Positive rational solutions to $x^y = y^{mx}$: a number-theoretic excursion

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1 Prologue

Late in the last millennium, the second author ran a seminar course for undergraduates which was intended to introduce them to problem-solving and question-asking in the context of mathematical research. He led them through the classic "difficult" equation

$$x^y = y^x, \qquad x, y > 0, \tag{1}$$

whose solution is much easier than one would think at first glance. Solutions were sought, first over $\mathbb{R}$, then over $\mathbb{Z}$ and finally over $\mathbb{Q}$. In the context of this seminar, it was natural to consider the variant equation

$$x^y = y^{2x}, \qquad x, y > 0, \tag{2}$$

which does not seem to have appeared in the literature. It turns out that there are solutions to (2) which do not fit the well-known parametric pattern of (1); c.f. (5) below. For example,

$$x = \left(\frac{4}{5}\right)^{128}, \quad y = \left(\frac{4}{5}\right)^{125} \tag{3}$$

is a solution to (2). This preposterous fact is trivial to verify: simply substitute (3) into (2), take logs and transpose:

$$\frac{2x}{y} = 2\left(\frac{4}{5}\right)^3 = \frac{128}{125} = \frac{\log x}{\log y}.$$

As we shall see, the ultimate explanation for this identity is that $2 \cdot 4^3 = 5^3 + 3$. Upon discovering (3), the second author realized he was in over his head and contacted the first author. This paper is the result.

2 $x^y = y^x$

First, we review the familiar, but beautiful solution to (1), reserving historical references to the last paragraph of the section. We acknowledge the solutions $x = y$ and now let $y = tx$, $t \neq 1$, so that

$$\left(x^t\right)^x = (tx)^x \Longleftrightarrow x^t = tx \Longleftrightarrow x = t^{\frac{1}{t-1}}. \tag{4}$$

The positive real solutions to (1) are thus

$$(x, y) = (u, u) \qquad \text{and} \qquad (x, y) = (t^{\frac{1}{t-1}}, t^{\frac{t}{t-1}}). \tag{5}$$

(We might equally well have set $y = x^r$ in (1), and drawn essentially the same conclusion.) Since (1) implies that $x^{1/x} = y^{1/y}$ and since $f(u) = u^{1/u}$ increases on $(1, e)$ and decreases on (e, ∞), for each $x \in (1, e)$, there is exactly one $y \in (e, \infty)$ so that (1) holds. There are at most two solutions to (1), one of which is $x = y$. In particular, the only integer solution to (1) with $x \neq y$ is $2^4 = 4^2$.

Euler already noted that if $t = 1 + \frac{1}{n}$ for integral n, then (4) gives a rational solution to (1); namely,

$$x_n = \left(1 + \frac{1}{n}\right)^n, \quad y_n = \left(1 + \frac{1}{n}\right)^{n+1}. \tag{6}$$

(Observe that, as $n \to \infty$, we have $t \to 1$ and (x_n) and (y_n) increase and decrease monotonically to e, as is familiar from calculus.)

To show that these are the only rational solutions, we need an elementary lemma, whose proof, relying upon the Fundamental Theorem of Arithmetic, we omit:

Lemma 1. *Suppose a, b, m, n are integers, with $\gcd(a, b) = \gcd(m, n) = 1$ and $b, n \neq 0$. Then $\left(\frac{m}{n}\right)^{a/b}$ is rational if and only if m and n are $|b|$-th powers of integers.*

Let us now proceed to find all rational solutions to (1) with $x \neq y$. By symmetry, we may assume that $y > x$, so $t > 1$. If x and y are rational, then so is $t = y/x$. Write t in lowest terms as

$$t = \frac{p}{q} := \frac{q + d}{q},$$

$(d, q > 0)$, so that $\frac{1}{t-1} = \frac{q}{d}$ and $\frac{t}{t-1} = \frac{q+d}{d}$. With this substitution, we have

$$x = \left(\frac{q + d}{q}\right)^{q/d}, \qquad y = \left(\frac{q + d}{q}\right)^{q/d+1}.$$

Since $\gcd(d, q) = \gcd(q, q + d) = \gcd(d, q + d) = 1$, by Lemma 1, the integers q and $q + d$ must both be $|d| = d$-th powers. This causes no problem when $d = 1$, of course, and, setting $q = n$, we recover (6). Suppose $d > 1$, and write $q = a^d$, $q + d = b^d$ for positive integers $a < b$, so that $b^d - a^d = d$. Observe that, for positive integers a and b,

$$b^d - a^d \geq (a+1)^d - a^d \geq 1 + da \geq 1 + d > d$$

and so there are no solutions with $d > 1$. Finally, we remark that if $n = -r$ is allowed to be negative in (6), then $(x_{-r}, y_{-r}) = (y_{r-1}, x_{r-1})$:

$$\left(1 - \frac{1}{r}\right)^{-r} = \left(1 + \frac{1}{r-1}\right)^{r}, \quad \left(1 - \frac{1}{r}\right)^{-r+1} = \left(1 + \frac{1}{r-1}\right)^{r-1}.$$

Thus, by removing the restriction that n be positive, we can eliminate the constraint $y > x$ in (6).

A historical discussion of $x^y = y^x$ occupies a paragraph in Chapter XXIII of Dickson [3, p.687]. The first reference to (1) was in a letter from D. Bernoulli to C. Goldbach, dated June 29, 1728. Bernoulli asserts, without proof, (see [2, p.263]) that this equation has only one solution in positive integers, and infinitely many rational solutions. The first person to write about (1) in detail was Euler (see [4, pp.340–341]). Euler made the substitution $y = tx$ and solved the equation over $\mathbb{R}_+$ and $\mathbb{Z}_+$, and presented the rational solutions (6), without claiming that they were the only ones.

Dickson mentions other writers who covered the same ground, and adds that "*A. Flechsenhaar and R. Schimmack discussed the rational solutions". The cited papers appeared in 1911 and 1912 in the expository journal *Unterrichtsblätter für Mathematik und Naturwissenschaften.* Dickson writes: "The reports in Chapters XI–XXVI have been checked by the original papers in case they are to be found in Chicago" ([3, p.xxi]); the asterisk means that the paper was "not available for report" ([3, p.xxii]). We were able to examine [6] at the magnificent Mathematics Library of the University of Illinois at Urbana-Champaign and can report that Flechsenhaar appears to deserve credit as the first author to solve (1) over positive rationals. In 1967, S. Hurwitz [9] gave the first readily accessible analysis of the rational solutions; subsequent work on this equation, including generalizations to algebraic solutions, can be found in e.g. [8], [14], [17], [18], [21] and [22].

3 $x^y = y^{mx}$

Let us now consider the generalization of Euler's equation to

$$x^y = y^{mx} \tag{7}$$

where $m > 1$ is a fixed positive integer. We will again restrict our attention to positive solutions (x, y). If $x = 1$ or $y = 1$, then necessarily $(x, y) = (1, 1)$.

Supposing that $x, y \neq 1$ and taking logarithms in (7), we find that

$$\frac{m \log y}{y} = \frac{\log x}{x},$$

so $x \neq y$. Write $y = x^r$, $x \neq 1$, $r \neq 1$. Then $x^{r-1} = mr$. Therefore, the positive real solutions to (7) are given by $x = y = 1$ and

$$x = (mr)^{\frac{1}{r-1}} \text{ and } y = (mr)^{\frac{r}{r-1}}.$$

We now restrict our attention to positive rational solutions. Since $y/x = mr$, it follows that $r > 0$, and, further, since x and y are rational, we have $r \in \mathbb{Q}$. Let us write $r = \frac{a}{b}$ where $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ and set $k = |a - b| \in \mathbb{Z}_+$. To have $x \in \mathbb{Q}$, we require

$$x = \left(\frac{ma}{b}\right)^{\frac{b}{a-b}} \in \mathbb{Q}.$$

Suppose that $\gcd(b, m) = d$, and write $b = db'$ and $m = dm'$. Then $\gcd(am', b') = 1$ and so we need am' and b' to be k-th powers of integers. If

$$am' = u^k \text{ and } b' = v^k,$$

it follows that

$$\left|u^k - mv^k\right| = |am' - mb'| = \left|\frac{am}{d} - \frac{mb}{d}\right| = \frac{m}{d} \cdot k = m'k. \tag{8}$$

For a given positive integer m, we will classify the set $S(m)$ of positive rational solutions to (7) as follows: write

$$S(m) = \bigcup_{k=0}^{\infty} S_k(m),$$

where $S_k(m)$ represents the set of solutions (x, y) corresponding to equation (8), for m' ranging over all positive integral divisors of m. Here, $S_0(m)$ denotes the solutions with $x = y$ (i.e. just the set $(x, y) = (1, 1)$ for $m > 1$).

The remainder of this paper will be devoted to analyzing $S(m)$. As a consequence, we will show how this set may be completely characterized for any given m. In case $m = 2$ or 3 (where all features of interest for general m may in fact be observed), we have the following results:

Theorem 2. *If x and y are positive rational numbers for which $x^y = y^{2x}$ then either*

(a) $x = \left(2 + \frac{2}{n}\right)^n$ and $y = \left(2 + \frac{2}{n}\right)^{1+n}$, and $n \in \mathbb{Z}$, $n \neq 0, -1$;

or

(b) $(x, y) = (1, 1), (2, 16)$ or $\left(\left(\frac{4}{5}\right)^{128}, \left(\frac{4}{5}\right)^{125}\right)$.

and

Theorem 3. *If x and y are positive rational numbers for which $x^y = y^{3x}$ then either*

(a) $x = \left(3+\frac{3}{n}\right)^n$ and $y = \left(3+\frac{3}{n}\right)^{1+n}$, and $n \in \mathbb{Z}$, $n \neq 0, -1$;

or

(b) $x = \left(\frac{3w_n}{v_n}\right)^{v_n^2}$ and $y = \left(\frac{3w_n}{v_n}\right)^{3w_n^2}$, $0 \leq n \in \mathbb{Z}$;

or

(c) $x = \left(\frac{w_n}{v_n}\right)^{3w_n^2}$ and $y = \left(\frac{w_n}{v_n}\right)^{v_n^2}$, $0 \leq n \in \mathbb{Z}$.

In (b) and (c), v_n and w_n are the integers defined by

$$v_n + w_n\sqrt{3} = (1+\sqrt{3})(2+\sqrt{3})^n.$$

Our proofs of Theorems 2 and 3 (perhaps rather surprisingly) involve techniques from transcendental number theory and Diophantine approximation. Moreover, they provide us with an opportunity to illustrate a fairly diverse grab-bag of methods for solving Diophantine problems.

4 The cases $k = 1$ and $k = 2$

The set $S_1(m)$ is easily computed. Indeed, we immediately find that $k = 1$ implies that either

$$x = \left(m + \frac{m}{n}\right)^n, \quad y = \left(m + \frac{m}{n}\right)^{1+n}$$

or

$$x = \left(m - \frac{m}{n}\right)^{-n}, \quad y = \left(m - \frac{m}{n}\right)^{1-n}$$

for n a positive integer (with $n \geq 2$ in the latter case). We note that these solutions correspond to the parametrized family (6) of solutions to Euler's original equation and provide us with part (a) of Theorems 2 and 3.

If $k \geq 2$, the situation becomes more interesting, though the set $S_2(m)$ is also not too difficult to describe: it is either empty or infinite. The following lemmata provide sufficient conditions for the former to occur. As usual, for $x \in \mathbb{N}$, let $\nu_2(x)$ be the largest integer such that $2^{\nu_2(x)}$ divides x.

Lemma 4. *If $\nu_2(m)$ is odd and k is even, then $S_k(m)$ is empty.*

Proof. We have $\gcd(a, b) = 1$, and since $k = |a - b|$ is even, a and b must both be odd. It follows that $d = \gcd(b, m)$ is odd as well. Write, as before, $b = db'$, $m = dm'$, $am' = u^k$, $b' = v^k$. Then $\nu_2(m) = \nu_2(d) + \nu_2(m')$ is odd, hence so is $\nu_2(am') = \nu_2(u^k) = k\nu_2(u)$, which contradicts k being even. □

Lemma 5. *If $m = 2^{\nu_2(m)} \cdot m_1$ is a positive integer for which $m_1 \equiv 1 \pmod 4$, then $S_2(m)$ is empty.*

Proof. If $\nu_2(m)$ is odd, this is a special case of Lemma 4. Since $k = 2$ is even, we may conclude as before that a and b are both odd. Suppose $\nu_2(m)$ is even, say $\nu_2(m) = 2t$. It follows from (8) that

$$u^2 - 2^{2t} m_1 v^2 = \pm 2^{2t+1} \cdot \tfrac{m_1}{d}$$

where u and v are coprime and d divides m. Since $d \mid b$, it follows that d is odd, and so $d \mid m_1$. We thus have that $2^t \mid u$, say $u = 2^t u_1$, whereby

$$u_1^2 - m_1 v^2 = \pm 2 \frac{m_1}{d}. \tag{9}$$

Since the right hand side of this equation is even, u_1 and v have the same parity, and, since u and v are coprime, are both necessarily odd. Since $m_1 \equiv 1 \pmod 4$, this implies that the left hand side of (9) is divisible by 4. Since the right hand side of this equation is congruent to 2 modulo 4, this yields the desired contradiction. □

It is a well-known fact that if m is a positive nonsquare integer and c is a nonzero integer, then a single solution in positive integers x and y to the equation $x^2 - my^2 = c$ implies the existence of infinitely many such solutions. In fact, one can find a finite collection of pairs of positive integers, say

$$(x_1, y_1),\ (x_2, y_2), \ldots, (x_r, y_r),$$

with $x_i^2 - my_i^2 = c$ for $i = 1, 2, \ldots, r$, such that *every* solution in positive integers (x, y) to the equation $x^2 - my^2 = c$ satisfies

$$x + y\sqrt{m} = \left(x_i + y_i\sqrt{m}\right) \cdot \left(u_1 + v_1\sqrt{m}\right)^k \tag{10}$$

where k is a nonnegative integer, $i \in \{1, 2, \ldots, r\}$ and (u_1, v_1) is the smallest positive integer solution to the equation $u^2 - mv^2 = 1$. The integer r here depends upon c and, potentially, upon m.

It follows that if $S_2(m)$ is not empty, then it is infinite. From the theory of Pell equations (or Fermat-Pell equations if one likes) these correspond to elements of finitely many recurrence sequences (see Nagell [15] for a nice, affordable exposition of such matters). This fact is also a pretty easy consequence of (10). We note that consideration of the case $m = 39$ demonstrates that the above lemmata do not in fact provide necessary conditions for $S_2(m)$ to be empty. A routine computation shows that for $2 \le m \le 50$, $S_2(m)$ is infinite precisely for

$$m \in \{3, 7, 11, 12, 15, 19, 23, 27, 28, 31, 35, 43, 44, 47, 48\}.$$

It is also not hard to provide sufficient conditions for $S_2(m)$ to be nonempty. An example of such a result is as follows:

Lemma 6. *If* $p \equiv 3 \pmod 4$ *is prime then* $S_2(p)$ *is infinite.*

Proof. From a venerable (after citing Euler, we can hardly call this old!) result of Petr [16], precisely one of

$$x^2 - py^2 = -2 \quad \text{or} \quad x^2 - py^2 = 2$$

is solvable in integers x and y. This implies, in either case, that $S_2(p)$ is infinite. □

We leave to our gentle reader the (rather nontrivial) task of deriving *necessary* conditions for $S_2(m)$ to be nonempty. In the next section, we will observe that the behavior of $S_k(m)$ is dramatically different when $k \geq 3$.

5 Thue equations

A famous theorem of the Norwegian mathematician Axel Thue [23] asserts, if θ is an algebraic number of degree $k \geq 3$ and $\epsilon > 0$, that the inequality

$$\left| \theta - \frac{x}{y} \right| < \frac{1}{|y|^{k/2+1+\epsilon}}$$

has at most finitely many solutions in integers x and y with $y \neq 0$. Note that, if $\theta = \sqrt[k]{m}$, then we have the algebraic identity

$$|x^k - my^k| = y^k \left| \theta - \frac{x}{y} \right| \cdot \left(\left(\frac{x}{y} \right)^{k-1} + \theta \left(\frac{x}{y} \right)^{k-2} + \cdots + \theta^{k-1} \right).$$

If m is not a perfect k-th power and $c \neq 0$ is an integer, then it follows that the equation

$$x^k - my^k = c \tag{11}$$

has at most finitely many solutions in integers x and y. Such equations are nowadays termed *Thue equations* (a nice exposition of this active area of research may be found in the book of Fel'dman and Nesterenko [5]).

In our situation, this immediately implies that $S_k(m)$ is finite for each fixed integer $k \geq 3$. As we shall see in the next two sections, it is possible to "effectively" determine each such $S_k(m)$, (and to derive an upper bound upon k that depends solely upon m).

6 Linear forms in logarithms

The sets $S_k(m)$ are defined by equalities of the shape

$$u^k - mv^k = \pm \frac{m}{d} k$$

for u and v positive integers. In all cases, we thus have

$$\left| u^k - mv^k \right| \leq mk.$$

It follows that

$$\left| m^{-1} \left(\frac{u}{v} \right)^k - 1 \right| \leq \frac{k}{v^k}$$

and hence the *linear form in logarithms* $|k \log(u/v) - \log(m)|$ is "small". A classical result of Gel'fond [7] (extending his work on Hilbert's 7th problem) indicates that, for any given nonzero algebraic numbers $\alpha_1, \alpha_2, \beta_1$ and β_2, with $\log \alpha_1$ and $\log \alpha_2$ linearly independent over the rationals, we have

$$|\beta_1 \log \alpha_1 - \beta_2 \log \alpha_2| \neq 0$$

and, moreover, provides lower bounds for such a form. Applying state-of-the-art versions of these bounds, say those due to Laurent, Mignotte and Nesterenko [11], we may conclude, as in Theorem 3 of Mignotte [13], that, in our situation, $k < 600$ (if $m = 2$) and, more generally, as in Theorem 2 of [13], that

$$k < 10676 \log m.$$

It follows that

$$\bigcup_{k=3}^{\infty} S_k(m)$$

is finite. In the remaining sections, we will describe a strategy for explicitly determining this set and illustrate it in the cases $m = 2$ and $m = 3$.

7 Solving the remaining equations

For small values of k, it is possible to use standard computational techniques based upon lower bounds for linear forms in logarithms, combined with lattice basis reduction, to solve the Thue equations that occur (a reasonably accessible book which covers this field is that of Smart [20]). If, however, k is moderately large, this becomes computationally infeasible due to the difficulty in finding systems of independent units in the ring of integers of $\mathbb{Q}(\sqrt[k]{m})$. (For instance, it is an interesting challenge to find the fundamental units in, say, $\mathbb{Q}\left(\sqrt[101]{2}\right)$.)

In our situation, though, we are able to find local obstructions to solvability (i.e. appropriate moduli m so that the equations have no admissible solutions modulo m) for virtually all values of k under consideration, obviating the need for extensive computations. For the sake of simplicity, let us restrict our attention to the case $m = 2$ (where, as mentioned previously, we may assume $k < 600$). Here the equations to be studied are of the shape

$$x^k - 2y^k = \pm 2^\delta k$$

where $\delta \in \{0, 1\}$. Lemma 4 also allows us to suppose that k is odd. For each such k, we consider primes of the form $p = 2nk + 1$ for $n \in \mathbb{N}$. For these p, there are at most $(2n+1)^2$ values of $x^k - 2y^k$ modulo p. If none of these are congruent to $\pm k$ or $\pm 2k$ modulo p, we have found a local obstruction to solvability and can thus conclude that $S_k(2)$ is empty. If, for example, $k = 13$, we consider the above equation(s) modulo 53. Noting that $x^{13} \equiv 0, \pm 1, \pm 23 \pmod{53}$, it follows that

$$x^{13} - 2y^{13} \equiv 0, \pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 8, \pm 16, \pm 21, \pm 23, \pm 25 \pmod{53}$$

and hence may conclude that $S_{13}(2)$ is empty. Similar arguments suffice to eliminate all equations (with $k < 600$) except for

$$x^3 - 2y^3 = \pm 3, \quad x^3 - 2y^3 = \pm 6, \quad x^5 - 2y^5 = \pm 10,$$

$$x^7 - 2y^7 = \pm 7, \quad x^7 - 2y^7 = \pm 14 \quad \text{and} \quad x^{11} - 2y^{11} = \pm 22$$

(to verify this, the reader may wish to write her own piece of code; we used Maple [12], being careful to note that the "msolve" routine is somewhat unreliable).

Similarly, when $m = 3$, we easily deal with all the equations encountered, with the exceptions of

$$x^3 - 3y^3 = \pm 3, \quad x^5 - 3y^5 = \pm 15 \quad \text{and} \quad x^7 - 3y^7 = \pm 21$$

(though we have many more k to treat – up to $k = 11728$, including even values).

8 Endgame

As mentioned in the last section, various techniques from Diophantine approximation may be used to solve these remaining equations. The symbolics package Kash [10], for example, has a built-in Thue solver that can handle all the equations under consideration in a matter of minutes on a Sun Ultra. In any case, nowadays it is a routine matter to verify that of the equations which have so far evaded our net, only those with $k = 3$ possess solutions, corresponding to $5^3 - 2 \cdot 4^3 = -3$, $2^3 - 2 \cdot 1^3 = 6$ and $3^3 - 3 \cdot 2^3 = 3$. By way of example, to solve the Diophantine equations $|x^3 - 2y^3| \in \{3, 6\}$, one can appeal to the inequality

$$\left| x^3 - 2y^3 \right| \geq \sqrt{|x|},$$

valid for all integers x and y (see e.g. [1]).

The equation $5^3 - 2 \cdot 4^3 = -3$ maps back to the solution (x, y) to $x^y = y^{2x}$ given in (3). Similarly, $2^3 - 2 \cdot 1^3 = 6$ yields the solution $(x, y) = (2, 16)$, also to $x^y = y^{2x}$. The equation $3^3 - 3 \cdot 2^3 = 3$ which potentially yields a solution to $x^y = y^{3x}$, leads to $a = 27, b = 24$, contradicting the coprimality of a and b.

These arguments suffice to completely solve $x^y = y^{mx}$ for, with a modicum of computation, all values of m up to 40 or so. As m increases, we are faced with the prospect of handling Thue equations of higher and higher degree, an apparently formidable task. We note that $S_k(m)$ can be nonempty for arbitrarily large k (for example, this is always the case for $S_k(2^k - k)$ if k is odd).

The reader might reasonably wonder about a generalization to $x^{ny} = y^{mx}$. This Diophantine equation was studied by Asher Kach, a student in that undergraduate seminar course, and currently a graduate student at the University of Wisconsin – Madison. He and the authors are preparing an article on this subject.

9 Postscript : On local-global principles

A standard heuristic employed in the field of Diophantine problems is that an equation should be solvable over $\mathbb{Q}$ precisely when it is solvable over $\mathbb{R}$ and over p-adic fields $\mathbb{Q}_p$, for all primes p. Such *local-global* or *Hasse* principles are known to be true in various settings, but false in general.

One of the first instances where local-global principles were shown to fail occurs in work of Skolem [19] of 1942, where he demonstrated that the equation $x^3 - 3y^3 = 22$ can be solved modulo p^α for every prime power p^α, but has no solution over the integers.

In our analysis, we encounter numerous equations for which a like conclusion holds, of degree up to (at least) 19. Such is the case, for instance, for the equations

$$x^3 - 5y^3 = 15, \quad x^5 - 2y^5 = 10, \quad x^5 - 3y^5 = 15, \quad x^7 - 2y^7 = 7,$$

$$x^7 - 2y^7 = 14, \quad x^7 - 3y^7 = 21 \quad \text{and} \quad x^{11} - 2y^{11} = 22.$$

To actually prove that the Hasse principle fails in these instances is an interesting exercise which we leave to our loyal (and, at this stage, perhaps rather fatigued) reader!

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