Research consists of question-asking, problem-solving, knowledge-finding and checking, checking, checking.

One **Introduction** to Mathematical Research

Bruce Reznick
University of Illinois, Urbana-Champaign

Overly general introduction

Few prospects are as daunting for a serious undergraduate math major as that of doing research. Mathematical research is harder than homework. You can't look for the answers to the chapter in the back of the book; you don't know which chapter, and you often don't even know *which* book, and it wouldn't be there anyway. You're not even sure what would constitute an acceptable answer.

But in some ways, mathematical research is easier than homework. The time limit is flexible, and since you have chosen the question yourself, you are more likely to find it interesting. And, if you invent your own problem, but solve a different one, you can always say that it was what you were trying to do in the first place!

The best research involves deep new ideas and you can't find all the secrets in any article. Nevertheless, there are a few tips that will put you in a more favorable position to start your research career.

The most important thing you need to do is to adopt an active attitude towards the mathematics you learn. When you see something new, don't ask "Will this be on the test?", ask "How can I make this work for me?" Be bold. To paraphrase the 1960s revolutionary Franz Fanon: "Knowledge cannot be given, it must be taken."

Learn as much mathematics as you can, in as many different areas as you can, including areas in which mathematics might be applied. Take hard classes. Go to as many mathematical lectures as you can, even if you won't understand it all. (Big secret revealed: most professors go to lectures and don't understand it all.)

In my experience, research consists of question-asking, problem-solving, knowledge-finding and checking, checking, checking. There are many excellent books on problem-solving; start with *How to Solve It* by George Pólya. Knowledge-finding is extremely dependent on your location—always get to know your librarians. Asking big questions requires inspiration. The rest of this article is devoted to the tabletop craft of asking little questions.

Arthur C. Clarke wrote that any smoothly running advanced technology is indistinguishable from magic. Great mathematics is magical. You can't learn how to make a metaphorical elephant disappear in plain view by reading about it. Don't be scared off by thinking that you don't know enough mathematics to do research—nobody ever knows enough mathematics. But you can always seem brilliant if you take an idea or a technique in one area and apply it elsewhere.

When I teach my undergraduate honors seminar, Introduction to Mathematical Research, I start with a single exercise. Repeat as often as desired.

- Present your favorite theorem and proof or problem and solution.
- Change your favorite theorem in some way, and prove or solve it again.
- Change your favorite theorem in another way, so that you no longer know how to prove or solve it.

You proved a theorem, now what?

Okay, so let's say things have gone well and you think you've proved something. The first thing to do is to enjoy the moment, whether or not you're right. After a decent interval, the next thing to do is to put away your notes and try to prove it again from scratch. (My undergraduate proof of Fermat's Last Theorem was fun while it lasted.) Make sure you understand all the definitions: it's very embarrassing to discover that you haven't proved what you thought you proved. (Turns out I got the Binomial Theorem wrong.) If you find yourself wanting to skip a portion of the argument as you check your work, be very careful: this is where mistakes usually hide.

Suppose everything is OK. Congratulations, you've proved a theorem! What you want to do now is see how you can water it and make it grow. Magic can be making the elephant disappear. That requires elaborate equipment and the work of dozens of technicians. But there's also magic done at a table with no equipment but a deck of cards, dextrous fingers and a few good ideas.

Return to your theorem. Try to *push* the conclusions. Have you made full use of your argument? Try to *pull* the hypotheses. Do you need every assumption? Does your argument apply to a more general class of object? Have you proved more than you thought? Isolate the mechanism that makes your proof work. Is it familiar? Are you really proving something else and applying it to your case? If so, maybe that's what you should study next. Most mathematicians like saying "every X satisfies condition A or condition B," so you should make as many "if and only if" propositions as possible. Does the converse of your theorem hold? What if you play with the hypotheses?

A theorem usually isn't very interesting unless it has some applications. Work out the details in at least one specific case. If you find that your theorem doesn't answer all the questions in this case, you know your next project.

A theorem has a natural rhythm. Does your proof remind you in any way of another proof you know, possibly in a different area? Can you show that two apparently different results are special instances of a more general theorem? (This is the concrete approach to abstraction.)

It always helps to try to explain your work to other people. This might be one of the most important tips here. Not only are listeners likely to have useful suggestions, but the process of preparing an explanation is extremely valuable in clarifying your own thoughts. You may even find a collaborator. Most mathematicians these days work on joint projects. Imagine giving a presentation of your result to a roomful of your classmates and teachers. It's possible that one of them asks questions in a characteristic manner. See if you can answer the question in your mind.

Consider the *Seinfeld Principle*. Take whatever you are doing and do the *opposite*. Switch the foreground and the background. For example, instead of finding the roots of a polynomial based on its coefficients (very hard), try to find the coefficients of a polynomial based on its roots (the door to symmetric polynomials). Perform even more elaborate permutations on the roles of these objects. There's also the *Oprah Principle*. Visualize a solution to your problem. What other properties would your solution have? This often makes the original problem easier. Finally, try the *Postmodern or Mad Magazine Principle*. Everything is in play. Change, one by one, each of your hypotheses and assumptions and see what you can prove. (If you change the rules of implication itself, you are a mathematical logician!)

Check for the presence or absence of certain familiar characteristics. For example: look for linearity in your problem. If your problem isn't linear, measure the deviation from linearity. The same thing applies to symmetry and commutativity. You can generalize by turning every number that appears in your problem into a parameter. (Remember that 0 and 1 are numbers.) Contrariwise, if some restriction on the parameters leads to an interesting outcome, see what happens if you enforce this as a hypothesis. A productive technique is to generalize in one direction and then specialize in another one.

Finally, some bad news. These techniques won't work all the time. Some of the new directions will be boring. You'll ask questions for which you can't find an interesting answer. You'll find problems that consume your interest for weeks at a time, and either not solve them, or be unable to persuade anyone else to care. Mathematical research journals are filled with hard-won and completely forgotten theorems. It's very hard in advance to predict other people's reaction to your work. One good reason for answering someone else's question is that it guarantees you a nonempty audience. Finally, don't throw out your old notes. There are a couple of questions I've been thinking about for almost thirty years. It's not plagiarism to steal from your old unpublished work.

The inevitable Fibonacci example

Let me illustrate the techniques of the last section with an example taken from Fibonacci numbers. (Caveat: Fibonacci numbers are the lab rats of the problem-solving literature. They are not particularly important, but they are easily studied because many of their properties are very close to the surface.)

Recall that the sequence $\{F_n\}, n \ge 0$, is defined by:

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},\ n \ge 2.$$

One familiar property satisfied by the Fibonacci sequence is the *addition* formula:

(1) $\quad F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$, for integers $m, n \ge 0$.

Proof. Fix m and consider the sequence $\{x_n\}$, defined by $x_n := F_{m+n+1}$. Then $\{x_n\}$ satisfies the Fibonacci recurrence $x_n = x_{n-1} + x_{n-2}$. So does $\{y_n\}$, defined by $y_n := F_{m+1} F_{n+1} + F_m F_n$. You show that $x_0 = y_0$ and $x_1 = y_1$ and then use the recurrence to prove (1) by induction.

The underlying mechanism of this proof is that the vector space V of sequences $\{a_n\}$ that satisfy the Fibonacci recurrence is two-dimensional. In fact, if you write out the terms of the sequence with $a_0 = r$ and $a_1 = s$:

$$(a_0, a_1, a_2, a_3, a_4, a_5, \cdots) = (r, s, r+s, r+2s, 2r+3s, 3r+5s, \cdots)$$

you can see that the general element of V can be immediately written as a linear combination of the two sequences

$$(1, 0, 1, 1, 2, 3, \cdots) \text{ and } (0, 1, 1, 2, 3, 5, \cdots),$$

which thereby form a basis of V. The first sequence in the basis is $\{F_{n+1} - F_n\}$ (we can't say F_{n-1} because we haven't defined F_{-1}, yet); the second sequence in the basis is $\{F_n\}$. Thus, any sequence $\{a_n\}$ in V has the general form

$$a_n = a_0(F_{n+1} - F_n) + a_1 F_n = a_0 F_{n+1} + (a_1 - a_0) F_n.$$

Continued on page 38

Continued from page 17

Taking $a_n = F_{n+m+1}$, we see that (1) follows almost immediately.

Another thing to observe is that, if $\lambda^2 = \lambda+1$, then the sequence $\{\lambda^n\}$ will lie in V. Thus, another basis for V is $\{\{\phi^n\}, \{\bar{\phi}^n\}\}$, where

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

By representing $\{F_n\}$ in terms of this basis, we get the familiar closed form

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \bar{\phi}^n\right).$$

(Plugging this into (1) gives a formal identity that holds for all m, n.)

The underlying proof mechanism immediately suggests that the vector space of sequences $\{a_n\}$ that satisfy any constant-coefficient linear recurrence

$$a_n = c_1 a_{n-1} + \cdots + c_r a_{n-r}, \qquad n \geq r$$

will be at most r-dimensional, and a_n for $n \geq r$ can be expressed in a fixed way in terms of the initial conditions $a_0, a_1, \ldots, a_{r-1}$. Any shifted sequence a_{m+n} for fixed m will also be in this vector space, and so can be expressed as a linear combination of the basis. Thus, addition formulas are inevitable, and there's nothing so amazing about (1).

Here are some additional small-step directions you could take in playing with (1).

1. Generalize (1) and find "symmetric" formulas for $m_1 + m_2 + m_3$, etc.

2. Specialize (1) to find formulas for F_{2n} and F_{2n+1}.

3. Combine 1. and 2. to find formulas for F_{3n}, and more generally, F_{kn}. One of the most common uses for (1) is as a lemma in proving that F_{kn} is always a multiple of F_n for $k, n \in N$.

4. Determine the addition formula for your favorite recurrence sequence with your favorite initial conditions.

5. Prove (1) using generating functions. That is, define

$$\Phi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{m+n+1} x^m y^n,$$

and explore the properties of Φ, with the goal of expressing it as a rational function of x and y. It will be helpful to recall that

$$\sum_{n=0}^{\infty} F_n t^n = \frac{t}{1-t-t^2},$$

from which the closed form can also be derived.

6. Seinfeld: find all sequences $\{a_n\}$ that satisfy (1). Hint: start with putting $m = 0, 1$ into (1), so that

$$a_{n+1} = a_1 a_{n+1} + a_0 a_n; \qquad a_{n+2} = a_2 a_{n+1} + a_1 a_n.$$

The first equation should be studied under the conditions: $(a_0, a_1) = (0, 1)$ and $(a_0, a_1) \neq (0,1)$. The second equation says that $\{a_n\}$ satisfies a second-order constant coefficient recurrence. You can also use this to guess that $a_n = \alpha\lambda^n + \beta\mu^n$, and use (1) to get conditions on $\alpha, \beta, \lambda, \mu$, which in turn will help nail down the recurrence.

7. We noted earlier that F_{-1} is undefined; but any "reasonable" definition would have $F_{-1} + F_0 = F_1$, so $F_{-1} = 1$. In this way, you can generalize the Fibonacci sequence to negative subscripts, find a nice pattern and compare it with the closed form. You can also generalize the Fibonacci sequence to F_t, $t \in R$, by defining an arbitrary function $F_t = g(t)$ on $[0,2)$ and then using the recurrence $g(t) = g(t-1) + g(t-2)$ to extend it to R. Can you do this in an "interesting way"? (I can't.) You can even define a Fibonacci function on C by combining the recurrence with a function on the strip $\{z: Re(z) \in [0,2)\}$.

8. One reason that you can't easily define $F_{1/2}$ is that $\bar{\phi} < 0$, so $\bar{\phi}^{1/2}$ in the closed form would not be real. More generally, show that there is no sequence a_n of real numbers, where $2n \in Z$ so that $a_{m+n+1} = a_{m+1} a_{n+1} + a_m a_n$. (Hints: mix up integers and half-integers; a sum of real squares can't be negative.)

9. When you hear "addition formula," you probably think of the formulas

$$\cos(t+u) = \cos t \cos u - \sin t \sin u;$$
$$\sin(t+u) = \sin t \cos u + \cos t \sin u.$$

Use the mechanism of the first proof to derive these, using as your two-dimensional vector space the solutions to the differential equation $y'' + y = 0$. Now generalize to other second-order differential equations.

Exercise • Rewrite this section using your favorite theorem or problem.

Final words

Human creativity is inevitably personal, and different techniques work well for different practitioners. I hope this article may prove of some value to its student readers as they start their research. I also hope it will irritate my research colleagues so much that they feel obligated to write their own articles about their own styles and techniques.

I thank all my students for their suggestions and feedback. I also want to thank Dan Grayson, Abbey Rechner and Robin Sahner, and the editors for their comments.

An expanded version of this article can be found at `www.math.uiuc.edu/~resnick/mhori.pdf`. ■