Moment Problems and PSD symmetric functions

Bruce Reznick

In the early 1980s, I wrote [2], a paper about inequalities satisfied by quotients of products of power-sums. By early March 2017, this paper had amassed a total of zero citations on MathSciNet, so I felt I could justifiably talk about its contents at this Oberwolfach Workshop. (The paper is freely available from Project Euclid: https://projecteuclid.org/euclid.pjm/1102723674.)

For a positive integer r and $x \in \mathbb{R}^n$, define the r-th power sum:

$$M_r(x) = M_{r,n}(x) = \sum_{j=1}^{n} x_j^r.$$

There are some standard inequalities for products of power sums, such as the Hölder and Jensen inequalities. Using elementary methods, [2] considers cases which are not covered by these; in particular, by determining the maximum and minimum (as a function of n) of these three homogeneous symmetric rational functions $f_i = f_{i,n}$:

$$f_1(x) = \frac{M_1(x)M_3(x)}{M_4(x)}; \qquad f_2(x) = \frac{M_1(x)M_3(x)}{M_2^2(x)}; \qquad f_3(x) = \frac{M_1^3(x)M_3(x)}{M_2^3(x)}.$$

Each of these achieves its extrema at points $x \in \mathbb{R}^n$ with at most two different coordinates: $x = (1, \dots, 1, t, \dots, t)$, with $n - k$ 1's and k t's.

At the time, Choi, Lam and I were working on writing psd symmetric forms as a sum of squares (see e.g. [1]) and these seemed like a potentially interesting source of examples, although it didn't work out back then. In view of the title of this workshop, I wanted to revisit the subject, especially since $\inf f_1$ can be more easily analyzed via the Moment Problem.

Here are the relevant answers:

$$\max f_1(x) = n, \quad \min f_1(x) = -\alpha_n n, \quad \text{where} \quad \alpha_n < \frac{1}{8},\ \alpha_n \to \frac{1}{8};$$

$$\max f_2(x) = \frac{3\sqrt{3}}{16} n^{1/2} + \frac{5}{8} + \mathcal{O}(n^{-1/2}),$$

$$\min f_2(x) = -\frac{3\sqrt{3}}{16} n^{1/2} + \frac{5}{8} + \mathcal{O}(n^{-1/2});$$

$$\max f_3(x) = \frac{(\sqrt{n-1}+1)^4}{8\sqrt{n-1}}, \qquad \min f_3(x) = -\frac{(\sqrt{n-1}-1)^4}{8\sqrt{n-1}}.$$

In the case of f_1, the convergence of (α_n) is not monotone:

$$\alpha_{15} \approx .124999536, \qquad \alpha_{16} \approx .124905705.$$

The reason for all of this, in some sense, is that the global minimum of $\frac{M_1 M_3}{n M_4}$ occurs when $\frac{n-k}{k} = 7 + 4\sqrt{3} \approx 13.93$, which is of course irrational. (The maximum is easily derived from the usual inequalities.) As $n \to \infty$, there exist k_n so that $\frac{n-k_n}{k_n} \to 7 + 4\sqrt{3}$, and this implies that $\inf \frac{M_1(x)M_3(x)}{nM_4(x)} = -\frac{1}{8}$.

From the moment theory point of view, note that

$$\frac{M_1(x)M_3(x)}{nM_4(x)} = \frac{(\int_{-\infty}^{\infty} t\ d\mu)(\int_{-\infty}^{\infty} t^3\ d\mu)}{(\int_{-\infty}^{\infty}\ d\mu)(\int_{-\infty}^{\infty} t^4\ d\mu)},$$

where μ is the measure with unit point masses at $t = x_1, \dots, x_n$. As is well known, a special case of the Hamburger moment problem says that, on writing

$$a_j = \int_{-\infty}^{\infty} t^j\ d\mu,$$

for any non-negative measure, the resulting Hankel matrix

$$H := \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

is positive semidefinite, and for any choice of a_j's making H psd, there exists a nonnegative measure satisfying

$$a_j = \int_{-\infty}^{\infty} t^j\ d\mu, \quad 0 \le j \le 3; \qquad a_4 \ge \int_{-\infty}^{\infty} t^4\ d\mu.$$

The computation of the lower bound becomes a nice undergraduate optimization problem: minimize $\frac{a_1 a_3}{a_0 a_4}$ given that $\det H \ge 0$. One finds that the extremal measure gives $-\frac{1}{8}$ for this minimum, and the measure has two atoms whose masses have ratio of $7 + 4\sqrt{3}$, at positions with a ratio of $-(2 + \sqrt{3})$. (See [2, p.462] for details.)

One application is that the symmetric quartic form $8M_1M_3 + nM_4$ is positive definite for all n. It's obviously then sos for $n = 2, 3$. What about larger n? Thirty years ago, this seemed beyond the reach of hand-computation. Here is a hand-computed sos representation for this symmetric quartic for all n, which I should have been able to find in 1981!

$$n^3(8M_1M_3 + nM_4) = \sum_{j=1}^{n}(-8M_1^2 + 4nx_jM_1 + n^2x_j^2)^2.$$

This representation can also be used to show that $8M_1M_3 + nM_4$ is a strictly positive definite n-ary quartic for all n.

Another new result was announced (but not proved) in the talk: if $a, b \in \mathbb{N}$ and a is odd, then

$$\frac{M_1^a M_2^{2b} M_3^a}{n^{a+b} M_4^{a+b}} = -\frac{1}{2^a} \cdot \frac{a^a(a+2b)^{a+2b}}{(2a+2b)^{2a+2b}};$$

note that $a = 1, b = 0$ recovers the bound $-\frac{1}{8}$.

Neither f_2 nor f_3 seems susceptible to the moment method, because the largest index r occurs in the numerator, not the denominator. But calculus shows that the extreme values of f_2 and f_3 must occur at a point with at most two different coordinates, and cubic equations arise (see [2] for details.) In particular, the extreme values for f_2 occur when $k = 1$ and $t = 1 + 2\sqrt{n}\cos\theta$, where $\cos 3\theta = n^{-1/2}$. There are three such values of $\cos\theta$: the one with $\theta \approx \frac{\pi}{6}$ gives the maximum, and the one with $\theta \approx \frac{5\pi}{6}$ gives the minimum. For f_3, the cubic has a linear factor and $t = \pm\sqrt{n-1}$ at the extrema.

I can report no progress in computing λ_n so that $M_2^2 + \lambda_n M_1 M_3$ is sos. On the other hand, a new result is that $\inf \frac{M_1(x)M_5(x)}{nM_6(x)}$ seems to be $-\frac{1}{4}$.

As I said at the Workshop, the standard American cultural approach to reviving [2] would involve a "reboot" with perhaps a more popular cast. I'm in talks to have Arnold Schwarzenegger make a special guest appearance as the Nullstellensatz.

References

[1] M. D. Choi, T. Y. Lam, B. Reznick, *Even symmetric sextics*, Math. Z. **195** (1987), 559–580

[2] B. Reznick, *Some inequalities for products of power sums*, Pacific. J. Math **104** (1983), 443–463.