1. 

Let \( f_n(x) = \frac{x^n}{n!} \).

Then \( f_n(0) = 0 \) for all \( n \)
and, if \( x > 0 \), then \( f_n(x) = 0 \)
if \( x \geq \frac{1}{n} \), i.e., \( n \geq \frac{1}{x} \), so
\( \lim_{n \to \infty} f_n(x) = 0 \).

But \( \frac{d}{dx} f_n(x) = \frac{1}{n} \cdot x^{n-1} \) \( n \cdot n^2 = n \)
(Using integral as area)

So the sequence \( \left( \frac{d}{dx} f_n(x) \right) \)
\( = (n) \) is unbounded.

2. \( \sqrt[3]{1-x} \)

Let \( f_n(x) \) be the power series
\[ 1 + ax + \frac{a(a-1)x^2}{2!} + \frac{a(a-1)(a-2)x^3}{3!} + \ldots \]

The \( n \)th term is \( \frac{a(a-1)\ldots(a-n+1)x^n}{n!} \).

So \( \left| \frac{f_{n+1}}{f_n} \right| = \left| \frac{a-n}{n+1} \right| \cdot \left|x\right| = \frac{n-a}{n+1} \cdot |x| \to |x| \).

Thus, if \( |x| < 1 \), the series converges.
Also, differentiating term-by-term,
\[ f_n(x) = a + \frac{a(a-1)}{1!} x + \frac{a(a-1)(a-2)}{2!} x^2 + \ldots \]

\( = a \cdot f_{n-1}(x) \)

\( (f_{n-1}(x))(1+x) \)

\( = (1+x)(1 + (a-1)x + (a-1)(a-2)x^2 + \ldots) \)

The coefficient of \( x^n \) is
The sum of the coefficients of \( x^{n-1} \) and \( x^n \) in \( f_{n-1}(x) \):
\( \frac{(a-1)(a-2) - (a-a-n)(a-1)\ldots(a-n)}{(n-1)!} \) + \( \frac{a(a-1)\ldots(a-n)}{n!} \)

In the original Sherlock Holmes Stories, The Villain Moriarity
was a mathematician.

"At the age of twenty-one, he wrote a treatise on the
binomial theorem..."
We use parts.
\[
\begin{align*}
\int x \log x \, dx &= x \log x - \int \frac{dx}{x} = x \log x - \log x + C \\
&= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C \\
\end{align*}
\]

So
\[
\int x \log x \, dx = \frac{x^2 \log x}{2} - \frac{x^2}{4} + C
\]

Thus
\[
\int x \log x \, dx = \left( \frac{x^2 \log x}{2} - \frac{x^2}{4} + C \right)
\]

Let \( a_n = \log x \), then
\[
A_n = 4e^{-3/4} \approx 1.89078
\]

4a. \( \frac{x}{2} \leq f(x) \leq 2x \)

Suppose \( x \in [0, 1] \). If \( x = 1 \), then
\[
f_n(1) = f(1) = \text{constant} = \text{constant}
\]

where \( \lim_{n \to \infty} f_n(1) = f(1) \).

If \( 0 \leq x < 1 \), then \( f_n(x) = f(x^n) \),

so \( \frac{1}{2} x^n \leq f_n(x) \leq 2x^n \)

so as \( n \to \infty \), \( f_n(x) \to 0 \).

4b. If \( x \in [0, 1] \), then

\( 0 \leq f_n(x) \leq 2x^n \) implies that

\[
\sum_{n=1}^{\infty} f_n(x) \text{ converges by the limit comparison test.}
\]

4c. For \( 0 \leq x \leq \frac{1}{n} \),

\[
\sum_{n=1}^{\infty} \left( \sum_{n=1}^{\infty} f_n(x) \right) \leq \sum_{n=1}^{\infty} 2x^n = \frac{2}{2^n} = 2 \cdot \frac{1}{2} \frac{1}{2n}, \text{ so for } x \in [0, \frac{1}{n}]
\]

choose \( N \) s.t. \( \frac{1}{2^n} \leq \varepsilon \). Then

\[
\sum_{n=N+1}^{\infty} f_n(x) \leq \varepsilon \text{ for all } x \in [0, \frac{1}{n}]
\]

However, for any fixed \( N \),

\[
\sum_{n=N+1}^{\infty} f_n(x) \geq \frac{1}{2} \sum_{n=N+1}^{\infty} x^n = \frac{1}{2} \cdot \frac{x^{N+1}}{1-x}, \text{ so for any given } \varepsilon, \text{ we can choose } x \text{ close to } 1 \text{ to make the sum larger than } \varepsilon.
\]

1 so that \( \frac{1}{2} \cdot \frac{x^{N+1}}{1-x} > \varepsilon \). This means that convergence is not uniform on \([0, 1]\).

5a. \( \sum_{n=0}^{\infty} \frac{10^n \cdot n!}{(2n)!} \)

Don't use Stirling, use ratio:

\[
\frac{10^{n+1} \cdot (n+1)!}{(2n+2)!} = \frac{10 \cdot n!}{(n+1)(2n+1)} = \frac{5}{2n+1}
\]

so it converges

5b. \( \sum_{n=0}^{\infty} \frac{(424)^n}{n!} \)

Again ratio:

\[
\frac{(424)^{n+1}}{(n+1)!} = \frac{424}{n+1} \to 0
\]

Here, we can identify this as \( e^{424} \) from the power series for \( e^x \).
5a. \[ \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 10^n}{2^n n^2} \]

Ratio test will work, as will root test:

\[
\left| \frac{(-1)^n \cdot 10^n}{2^n n^2} \right|^\frac{1}{n} = \frac{10}{2^n} \to 0
\]

so it converges.

5b. \[ \sum_{n=2}^{\infty} \frac{\cos(e^n)}{n^{1.01}} \]

Don't know much about \( \cos(e^n) \) except that it's in \([-1,1] \).

\[
\left| \frac{\cos(e^n)}{n^{1.01}} \right| \leq \frac{1}{n^{1.01}}
\]

converges by comparison with \( \frac{1}{n^{2}} \).

6a. \[ f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{1.4} n^2} \]

\[ C_n = \frac{1}{n^{1.4} n^2}, \quad C_n^{1/2} = \frac{1}{n^{1.4} n} \to \frac{1}{4} \]

so the radius of convergence is 4. Converges \( (x < 4) \)

Diverges \( (x > 4) \).

6b. \[ \sum_{n=1}^{\infty} \frac{\Delta x^{n-1}}{n^{1.4} n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(x/4)^n}{n!} = \frac{1}{4} \cdot \frac{1}{1 - \frac{x}{4}} = \frac{1}{4 - x} \]

\[ \text{as } x = n - 1 \]

6c. \[ f(1) = \sum_{n=1}^{\infty} \frac{1}{n^{1.4} n^2} \]

\[
f(1) = \int_0^1 f'(x) \, dx = \int_0^1 \frac{1}{4-x} \, dx = -\log(4-x)|_0^1 = \log 4 - \log 3 = \log \frac{4}{3}.
\]

Bonus to 5b.

Suppose you want to use Stirling's formula and the root test:

\[
(n^{424})^{1/4} = \frac{424}{(n^{1.01})}
\]

Since \[ \frac{n!}{n^{(n+1)/2}} \to 1 \text{ as } n \to \infty \]

\[
\frac{(\pi n/2)^{n/2}}{e^n} \to 1 \quad \text{as } n \to \infty
\]

as \( n \to \infty \) so you can also say that

\[
\frac{424}{(n^{1.01})^{1/4}} \to 0.
\]

A cruder (estimate is)

\[ n! = (1 \cdot n) \cdot (2 \cdot n-1) \cdots (\frac{n}{2} \cdot n-\frac{n}{2}) \]

\[ > n \cdot n \cdots n = n^{n/2}, \quad \text{so}
\]

\[ (n!)^{1/4} > n^{n/8} \to \infty \]