

Quotients of sums of distinct powers of three

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Enumerative Algebraic and Geometric Combinatorics
2015 Fall Central Section Meeting of AMS
Chicago, October 3, 2015

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Based on the abstract, this doesn't seem like the other talks in the Special Session. I assure you that it is combinatorial, enumerative, algebraic and geometric, just not all at the same time.

My interest in the problem of the title comes from the usual “middle third” Cantor set:

$$C = \left\{ \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} \mid \alpha_n \in \{0, 2\} \right\}.$$

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Here is the proof. For $u \in [0, 1]$, define $x, y \in C$ by

$$u = \sum_{n=1}^{\infty} \frac{d_n}{3^n}, \quad d_n \in \{0, 1, 2\}; \quad x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}, \quad y = \sum_{n=1}^{\infty} \frac{\beta_n}{3^n};$$

$$\text{where } d_n = 0 \implies (\alpha_n, \beta_n) = (0, 0),$$

$$d_n = 1 \implies (\alpha_n, \beta_n) = (2, 0),$$

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$$d_n = 1 \implies (\alpha_n, \beta_n) = (2, 0),$$

$$d_n = 2 \implies (\alpha_n, \beta_n) = (2, 2).$$

In each case, $d_n = \frac{\alpha_n + \beta_n}{2}$, so $u = \frac{x+y}{2}$.

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Using methods which are indistinguishable from questions on a graduate qualifying exam in analysis, we have analyzed the image of C^2 under functions such as $f(x, y) = x + \lambda y$, $f(x, y) = xy$, $f(x, y) = x^2y$, $f(x, y) = x/y$.

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Since $C \subset [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, it is easy to see that $u \in (\frac{1}{3}, \frac{4}{9})$ cannot be written as xy with $x, y \in C$.

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We have proved however, that u can be written as x^2y with $x, y \in C$, so u is a product of *three* elements of the Cantor set. It also turns out that the measure of the set of products of two elements is, to the first 8 digits, .80955358...

What's relevant to this talk is what happens with quotients:

$$\left\{ \frac{u}{v} \mid u, v \in \mathbb{C} \right\} = \bigcup_{m=-\infty}^{\infty} \left[\frac{2}{3} \cdot 3^m, \frac{3}{2} \cdot 3^m \right] = \left(\bigcup_{m=-\infty}^{\infty} \left(\frac{3}{2} \cdot 3^m, 2 \cdot 3^m \right) \right)^c$$

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$$x = \sum_k \frac{2}{3^{a_k}}, \quad a_k \geq a_{k-1} + 2;$$
$$4x = \sum_k \frac{2}{3^{a_k-1}} + \frac{2}{3^{a_k}}$$

The sum can be finite or infinite, and the gap in exponents ensures no overlap.

The theorem about $4x$ made me start thinking about ratios of left-hand endpoints of the sets whose intersection becomes the Cantor set:

$$\frac{\frac{2}{3^{a_1}} + \cdots + \frac{2}{3^{a_r}}}{\frac{2}{3^{b_1}} + \cdots + \frac{2}{3^{b_s}}}$$

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After cancelling the 2's, pulling out factors of 3 and clearing the denominators, we get

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$$\frac{3}{2} * 3^n > 3^n + 3^{m_{r-1}} + \cdots + 3^{m_1} + 1 > 3^n,$$

so

$$\frac{3^{m_r} + \cdots + 3^{m_1} + 1}{3^{n_s} + \cdots + 3^{n_1} + 1} \in \left(\frac{2}{3} \cdot 3^{m_r - n_s}, \frac{3}{2} \cdot 3^{m_r - n_s} \right).$$

To sum up,

$$N = \frac{3^{m_r} + \cdots + 3^{m_1} + 1}{3^{n_s} + \cdots + 3^{n_1} + 1} \implies$$
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The first exception is $N = 529$, provably. I found other exceptions: 592, 601, 616 and expected a deluge after that. Sakulbuth programmed an algorithm I’ll describe below, and found that the only other exception less than 10^5 is 5368.

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There is some difficulty in describing which integers can occur.

That completes the literature, as far as I have been able to find it.

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Our question then transforms into writing

$$N = \frac{p(3)}{q(3)}, \quad \text{where } p, q \in A.$$

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Otherwise, N appears *the hard way*.

Time for some examples. Crucial observation: for $p \in A$, the summation form of $p(3)$ gives its (unique) standard base 3 representation. In the theorems below, assume $p, q \in A$:

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$$1 - x^2 + x^3 = \frac{1 + x + x^5}{1 + x + x^2}.$$

$$22 = \frac{22 * 37}{37} = \frac{3^6 + 3^4 + 3 + 1}{3^3 + 3^2 + 1}.$$

However, $x^3 + x^2 + 1$ is not a factor of $x^6 + x^4 + x + 1$. In fact, THM: 22 can only appear the hard way.

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So what is known so far? There is a natural heuristic. Suppose $3^{r-1} < N < 3^r$. There are 2^k integers of the form $\sum 3^{a_i}$ with $0 \leq a_1 < a_2 < \dots \leq k-1$, giving 2^k multiples of N less than 3^{r+k} . Each of these has at most $r+k$ digits in base 3, and the “probability” that every digit is 0 or 1 is $(\frac{2}{3})^{r+k}$.

Still, $2^k \cdot (\frac{2}{3})^{r+k} = (\frac{2}{3})^r \cdot (\frac{4}{3})^k \rightarrow \infty$ as $k \rightarrow \infty$, so each N should be represented.

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What's wrong? Two things. This works even when N has been forbidden. More significantly, there is considerable structure in the digital representations of the product. They are not random.

Time for some proofs.

Lemma

Suppose

$$p(x) = \sum_{i=0}^m a_i x^i, \quad q(x) = \sum_{j=0}^n b_j x^j, \quad a_i, b_j \in \{0, 1, 2\}.$$

If $p(3) = q(3)$, then $p = q$.

Proof.

If $p(3) = q(3) = t$, then there are two valid standard base 3 representations of t :

$$t = \sum_{i=0}^m a_i 3^i = \sum_{j=0}^n b_j 3^j, \quad a_i, b_j \in \{0, 1, 2\}.$$

These must be formally identical, hence $a_i = b_i$ for all i . □

Theorem

$$4 = \frac{p(3)}{q(3)}, \quad p, q \in A \implies p(x) = (1+x)q(x).$$

Proof.

Let $r(x) = (1+x)q(x)$; the coefficients of r are in $\{0, 1, 2\}$. Since $r(3) = 4q(3) = p(3)$, it follows from the Lemma that $r = p$. \square

The similar theorem when $N = 7$ compares $(1+x^3)q(x)$ and $(1+x)p(x)$, which are equal at 3 and both have coefficients in $\{0, 1, 2\}$. For 19, $(1-x^2+x^3)q(x)$ has coefficients in $\{-1, 0, 1, 2\}$, but p has coefficients in $\{0, 1\}$ and an iterative argument shows that $-1, 2$ never occur. One can find similar results for integers of the following kind:

$$3^m + 1, \quad 3^{mr} + 3^{(m-1)r} + \dots + 1, \\ 3^{2mr} - 3^{(2m-1)r} \pm \dots \pm 1, \quad 3^m - 3^n + 1.$$

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Let's turn to enumeration questions. When can $(1+x)q(x) = p(x)$ for $p, q \in A$? It is not hard to see that we must have

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Ben has looked at this question for $(1+x^2)$, and since you need to consider the even exponents and odd exponents separately, the enumeration is a product of two consecutive Fibonacci numbers of index roughly $n/2$, depending on the parity of n .

What happens with 22? If $22 = \frac{p(3)}{q(3)}$, then $22 \in (\frac{2}{3} \cdot 3^3, \frac{3}{2} \cdot 3^3)$ implies that the degree of p is exactly 3 more than the degree of q . If $q \mid p$, then the quotient $r = p/q$ must be a cubic. But we know a lot about the coefficients of r , since $p, q \in A$, in fact

$$r(x) = x^3 + c_2x^2 + c_1x + 1, \quad c_i \in \{-1, 0, 1\}.$$

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Since $r(3) = 22$, we must have $r(x) = x^3 - x^2 + x + 1$. But now suppose that $q(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \dots \in A$. The coefficients of $p(x) = q(x)r(x)$ must also be in $\{0, 1\}$. Here they are for x, x^2, x^3 :

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A similar theorem hold for $N = 34$ and other larger integers.

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Given N we find r_n , a sum of powers of 3, less than 3^n , so that

$$r_n N = a_n \cdot 3^n + b_n,$$

where $0 \leq b_n < 3^n$ is a sum of powers of 3. The question then is: do we take $r_{n+1} = r_n$ or $r_{n+1} = r_n + 3^n$? The answer depends on $a_n \pmod 3$. If this is 0, we can take 3^n or leave it; if this is 1, we must skip 3^n , if this is 2, we must take 3^n .

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This algorithm determines a directed graph on non-negative integers $< N/2$. Integers not congruent to 0 mod 3 have out-degree 1; multiples of 3 have out-degree 2. We want the component of 0 to contain a vertex which is a sum of powers of 3.

Let's apply it to $N = 22 = [211]_3$: $r_1 = 1$, as always, so
 $a_1 = 7 = [21]_3$ and $b_1 = 1 = [1]_3$. So we pass on 3, and $r_2 = 1$,
 $a_2 = [2]_3$ and $b_2 = 4 = [11]_3$.

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Because of the last digit 2 of $[a_2]_3$, we need $r_3 = 3^2 + r_2 = 10$; $r_3 * 10 = [22011]_3$, so $a_3 = 8 = [22]_3$ and $b_3 = 4 = [011]_3$.

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Again, because of the last digit, we must have $r_4 = 3^3 + r_3 = 37$, $r_3 * 37 = [1010011]_3$, $a_4 = 10 = [101]_3$ and $b_4 = [0011]_3$. We can stop here, because a_4 is a sum of powers of 3, but we could also continue, and find all the possible multipliers for 22.

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When you apply this to 529, it takes about 20 steps to loop back to earlier states without ever hitting a sum of powers of 3. A similar thing happens to 592. As I said, Sakulbuth's program shows that there are only a few others up to 100,000. Anyone know how to deal with this in general?

It's also worth mentioning that if N appears once, it appears infinitely often. Suppose

$$N = \frac{p(3)}{q(3)}$$

and $\deg p, \deg q \leq T$. If $f(x)$ is any polynomial in A for which the gaps in exponents are greater than T , then $fp, fq \in A$, because there are no possible cross-terms. In this case

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It doesn't really fit anywhere, but it is somewhat reasonable to assume that if a number N appears the easy way, then the quotient $\frac{p}{q}$ is always the same. This is false: 551080 (probably not minimal) has two such representations, with p/q equal to:

$$r_1(x) = x^{12} + x^9 - x^7 + 2x^6 - x^5 + x^3 + 1; \quad r_2(x) = r_1(x) + x^6(x-3).$$

This leads into my final topic. How large can the coefficients of $\frac{p}{q}$ be, when $p \mid q$ and $p, q \in A$? Consider

$$1 - x + 2x^2 - 2x^3 + 3x^4 - 3x^5 + 3x^6 - 2x^7 + 2x^8 - x^9 + x^{10} \\ = \frac{1 + x^2 + x^4 + x^7 + x^9 + x^{11}}{1 + x}.$$

By extending the obvious pattern, one can get roughly $\frac{1}{4}$ times the degree. This is optimal for $1 + x$. What about other denominators?

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Finally, what if you just throw in the towel and look at quotients of power series with 0/1 coefficients? Some experimentation suggests that this is the best example:

$$\begin{aligned} \frac{1 + x^2 + x^4 + x^6 + \dots}{1 + x + x^3 + x^5 + \dots} &= \frac{1/(1 - x^2)}{1 + x/(1 - x^2)} \\ &= \frac{1}{1 + x - x^2} = \sum_{n=0}^{\infty} (-1)^n F_{n+1} x^n. \end{aligned}$$

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Thanks also to Steven Klee and Kyle Peterson for organizing this session, and for having the whimsical notion of inviting me to speak.