ON THE NON-MONOTONICITY OF $(|Im(z^n)|)$

Bruce Reznick

University of Illinois at Urbana-Champaign

June 12, 1998

Abstract. We show that the sequence $\big(|Im(z^n)|\big)$ is never monotone when z is a non-real complex number, and $\big(|Re(z^n)|\big)$ is never monotone, unless $z = re^{i\pi\theta}$, where $\theta = \frac{k}{m}$ for some odd m, and r is sufficiently large or sufficiently small. Similar results hold for the length of the projection of z^n onto any line through the origin.

The purpose of this note is to prove a simple, amusing and apparently new application of some familiar number theory. The most interesting special case is the following: if z is a non-real complex number, then the sequence $\big(|Im(z^n)|\big)$ is not monotone; in fact, it increases and decreases infinitely often. This oscillation contrasts with the familiar fact that $\big(|x^n|\big)$ is monotone for real x.

Recall that if ℓ_β denotes the line in $\mathbb{C}$ through the origin with angle β to the real axis, then the distance from a point w to ℓ_β is equal to $|Im(e^{-i\pi\beta}w)|$. This distance is $|Im(w)|$ when $\beta = 0$ and $|Re(w)|$ when $\beta = \frac{1}{2}$.

Theorem. *Let $z = a + bi = re^{i\pi\theta}$, be a non-real complex number ($b \neq 0$), and let $u(n) = |Im(e^{-i\pi\beta}z^n)| = |r^n \sin\pi(n\theta - \beta)|$. Then the sequence $\big(u(n)\big)$ increases and decreases infinitely often, (indeed, each event occurs with positive probability), unless the following conditions are met: $\theta = \frac{k}{m}$, $\gcd(k,m) = 1$, $m\beta \notin \mathbb{Z}$ and r is sufficiently large or sufficiently small. In these cases $\big(u(n)\big)$ is monotone increasing or monotone decreasing.*

Proof. Note first that $u(n) = 0$ if and only if $n\theta - \beta \in \mathbb{Z}$; under our hypotheses, $\theta \notin \mathbb{Z}$, so $(n \pm 1)\theta - \beta \notin \mathbb{Z}$, hence $u(n \pm 1) > 0$.

If $u(n) > 0$, then, since $a = r\cos\pi\theta$ and $b = r\sin\pi\theta$,

$$
\begin{aligned}
\frac{u(n+1)}{u(n)} &= \frac{|r^{n+1}\sin\pi\big((n+1)\theta - \beta\big)|}{|r^n \sin\pi\big(n\theta - \beta\big)} = r\big|\cos\pi\theta + \sin\pi\theta\cot\pi(n\theta - \beta)\big| \\
&= |b| \cdot \big|\tfrac{a}{b} + \cot\pi(n\theta - \beta)\big| = |b| \cdot \big|\cot\pi(n\theta - \beta) - \cot\big(-\pi\theta\big)\big|.
\end{aligned}
\tag{1}
$$

The author was supported in part by NSF grant DMS 95-00507

We conclude that $u(n+1) < u(n)$ (resp. $u(n+1) > u(n)$) if and only if

$$\left|\cot \pi(n\theta - \beta) + \frac{a}{b}\right| < \left|\frac{1}{b}\right|, \quad \left(\text{resp. } \left|\cot \pi(n\theta - \beta) + \frac{a}{b}\right| > \left|\frac{1}{b}\right|\right). \tag{2}$$

Since the function $f(x) = \cot(\pi x)$ is periodic with period 1, we can write these implications in terms of $\{n\theta - \beta\}$, the fractional part of $n\theta - \beta$, and the inverse cotangent, taken with range $(0, \pi)$. Define $r > 0$ and $s < 1$ by

$$r = \frac{1}{\pi}\cot^{-1}\left(-\frac{a}{b} + \left|\frac{1}{b}\right|\right), \qquad s = \frac{1}{\pi}\cot^{-1}\left(-\frac{a}{b} - \left|\frac{1}{b}\right|\right).$$

Since $u(n) > 0$, we have $\{n\theta - \beta\} > 0$, hence $u(n+1) < u(n)$ if $\{n\theta - \beta\} \in (r, s)$ and $u(n+1) > u(n)$ if $\{n\theta - \beta\} \in (0, r) \cup (s, 1)$. As noted earlier, if $u(n) = 0$, then $u(n+1) > u(n)$ and $\{n\theta - \beta\} = 0$. Thus, we have the unconditional criterion:

$$\begin{aligned} u(n+1) < u(n) &\iff \{n\theta - \beta\} \in (r, s), \\ u(n+1) > u(n) &\iff \{n\theta - \beta\} \in [0, r) \cup (s, 1). \end{aligned} \tag{3}$$

Let

$$\mathcal{S}_\beta^-(z) = \{n \ge 0 : u(n+1) < u(n)\}; \qquad \mathcal{S}_\beta^+(z) = \{n \ge 0 : u(n+1) > u(n)\}.$$

We now distinguish three cases.

First, if θ is irrational, then Weyl's Uniform Distribution Theorem [1, p. 390] implies that the sequence $\big(\{n\theta - \beta\}\big)$ is equidistributed in $(0,1)$. Hence,

$$\lim_{N\to\infty} \frac{|\mathcal{S}_\beta^-(z) \cap \{1, \dots, N\}|}{N} = s - r, \qquad \lim_{N\to\infty} \frac{|\mathcal{S}_\beta^+(z) \cap \{1, \dots, N\}|}{N} = 1 - (s - r).$$

This proves the Theorem for irrational θ.

Otherwise, write $\theta = \frac{k}{m}$, where $gcd(k, m) = 1$ and, necessarily, $m \ge 2$. Observe that $\{n\theta - \beta\}$ can attain one of m possible values:

$$\{-\beta\}, \left\{\tfrac{1}{m} - \beta\right\}, \dots, \left\{\tfrac{m-1}{m} - \beta\right\}.$$

Hence by (3), $\mathcal{S}_\beta^-(z)$ and $\mathcal{S}_\beta^+(z)$ will each be a union of arithmetic progressions with step m, and so the limiting densities of $\mathcal{S}_\beta^\pm(z)$ exist. The remaining question is whether these densities are both positive.

If $\beta = \frac{\ell}{m}$, let $n_0 \equiv k^{-1}\ell \ (mod\ m)$. Then the congruence $kn \equiv \ell \ (mod\ m)$ will hold on an arithmetic progression of the form $n = n_0 + tm$ for all integers t, hence $u(n_0 + tm) = 0$ and $u(n_0 + tm \pm 1) > 0$. Thus, each of $\mathcal{S}_\beta^-(z)$ and $\mathcal{S}_\beta^+(z)$ is infinite and has density $\ge \frac{1}{m}$.

Finally, suppose $m\beta \notin \mathbb{Z}$. Then $m\{n\theta-\beta\} \notin \mathbb{Z}$; in particular, $\{n\theta-\beta\} \neq 0, \{-\theta\}$. It follows that $u(n) \neq 0$ for all n, and $\cot\pi(n\theta-\beta) \neq \cot\pi(-\theta) = -\frac{a}{b}$. Let

$$\begin{aligned} 0 < d &= \min_n \left\{ \left| \cot\pi(n\theta-\beta) - \cot(-\pi\theta) \right| \right\}, \\ D &= \max_n \left\{ \left| \cot\pi(n\theta-\beta) - \cot(-\pi\theta) \right| \right\} < \infty. \end{aligned}$$

Then $\big(u(n)\big)$ is monotone decreasing if $D < \left|\frac{1}{b}\right|$ — that is, if $r < \frac{1}{D|\sin\theta|}$, and $\big(u(n)\big)$ is monotone increasing if $d > \left|\frac{1}{b}\right|$ — that is, if $r > \frac{1}{d|\sin\theta|}$. $\square$

Remark 1. The case $\beta = 0$ corresponds to $u(n) = |Im(z^n)|$. Since $0 \cdot m \in \mathbb{Z}$, the exceptional case does not occur, and $|Im(z^n)|$ is never monotone (unless z is real). Note that, $u(0) = 0$ in this case, so if we extend the definition of $\mathcal{S}^{\pm}_{\beta}(z)$ from $\mathbb{N}$ to $\mathbb{Z}$, then $-1 \in \mathcal{S}^{-}_{\beta}(z)$ for all (non-real) z.

Remark 2. The case $\beta = \frac{1}{2}$ corresponds to $u(n) = |Re(z^n)|$. In this case, $m\beta \notin \mathbb{Z}$ if and only if m is odd; that is, if and only if z^n can be real but never imaginary. And $\big(|Re(z^n)|\big)$ can be monotone for non-real z only if $z = re^{i\pi\frac{k}{m}}$ for odd m, where r is sufficiently large or sufficiently small. It is an interesting exercise to show that $dD\sin^2\theta = 1$ in this case, hence the transitional values of r are reciprocal. For example, if ω is a primitive cube root of unity and $z = r\omega$, then the sequence $\big(|Re(z^n)|\big)$, which begins $1, \frac{1}{2}r, \frac{1}{2}r^2, r^3, \frac{1}{2}r^4, \frac{1}{2}r^5, \ldots$, is increasing if $r > 2$ and decreasing if $0 < r < \frac{1}{2}$.

Remark 3. We have thus far ignored the possibility that $u(n) = u(n+1)$. If $\theta = \frac{k}{m}$, then equality either will never occur, or do so on arithmetic progressions of step m. If $\theta \notin \mathbb{Q}$, then $\{n\theta-\beta\}$ will achieve different values for different n and so can equal r and s at most twice. Thus, the most one can hope for is $u(n_1) = u(n_1+1)$ and $u(n_2) = u(n_2+1)$. In certain cases, one may even have $u(n) = u(n+1) = u(n+2)$. For example, if $z = \frac{1}{2}(1+i\sqrt{7})$, then $Im(z) = Im(z^2) = -Im(z^3)$.

Remark 4. If θ is irrational, then there is a natural interpretation for $g(z)$, the density of $\mathcal{S}^{-}_{\beta}(z)$. If $b > 0$, then

$$g(z) = g(a+bi) = s - r = \frac{1}{\pi}\left(\cot^{-1}\left(\frac{a^2+b^2-1}{2b}\right)\right) = \frac{1}{\pi}\arg\left(\frac{z-1}{z+1}\right).$$

It is easily seen that this function is the solution to the Dirichlet Problem in the upper half plane with boundary values $g(x) = 1$ if $|x| < 1$ and $g(x) = 0$ if $|x| > 1$. These conditions may be construed as the probabilities that $|x|^{n+1} < |x|^n$. (A similar interpretation may be made in the lower half plane.) It is not hard to show that the random variable $X_n = \frac{u(n+1)}{u(n)}$ has the following distribution for $t > 0$:

$$Prob(X_n \le t) = \frac{1}{\pi}\left(\cot^{-1}\left(\frac{a^2+b^2-t^2}{2bt}\right)\right),$$

and it is not hard to show that X_n has infinite expectation.

Remark 5. Florek's generalization of Slater's Theorem on gaps (see [3], especially p. 1118) says that θ is irrational and $s - r < \frac{1}{2}$, and if $\{n_1 < n_2 < n_3 < \dots\}$ is the set of n so that $\{n\theta\} \in (r, s)$, then there exist two integers A and B, depending only on θ and $s - r$, so that $n_{k+1} - n_k \in \{A, B, A + B\}$. Since $s - r < \frac{1}{2} \iff |z| > 1$, it follows from (3) that if θ is irrational and $|z| > 1$, then the gaps in successive elements of $\mathcal{S}_\beta^-(z)$ can take one of three values $\{A, B, A + B\}$, where A and B depend on z, but not on β.

Remark 6. Niven's Theorem [2, p.41] says that if $z = a + bi = re^{i\pi\theta}$, where $a, b \in \mathbb{Q}$ and $ab(a^2 - b^2) \neq 0$, then θ is irrational. Thus, the previous remark applies to $z = 19 + 98i = \sqrt{9965}e^{i\pi\theta}$. A calculation shows that $g(19 + 98i) \approx (159.73)^{-1}$. The first decrease for $|Im(19 + 98i)^n|$ occurs for $n = 40$:

$$\begin{aligned}(19 + 98i)^{40} &= 1.7965 \times 10^{79} - 9.1481 \times 10^{79} i;\\ (19 + 98i)^{41} &= 9.3064 \times 10^{81} + 2.2440 \times 10^{79} i.\end{aligned}$$

The first decrease for $|Re(19 + 98i)^n|$ occurs for $n = 483$:

$$\begin{aligned}(19 + 98i)^{483} &= 4.2176 \times 10^{965} + 7.7468 \times 10^{964} i;\\ (19 + 98i)^{484} &= 4.2157 \times 10^{965} + 4.2804 \times 10^{967} i.\end{aligned}$$

In fact,

$$\begin{aligned}\mathcal{S}_0^-(19 + 98i) &= \{40,\ 81,\ 122,\ 163,\ 1172,\ 1213,\ 1254,\ 1295,\ 1336,\ \dots\},\\ \mathcal{S}_{\frac{1}{2}}^-(19 + 98i) &= \{483,\ 524,\ 565,\ 606,\ 647,\ 688,\ 729,\ 770,\ 811,\ 1820,\ \dots\};\end{aligned}$$

so $A = 41$ and $B = 1009$.

A more careful analysis shows that the gaps in $\mathcal{S}_\beta^-(19 + 98i)$ always consist of blocks of seven or eight "41"s separated either by "1009" or "1050". One can precisely compute the probabilities of the various gaps: "41" occurs about 87.7% of the time, "1009" occurs about 12.1% of the time, and "1050" occurs about once every 686.8 gaps.

Remark 7. Clearly, the underlying density g is continuous, but there seems to be a very sensitive dependence of $\mathcal{S}_\beta^\pm(z)$ on z. We have $g(19 + 99i) \approx (161.26)^{-1}$, which is pretty close to $g(19 + 98i)$. However, the first declines for $|Im(19 + 99i)^n|$ and $|Re(19 + 99i)^n|$ occur at $n = 115$ and $n = 57$, instead of $n = 40$ and $n = 483$, as above, and $(A, B) = (116, 323)$, instead of $(41, 1009)$. We hope to discuss the properties of these gaps in a future paper.

References

[1] Hardy G. H. and E. M. Wright, *An introduction to the theory of numbers*, Fifth edition, Oxford at the Clarendon Press, 1979.

[2] Niven, I., *Irrational numbers*, Carus Mathematical Monograph No. 11, Mathematical Association of America, 1956.

[3] Slater, N. B., *Gaps and steps for the sequence $n\theta$ mod 1*, Proc. Camb. Phil. Soc. **63** (1967), 1115–1123.

1409 W. Green Street Urbana, IL 61801
E-mail address: `reznick@math.uiuc.edu`