2}$. By Lemma 5.1, $\Phi_1^2(a^2 + b^2)$ is padic $\frac{1}{2} \leq \lambda \leq 3$.

Thus, if $\lambda$ or $\lambda^2$ are in $(3, 17 + 12\sqrt{2})$, then $\Phi_1^2(a^2 + b^2)$ is not padic when $\lambda^* = (\lambda + a)^{k-1} \in \mathcal{Q}_{1,2}$.

It can be shown that $q_k(x, y) = (a^2 + b^2)(x^2 + y^2)^{k-1} \in \mathcal{Q}_{1,2}$ for $\lambda \in [M_{k-1}, M_{k}]$, where $M_{k}$ is a zero of the catalecticant of $q_k$. Theorem 5.2 with $n = 2$ implies $M_k \geq 2k + 1$. Calculations using Mathematica show that $M_2 = 13 + 6\sqrt{5} \approx 26.42$, $M_3 = 15 + 6\sqrt{3} \approx 25.23$, $M_4 \approx 25.64$, $M_5 \approx 26.69$, $M_6 \approx 28.05$, $M_7 \approx 29.60$, $M_8 \approx 31.26$, $M_{10} \approx 32.99$, $M_{11} \approx 34.78$ and $M_{12} \approx 36.61$. If $k \geq 13$, then $M_k \geq 2k + 1 \geq 27$ by Theorem 5.2, hence we can draw the following conclusion.

**Proposition 5.13** Let $M_{k} = \frac{15 + 6\sqrt{3}}{2}$. If $a, b > 0$ and $1/M_k \leq a/b \leq M_k$, then for all $k \geq 1$, $a^{k/b}$ is a sum of $2k$-th powers of rational functions with denominator $b + r$.

6 A self-contained proof of Hilbert's theorem

In this section, we give a self-contained proof of a generalization of Proposition 2.6. This requires more background from [R1]. We begin with Carathéodory's Theorem, which can be traced back to Hilbert [H4] (see [R1, p.27]).

**Proposition 6.1** (Cara-théodory's Theorem) Suppose $V$ is a vector space over a field $K \subseteq \mathbb{R}$ and $\dim V = N$. Suppose there exist $x_0 \in V$ and $y_0 \in K$, $\gamma_0 > 0$, so that $y_0 = \sum_{i=1}^{N} \lambda_i x_i$. Then there exist $\lambda_i \in K$, $\lambda_i \geq 0$, so that $y_0 = \sum_{i=1}^{N} \lambda_i x_i$.

Proof. There is nothing to prove if $L \leq N$, and it suffices by induction to consider $L = N + 1$. Since $\{x_1, \ldots, x_N\}$ is linearly dependent, there exist $\alpha \in K$, not all 0, such that $\sum \alpha_i x_i = 0$. Let $\beta_i = \alpha_i/\alpha$. We may assume that $0 < |\beta_i| < 1$. Then

$$y = \sum_{i=1}^{N} \lambda_i x_i = \sum_{i=1}^{N} \left( 1 - \beta_i \right) x_i = \sum_{i=1}^{N} \left( 1 - \beta_i \right) \left( 1 - \frac{\beta_i}{\beta_i} \right) x_i,$$

and $\lambda_i = \left( 1 - \beta_i \right) \lambda_i \gamma_i \geq 0$.

**Proposition 6.1.** If applied to $(K, V) = (\mathbb{R}, H_n(\mathbb{R}^n))$, retrieves Hilbert's result that every $p \in \mathcal{Q}_{a,n}$ is a sum of at most $N(n, m)$ $m$-th powers of linear forms. Applied to $(K, V) = (K, H_n(K^n))$, we see that if $p = \sum \gamma_i (\alpha_i x_i)$, $\alpha_i \in \mathbb{Q}^n$ and $\gamma_i \in K$, then $p$ is a non-negative $K$-linear combination of $N(n, m)$ $m$-th powers of linear forms in $(x_i)_{i=1}^n$.

**Proposition 6.2** For all $(n, m)$, $\mathcal{Q}_{n,m}$ is a closed convex cone.

Proof. We already know that $\mathcal{Q}_{n,m}$ is a convex cone. Suppose $p^{\alpha} \in \mathcal{Q}_{n,m}$ and $p^{\alpha} \to p$. By Proposition 6.1, we can write

$$p^{\alpha}(x_1, \ldots, x_n) = \sum_{i=1}^{n} (\alpha_i x_i + \cdots + \alpha_i^{(\alpha)} x_i).$$

(The linear forms are over $\mathbb{R}$, so the coefficients can be absorbed into the linear forms.) Let $e_i = (0, \ldots, 1, \ldots, 0)$ denote the $i$-th unit vector, so $p^{\alpha}(e_i)$ the $i$-th coefficient of $p^{\alpha}$, equals $\sum \alpha_i^{(\alpha)} e_i$, and $\sum \alpha_i^{(\alpha)} e_i$ converges to $p(e_i)$. Thus, for fixed $(k, \ell)$ and $j$ sufficiently large, $|\alpha_i^{(j)}| \leq \alpha_i^{(j)} e_i^{(j)}$. Hence $|\alpha_i^{(j)}| \leq \alpha_i^{(j)} e_i^{(j)}$. And there exists a sequence $(j)$ so that $\alpha_i^{(j)}$ converges for each $(k, \ell)$, say to $\alpha_i$. Then

$$p(x_1, \ldots, x_n) = \sum_{i=1}^{n} (\alpha_i x_i + \cdots + \alpha_i^{(\alpha)} x_i)^{\alpha} \in \mathcal{Q}_{n,m}.$$

The following material can be found in [R1, p.26]. If $[1, \cdots, 1]$ is an inner product on $\mathbb{R}^n$, then the dual cone to a closed convex cone $C \subseteq \mathbb{R}^n$ is defined by $C^* = \{ x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \} \forall y \in C \}$; $C^*$ is also a closed convex cone. Minkowski's Separation Theorem implies that $(C^*)^* = C$. Now show that $P_{a,m}$ and $Q_{a,n}$ are dual closed convex cones in $H_n(\mathbb{R}^n)$, using (2.3) as the underlying inner product; see [R1, p.38] for more discussion.

**Proposition 6.3** For all $(n, m)$, $P_{a,m}$ and $Q_{a,n}$ are dual cones.

Proof. By definition, $p \in Q_{a,m}^*$ if and only if $[p, q] \geq 0$ for every $q \in Q_{a,n}$. Using Lemma 2.4.6, we see that this means $[p, q] = [p, \sum \alpha_i x_i] \geq 0$ for any set $\{x_i\} \in \mathbb{R}^n$, hence $p(\alpha) = 0$ for all $\alpha \in \mathbb{R}^n$, that is, $p \in P_{a,m}^*$. □
There is a simple geometric criterion for \( x \in C \) to be in the interior of \( C \).

**Proposition 6.4** Suppose \( C \subset \mathbb{R}^2 \) is a closed convex cone with inner product \([\cdot, \cdot]\). Then \( x \in C \) is in the interior of \( C \) if and only if \([x, y] > 0 \) for all non-zero \( y \in C^* \).  

**Proof.** Suppose \( x \in C \) and \([x, y] = 0 \) for some non-zero \( y \in C^* \). Then for every \( \eta > 0 \), \([x - \eta y, y] = \eta [y, y] < 0 \), hence \( x - \eta y \not\in C^* \) and so \( x \) is not in the interior of \( C \). Conversely, if \([x, y] > 0 \) for all non-zero \( y \in C^* \), then \([x, u] \geq \epsilon > 0 \) on the compact set \( \{ u \in C^* : ||u|| = ||x|| = 1 \} \). By linearity, \([x, y] \geq \epsilon ||y|| \) for all \( y \in C^* \). Now suppose \( z \in \mathbb{R}^2 \) and \( ||x - z|| \leq \delta \). Then by \( [x, y] = [x - z, y] + \delta ||y|| \geq [x, y] - ||x - z|| \cdot ||y|| \geq 0 \) for \( y \in C^* \), hence \( z \in C \). Thus \( C \) contains a \( \delta \)-ball around \( x \). □

It is easy to show that the \( d \)-th powers span \( H_d(R^2) \). This is known (see [R1, p.30] for references) and explains why \( Q_{n,d} \) is not interesting for odd \( d \): \( (-\alpha)^d = -\alpha^d \), so every linear combination of \( d \)-th powers is a sum of \( d \)-th powers, and so \( Q_{n,d} = H_d(R^2) \).

**Proposition 6.5** For all \((n,d), H_d(R^2) \) is spanned by \( \{(-\alpha)^d : \alpha \in \mathbb{Q} \} \).

**Proof.** Let \( V \) be the subspace of \( H_d(R^2) \) spanned by \( \{(-\alpha)^d : \alpha \in \mathbb{Q} \} \) and suppose \( q \in V \). Then \( q = \sum \alpha \cdot \alpha^d \) for all \( \alpha \in \mathbb{Q} \). By continuity, \( q(\alpha) = 0 \) for all \( \alpha \in \mathbb{R}^2 \), so \( q = 0 \). Since \( V \) is a subspace of \( H_d(R^2) \), we have \( V = H_d(R^2) \).

We say that \( \{e_1, \ldots, e_{n(d)} \} \subset \mathbb{R}^2 \) is a basic set of nodes for \( H_d(R^2) \) if \( \{\alpha e_j \} \) is a basis for \( H_d(R^2) \). (This is a classical term in numerical analysis.) Proposition 6.5 asserts the existence of \( \mathbb{Q}^2 \) as a basic set of nodes for every \((n,d) \). In fact, Biermann's Theorem (see [R1, p.31]) states that \( \mathcal{F}(n,d) \subset \mathbb{Z}^2 \) is a basic set of nodes for \( H_d(R^2) \). The following integral representation for \( G_d(x) \) can be found in [R1, p.105], where several other proofs are given.

**Proposition 6.6** Suppose \( p \in L^2(R^n) \) and let \( d \mu = \text{Lebesgue measure on } S^{n-1} \subset \mathbb{R}^n \), normalized so that \( \int_{S^{n-1}} f(x)^2 \, d\mu = 1 \). Then,

\[
[p, G_d(x)] = \int_{S^{n-1}} p(x) \, d\mu.
\]

**Proof.** Since (6.7) is bilinear, it suffices to verify on a spanning set for \( H_d(R^n) \); namely, \( \{ (-\alpha)^{d_2} : \alpha \in \mathbb{R}^2 \} \). That is, we must show that for \( p = (-\alpha)^{d_2} \),

\[
\int_{S^{n-1}} (-\alpha_1 x_1 + \ldots + \alpha_n x_n)^{d_2} \, d\mu.
\]

But the left-hand side of (6.8) is \( G_d(x) = \sum (\sum \alpha_i)^2 = |\alpha|^2 \) by Lemma 2.4(a).

On the right-hand side, make any orthogonal change of variables in which \( x_1 = \ldots = x_{d_2} \). Then

\[
\int_{S^{n-1}} p(x) \, d\mu \]

Theorem 6.9 Suppose \( \phi \geq 0 \) and \( f \neq 0 \) in \( P_{n,d} \). Then \( \phi \cdot f \cdot G_d(x) \) is interior to \( Q_{n,d+2} \). In particular, taking \( s = 0 \) and \( f = 1 \), \( G_d(x) \) is interior to \( Q_{n,d} \).

**Proof.** If \( p \in P_{n,d+2} \), then \( p \in P_{n+1,d+2} \), and by Proposition 6.6,

\[
0 \leq \int_{S^{n+1}} p(x) \, d\mu \leq \int_{S^{n+d+2}} p(x) \, d\mu = \int_{S^{n+1}} p(x) \, d\mu = \int_{S^{n+d+2}} p(x) \, d\mu,
\]

with equality if and only if \( p(\phi(x)) < 0 \) almost everywhere on \( S^{n-1} \). By continuity, this implies \( p = 0 \), and \( f \neq 0 \), so \( p = 0 \). But then by Lemma 2.4 and (6.10),

\[
0 \leq \int_{S^{n+1}} p(x) \, d\mu \leq \int_{S^{n+d+2}} p(x) \, d\mu = \int_{S^{n+1}} p(x) \, d\mu = \int_{S^{n+d+2}} p(x) \, d\mu,
\]

with equality if and only if \( p = 0 \). This is true for all \( p \in P_{n,d+2} \), hence (6.10) and Propositions 6.3 and 6.4 imply that \( \phi \cdot f \cdot G_d(x) \) is interior to \( Q_{n,d+2} \). If \( f = 1 \), then this simplifies and \( \phi \cdot G_d(x) \) is interior to \( Q_{n,2} \).

If \( 0 \neq \lambda \) and \( f \in H_d(R^n) \), define \( f_\lambda \in H_d(R^n) \) by \( f_\lambda(x_1, \ldots, x_n) = f(\lambda x_1, \ldots, x_n) \). Then it is easy to see that \( p_\lambda = p(f) \). It follows by Proposition 6.4 that, if \( 0 \neq \lambda \in K \) and \( h \in H_n(K^n) \) is interior to \( Q_{n,d} \), then so is \( h_\lambda \). It is not hard to prove a substantial generalization of this result: If \( T \) is any invertible linear transformation in \( R^n \) and \( p \) is interior to \( Q_{n,d} \), then so is \( p \circ T \). See also [R1, p.40] and [R2, p.1065].

Our last major theorem, which is implicit in [HS], combines with Theorem 6.9 to prove Proposition 2.6. A sketchier proof of this result (applying only to \( G_d(x) \) and \( K = \mathbb{Q} \)) can be found in Ellison [E1].

**Theorem 6.12** Suppose \( h \in \mathbb{K} \) and \( h \) is in the interior of \( Q_{n,d} \). Then there exist \( 0 \leq \lambda \in K \) and \( \alpha \in \mathbb{Q} \) so that

\[
h(x_1, \ldots, x_n) = \sum_{k=1}^{N(n,m)} \lambda_k (\alpha_1 x_1 + \ldots + \alpha_n x_n)^m.
\]
Proof. Let \( \{a_k : 1 \leq k \leq N(n,m) \} \subset \mathbb{Q}^* \) be a basic set of nodes for \( H_m(R^n) \). Since \( \{a_k \} \) spans \( H_m(R^n) \) and is contained in \( H_m(K^n) \), it spans that space as well. Thus, if \( g \in H_m(K^n) \), then there exist unique \( \lambda_k \in K \) so that \( g = \sum \lambda_k a_k \).

Since \( h \) is interior to \( Q_{n,m} \), there exists \( c > 0 \) so that \( h - c \sum \lambda_k a_k \in Q_{n,m} \). Thus there exist \( \beta_k \in \mathbb{R}^n \) such that

\[
(N_{n,m})^m_h - c \sum \lambda_k a_k = \sum \beta_k \in Q_{n,m}.
\]

(6.13)

Now take \( \beta_k \in \mathbb{Q}^* \) close to \( \beta_k \) and express \( (\beta_k)^m - (\beta_k)^m \) in terms of the basis \( \{a_k \} \):

\[
(\beta_k)^m - (\beta_k)^m = \sum \delta_k \epsilon(a_k a_k)^m.
\]

(6.14)

We may choose \( \beta_k \) so close to \( \beta_k \) that \( |\delta_k| < N(n,m)^{-1} \). Sum (6.14) over \( k \):

\[
\sum \delta_k (\beta_k)^m = \sum \sum \delta_k (\beta_k)^m = \sum \sum \delta_k (\beta_k)^m,
\]

where \( \delta_k = \sum \delta_k \) and \( |\delta_k| < e \). Combine (6.13) and (6.15) to get

\[
h = \sum \delta_k (\beta_k)^m.
\]

(6.15)

By construction, \( \epsilon + \delta_k > 0 \); we must check that \( \epsilon + \delta_k \in K \). But \( \beta_k \in \mathbb{Q} \), so \( h - \sum \delta_k (\beta_k)^m \in H_m(K^n) \), and by the remark of the first paragraph, \( \epsilon + \delta_k \in K \). Thus, (6.16) gives \( h \) as a non-negative \( K \)-linear combination of \( 2N(n,m) \) \( m \)-th powers of linear forms, each of which is in \( Q_{n,m} \). It follows from Proposition 6.1 that \( h \) is a non-negative \( K \)-linear combination of \( N(n,m) \) of these \( m \)-th powers.

\( \square \)

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